Dynamics of Differential Equations on Invariant Manifolds

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Abstract:-

The simplification resulting from reduction of dimension involved in the study of invariant manifolds of differential equations is often difficult to achieve in practice. Appropriate coordinate systems are difficult to find or are essentially local in nature thus complicating analysis of global dynamics. This paper develops an approach which avoids the selection of coordinate systems on the manifold. Conditions are given in terms compound linear differential equations for the stability of equilibria and periodic orbits. Global results include criteria for the nonexistence of periodic orbits and a discussion of the nature of limit sets. As an application, a global stability criterion is established for the endemic equilibrium in an epidemiological model. 2000 Academic Press Key Words: differential equations; invariant submanifolds; Bendixson conditions; periodic orbits; compound matrices; compound equations.Letx[f(x) beaC1 functionwith opendomaininRn andrangeinRn and let x(t)=.t(x) be the solution of x v= f(x) (1.1) such that x(0)=x. Ifx[g(x) is aRm-valued C1 function with the same domain and 7 denotes the subset of Rn where g(x)=0, then 7 is a

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manifold of dimension n&m if rk g x(x)=m when g(x)=0. It is an invariant manifold with respect to (1.1) if g(x)=0 implies g(.t(x))=0 for all t such that .t(x) exists. For notational convenience, the situation where there is no invariant manifold in consideration will be denoted as the case m=0. An important special case occurs when g(.t(x))=g(x) for all x and every manifold g(x)=c is invariant. The system (1.1) is then said to have m first integrals. In many scientific models, first integrals appear as conservation laws for quantities such as energy or population and provide important tools in analysis of the dynamics. The existence of first integrals effectively reduces the dimension of the system and the reduced problem may be studied by changes of variable. However, the changes of variable may be difficult to implement or may not be optimal for the study. This paper investigates the flow due to (1.1) on an invariant manifold without resort to a reduced system. Invariant manifolds also arise as the stable, unstable or centre manifolds associated with equilibria or other invariant structures. Frequently, only the existence of these manifolds and the nature of the dynamics nearby are known so techniques which analyze the dynamics in the manifold with incomplete information are desirable. Invariant manifolds may also arise from application of the LaSalle invariance principle and related results. For example, if x[v(x) isC1, real and such that g(x)=v$(1.1)(x)=v x(x)f(x) satisfies g(x)0 in the domain of f, then every non-wandering point in general and every equilibrium, periodic orbit and omega limit set in particular, lies in the set where g(x)=0. This is an (n&1)-dimensional manifold if rkg x(x)=1. All of the interesting dynamics then occur in this manifold and it is useful if projects such as stability analysis, existence or non-existence of periodic orbits and so forth can be conducted without tedious calculations in coordinate systems on the manifold. When n=2, it is well known that (1.1) has no non-equilibrium periodic solution whose orbit lies entirely in a simply connected region where div f{0. This is no longer true when n>2. However it is shown by Demidowitsch [2] that, if n=3 and (1.1) has a first integral, the Bendixson condition div f{0 in a simply connected region precludes periodic orbits there. In the case that n>2 and m=0, Muldowney [15] gives a generalization of Bendixson's criterion and shows that if the flow of (1.1) diminishes some measure of 2-dimensional surface area in a simply connected region, then the region does not contain a periodic orbit. It is shown by Li in the Ph.D. dissertation [7] that there is a relaxation in these conditions in the presence of first integrals. Essentially, if (1.1) has m independent first integrals and the flow decreases (m+2)-dimensional surface areas, then there are no periodic orbits. In the spirit of Demidowitsch, if (1.1) has m=n&1 first integrals, then div f{0 is still a valid Bendixson condition.

Li also investigates similar questions relative to invariant affine manifolds in [8] and discusses some biological implications in [7, 9]. M. Feckan pointed out to us in a private communication that the original proof of Demidowitsch in [2] contained a gap and a correction was given by Feckan. As we remarked earlier, Demidowitsch's result in [2] follows from Theorem 5.2 of the present paper. The discussion in this paper is applicable to any invariant manifold but, when it is not associated with first integrals, some information on the dynamics near the manifold is required. When the function g is explicitly known, the required behaviour may be computed from g and f as shown in Section 3. Section 7 considers a 4-dimensional epidemiological model where the dynamics of interest occur in an invariant manifold of dimension 3. A new criterion for the global stability of the endemic equilibrium is established using the techniques developed in the earlier sections.

1. INTRODUCTION

Letx[f(x) beaC1 functionwith opendomaininRn andrangeinRn and let x(t)=.t(x) be the solution of x v= f(x) (1.1) such that x(0)=x. Ifx[g(x) is aRm-valued C1 function with the same domain and 7 denotes the subset of Rn where g(x)=0, then 7 is a

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2.2. The subsymbol. Our first main theorem gives a new contact differential invariant. Theorem A. If δ is not contact-resonant, then there exists a unique K(M)-equivariant linear map sσkλ,μ : Dkλ,μ(M) → Σk−1, 2(k−1) δ (M) whose restriction to Dk−1 λ,μ (M) is fσk−1, 2(k−1) λ,μ . We refer to sσkλ,μ as the contact subsymbol. We will give an explicit formula for it in Proposition 7.4. It may be regarded as a K(M)-equivariant projection from Dkλ,μ(M) to Fδ− k−1 ℓ+1 (M). We remark that in the general self-adjoint case, where λ+μ = 1 and k is arbitrary, the existence of such a differential invariant is obvious. Indeed, for T in Dkλ,μ(M), the operator T −(−1)kT∗ is in Dk−1 λ,μ (M), and so can be projected to Σk−1, 2(k−1) δ (M). Since F− 1 ℓ+1 (M) is equivalent to K(M), the case that k = 2 and μ = λ is of particular interest, as there the differential invariant given by the contact subsymbol may be viewed as a contact vector field. In other words, for all λ ∈ C, the subsymbol sσ2λ,λ defines a K(M)-equivariant projection from D2λ,λ(M) to K(M). In order to give an intrinsically defined and manifestly contact-invariant formula for sσ2λ,λ, observe that any second order differential operator can be represented as a linear combination of compositions of vector fields. On contact manifolds, contact vector fields and tangential vector fields are intrinsically distinguished. Thus we are led to express an arbitrary second order operator on Fλ(M) as a linear combination of operators of the form

(2)

T = Lλ(Xϕ1)◦Lλ(Xϕ2) + Lλ(Xϕ3)◦Lλ(Y1) + Lλ(Y2)◦Lλ(Y3) +Lλ(Xϕ4) + Lλ(Y4) + f, where the ϕi are arbitrary contact Hamiltonians, the Yi are tangential vector fields, and f is a function. Theorem B. The subsymbol sσ2λ,λ(T) is the contact vector field 1 2Xϕ1,Xϕ2————————————−ℓ+1 ℓ+2λ− 1 2XL(Y1)ϕ3 + 1 2πY2,Y3————————————+ Xϕ4,where L(Y1)ϕ3 denotes the natural action of Y1 on the − 1 ℓ+1-density ϕ3. Let us comment on this formula. It only contains natural operations, so it is clearly contactinvariant. Conversely, equivariance with respect to K(M) (in fact the affine subalgebra suffices) implies that sσ2λ,λ(T) has to be of the form c12Xϕ1,Xϕ2————————————+ c13XL(Y1)ϕ3 + c23πY2,Y3————————————+ c4Xϕ4,where the c's are constants. The normalization condition onD1λ,λ gives c4 = 1. Skew-symmetrizing the expression then yields c12 = c23 = 1 2. Symmetrizing the expression implies that c13 vanishes in the self-adjoint case λ = 1 2, but its exact form must be deduced by computation.

The main content of the theorem is that the formula is actually well-defined. Indeed, the choice of the ϕi and Yi in (2) is not unique: one can write an operator as a linear combination of such expressions in many different ways. However, the formula is independent of the choice. Moreover, the uniqueness statement of Theorem A implies that, up to a scalar, this is not true for any other choice of the c's.

2.3. The fine filtration. In order to explain the significance of our next theorem, consider the following arrangement of the fine symbol modules (we have omitted M and δ for clarity):

Σ6,12

Σ5,10 Σ6,11

Σ4,8 Σ5,9 Σ6,10 Σ3,6 Σ4,7 Σ5,8 Σ6,9 ··· Σ2,4 Σ3,5 Σ4,6 Σ5,7 Σ6,8

Σ1,2 Σ2,3 Σ3,4 Σ4,5 Σ5,6 Σ6,7

Σ0,0 Σ1,1 Σ2,2 Σ3,3 Σ4,4 Σ5,5 Σ6,6

Observe that the graded module of Sk δ (M) defined by the bifiltration Dk,d λ,µ(M) is the "vertical" sum grSk δ (M) = M k≤d≤2k Σk,d δ (M). The graded module of Pdλ,µ(M)/Pd−1 λ,µ (M) is the "slope −1" sum grPdλ,µ(M)/Pd−1 λ,µ (M)= M ⌈d 2 ⌉≤k≤d Σk,d δ (M). The content of our next theorem is that there exists aK(M)-invariant filtration that strengthens the filtration Pdλ,µ(M). The graded modules of its subquotients are the "slope −1 2" sums. Theorem C. Assume that δ is not contact-resonant. Then there is a unique K(M)-invariant filtration of Dλ,µ(M), D(0) λ,µ(M) ⊂···⊂D(b) λ,µ(M) ⊂D(b+1) λ,µ (M) ⊂··· , such that the graded module of D(b) λ,µ(M) is given by grD(b) λ,µ(M) = M 2d−k≤b Σk,d δ (M). For example, grD(6) λ,µ(M)/D(5) λ,µ(M) = Σ6,6 δ (M) ⊕ Σ4,5 δ (M) ⊕ Σ2,4 δ (M), as indicated by the boundaries in the diagram above. We will define D(b) λ,µ(M) via the projective quantization: see Section 6.3.

EQUILIBRIA AND PERIODIC ORBITS

Suppose that x0 #7. Iff(x0)=0, the equilibrium is stable hyperbolic with respect to the dynamics on 7 if every eigenvalue *j of f x (x0) corresponding to the invariant subspace T x0 satisfies Re *j<0; see Szlenk [19] page 58. Then all orbits in 7 near x0 are attracted to x0 exponentially in time. Similarly x0 #7 is |-periodic if .(t+|)=.(t) for some minimal |>0, where .(t)=.t(x0). Then x0 is a fixed point of the diffeomorphism x[.|(x). Since .| x (x0) T x0=T .|(x0)=T x0, T x0 is an invariant subspace of .| x (x0). Moreover .* (0)#T x0 and .| x (x0) .* (0)=.* (|)=.* (0) so +n=1 is an eigenvalue of .| x (x0)|Tx0. If all remaining eigenvalues +j of this matrix satisfy |+j|<1, then the periodic orbit #+(x0) isstable hyperbolic; see for example Szlenk [19], Section 1.9. Each orbit in 7 near #+(x0) is then attracted exponentially to #+(x0) with a certain phase with respect to .(t); see Coppel [1] page 82 and Hartman [5] page 255. We will prove the following theorems.

Theorem 4.1. Let x0 #7 be an equilibrium of (1.1). (a) A sufficient condition for x0 to be stable hyperbolic with respect to the dynamics of (1.1) on 7 is that

z v=\f x

[m+1] (x0)&&(x0) I+z (4.1)

be asymptotically stable. (b) The sufficient condition of (a) is also necessary if the system u* =&N*(x0) u is stable.

Theorem 4.2. Let x0 #7 be |-periodic with .(t)=.t(x0). (a) A sufficient condition for #+(x0) to be stable hyperbolic with respect to the dynamics of (1.1) on 7 is that

z v=\f x

[m+2] (.(t))&&(.(t))I+z (4.2)

be asymptotically stable. (b) The sufficient condition of (a) is also necessary if the system u* =&N*(.(t))u is stable. Remark. When g is a first integral, N(x)=0 and the condition (a) of each of these theorems is both necessary and sufficient for the hyperbolic stability considered.

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These theorems provide a mechanism for testing the stability of equilibria and periodic orbits with respect to the dynamics of (1.1) on 7 without the use of any particular coordinate system. The matrix f x [k], k=m+1,m+2, are concrete entities and &(x0) may always be calculated if x0, g(x) are known explicitly. Even this information is not always necessary; for example if g(x) is a system of first integrals then &(x)=0 for all x. More generally, if it is known that 7 does not attract nearby orbits, it can often be inferred that &(x)-0 in (4.1), (4.2). If we consider V(z)=|z| as a Lyapunov function in (4.1), k=m+1 and (4.2), k=m+2, we find that V4 (+(f x [k])&&) V, where +(A)= limh0 1 h(|I+hA|&1) is the Lozinski@$ measure of the square matrix A and | } | denotes both the vector norm and the matrix norm it induces; see Coppel [1] page 41. When | } | is the l, l1 or l2 norm on R(n k), +(f x [k]) is, respectively, the expression (i), (ii), or (iii),

(i) sup (i) {fi1 xi1

+}}}+ fik xik

+7j (i)\} fj xi1}+}}}+} fj xik}+=,

(ii) sup (i) {fi1 xi1

+}}}+ fik xik

+7j (i)\}fi1 xj}+}}}+}fik xj}+=, (4.3)

(iii) :1+}}}+:k, where the suprema are taken over all k-tuples (i)=(i1, ..., ik), 1i1<}}}<ikn, and:1-:2-}}}-:n are the eigenvalues of 1 2(f x *+ f x). Thus we find the following corollaries.

Corollary 4.3. An equilibrium x0 #7 is asymptotically stable with respect to the flow of (1.1) on 7 if +\f x [m+1] (x0)+&&(x0)<0.

Corollary 4.3. An |-periodic solution .(t)#7 is orbitally asymptotically stable with asymptotic phase with respect to the flow of (1.1) on 7 if || 0 _+\f x [m+2] (.(s))+&&(.(s))&<0. Proof of Theorem 4.1. The eigenvalues of f x [m+1](x0) are *i1+}}}+ *im+1, 1i1<}}}< im+1n, where *1, ..., *n are the eigenvalues of f x (x0). The asymptotic stability of the constant coefficient system (4.1) is therefore equivalent to Re(*i1+}}}+*im+1)&&(x0)<0. In particular, since

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&(x0)=*1+}}}+*m from Proposition 3.5, Re *j=Re(*1+}}}+*m+*j)& &(x0)<0, j=m+1, ..., n, if (4.1) is asymptotically stable. Thus all eigenvalues *j of f x (x0) corresponding to the invariant subspace T x0 satisfy Re*j<0 and x0 is stable hyperbolic with respect to the flow on 7 as asserted in (a); see Szlenk [19] page 58, Theorem 1.7.2. To prove part (b), note that &*1, ..., &*m are the eigenvalues of &N*(x0) and that Re *i-0, i= 1,..., m, if the system in (b) is stable. Therefore, Re(*i1+}}}+*im+1)&&(x0)= Re(*i1+}}}+*im+1&*1&}}}&*m)<0 if Re*j<0, j=m+1, ..., n, establishing the asymptotic stability of (4.1) as asserted when x0 is stable hyperbolic with respect to the flow on 7. K Proof of Theorem 4.2. First we note that y* = f x (.(t)) y is an |-periodic system. The eigenvalues +1, ..., +n of .| x (x0) are the Floquet multipliers of this system. As remarked previously, T x0 is an invariant subspace of .| x (x0); the multipliers +m+1, ..., +n corresponding to this subspace are thus the eigenvalues of .| x (x0)|T x0. Since +n=1, we must show that the asymptotic stability of (4.2) implies |+j|<1, j=m+1, ..., n&1 to deduce the hyperbolic stability of #+(x0). With U(t) as in Proposition 3.2, U*(t) g x (.(t)) .t x (x0)=g x (.(0)) from that proposition. Since x0=.(0) =.(|), g x (x0) .| x (x0)=U*&1(|) g x (x0). (4.4) Referred to an orthogonal basis [u1, ..., un] of Rn where [un&m, ..., un] span T x0, the matrices in (4.4) have the form g x (x0)=[Gm_m 0m_(n&m)], .w x (x0)=_ Am_m C(n&m)_m 0m_(n&m) B(n&m)_(n&m)& since the row space of g x (x0) is orthogonal to T x0. The eigenvalues of A are +1, ..., +m and those of B are +m+1, ..., +n and (4.4) implies +1 }}} +m=det U*&1(|)=exp\|| 0 &(.)+. (4.5) Now the system (4.2) is |-periodic. Its Floquet multipliers are eigenvalues of

m+2 .| x

(x0) exp\&|| 0

&(.)+,

which are :(i)=+i1 }}}+im+2 exp(&-| 0 &(.))=+i1 }}}+im+2+1 }}}+m, 1i1< }}}< im+2n, from (4.5). The asymptotic stability of (4.2) implies

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|:(i)|<1. In particular, since +n=1, |+j|=|+1 }}}+m+j +n+1 }}}+m|<1, j=m+1, ..., n&1, which implies part (a) of Theorem 4.2. To prove part (b), note that the hyperbolic stability of #+(x0) implies |+j|<1, j=m+1, ..., n&1, which in turn implies |:(i)|<1, and hence the asymptotic stability of (4.2) as asserted, when u* =&N*(.(t))u is stable. This follows from the fact that the Floquet multipliers of this system are 1+1, ..., 1+m and its stability implies |1+j|1 and therefore |+j|-1, j=1, ..., m. K Theorem 4.2 reduces, when m=0, to a result of Muldowney [14] which in turn generalizes a result of Poincare that, when n=2, an |-periodic solution .(t) of (1.1) is orbitally asymptotically stable with asymptotic phase if -| 0 div f(.(t))dt<0, which is equivalent to the asymptotic stability of z* = f x [2] (.(t))z since f x [2] =div f when n=2; see Coppel [1] page 85. The present theorem is motivated by a result of Li [8] in which x[g(x) is an affine function and the m_m matrix N(x)=:(x) I where :(x) is real and, in the notation of this paper, &(x)=m:(x).

It was shown by R. A. Smith [18] that his Bendixson condition for dissipative systems in Rn has an even stronger implication than the nonexistence of periodic orbits other than equilibria. The alpha or omega limit set of any precompact semi-orbit in such a system is a single equilibrium. Li and Muldowney [13] extend this result to general systems in Rn satisfying their BendixsonDulac conditions and further show that the Hausdorff dimension of any compact invariant set in such a system is at most 1. McCluskey and Muldowney [14] give an elementary proof that the classical Bendixson condition for planar systems implies that every bounded solution converges to an equilibrium. Here we will prove a similar assertion for systems that satisfy a Bendixson condition (5.2), (5.3), (5.4) on an invariant manifold 7.

Theorem 6.1. Suppose that the invariant manifold 7 is simply connected and that the system (1.1) satisfies a global Bendixson condition on 7. Then, if x0 #7 and #+(x0) is precompact, limt .t(x0)=p where p is an equilibrium whose stable manifold with respect to the flow on 7 has codimension 1 at most. The restriction of the C1 vector field f to the invariant manifold 7 will also be denoted by f. A point p#7 is wandering with respect to the flow (t, x)[.t(x) on7 if there exists a neighbourhood N in 7 of p and T>0 such that .t(N)&N is empty if t-T. Any alpha or omega limit point, for example, is non-wandering. The C1 Closing Lemma of Pugh [16], as proved by Pugh and Robinson [17] and formulated by Hirsch in [6], states that, if a non-equilibrium p is non-wandering with respect to the flow of a C1 vector field f on a manifold 7 and the orbit of p has compact closure, then every neighbourhood of f in the space of C1 vector fields on 7 contains a field fhaving a periodic orbit through p. Moreover, fcan be chosen to agree with f outside a given neighbourhood N of p.

AN APPLICATION TO AN EPIDEMIC MODEL

Consider the following system of differential equations

s$=b&bs&*is+:is+$r e$=*is&(=+b) e+:ie

(7.1)

i$==e&(#+:+b) i+:i2 r$=#i&(b+$) r+:ir,

which arises from the study of a mathematical model for the spread of an infectious disease in a population with a varying total size. For the biological background and the derivation of the system, we refer the reader to [4], and to [10] for a special case. The variables s, e, i, and r represent fractions of the population that are susceptible, exposed (in the latent period), infectious, and recovered, respectively. All parameters are assumed to be nonnegative, and we assume that =>0 and #>0. The biological feasible region for system (7.1) is the following invariant simplex in the positive cone of R4

1=[(s, e, i, r)#R4+ : s+e+i+r=1] (7.2)

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including all of its lower dimensional boundaries. Mathematically, system (7.1) will be regarded as a system in R4+ with an invariant manifold 1 of dimension 3. The invariance of 1 follows from

(s+e+i+r&1)$=(:i&b)(s+e+i+r&1). (7.3)

It is also clear from (7.3) that g(x)=s+e+i+r&1, N(x)=&(x)=:i&b, where x=(s, e, i, r)#R4+, and m=rk(g x)=1. Let 1and 11 denote the closure and the interior of 1 in the hyperplane s+e+i+r=1, respectively. Set

R0=

*= (=+b)(#+b+:)

.

The following result follows from Theorem 2.3 of [4].

Proposition 7.1. (a) If R0<1, then the equilibrium P0=(1,0,0,0) of (7.1) is globally stable in 1. (b)If R0>1 and if $<min[=, #], then P0 is unstable, and there exists a unique interior equilibrium P*=(s*, e*, i*, r*)#11 and P* is locally asymptotically stable. Moreover, (7.1) is uniformly persistent in 11 if R0>1. The equilibrium P0 corresponds to the population being disease-free, and P* to the disease being endemic. The uniform persistence assertion in Proposition 7.1 follows from the part (b) of Proposition 7.1 and can be proved using the same argument as in the proof of Proposition 3.3 in [10]. The uniform persistence and the boundedness of 1 implies the existence of a compact absorbing set K/D such that each compact subset K1 of 11 satisfies .t(K1)/K for sufficiently large t. Equivalently, there exists a constant c>0 such that s(t)>c, e(t)>c, i(t)>c, r(t)>c (7.4)

if t>T=T(K1), for all solutions x(t)=(s(t), e(t), i(t), r(t)) such that x(0)#K1. The question of whether P* is globally stable with respect to 11 when R0>1 is of great biological interest and was left unresolved in [4]. Using the theory developed in the previous sections and Theorem 6.1 in particular, we prove the following global stability result. Note that R0>1 implies *>:.

Theorem 7.2. Assume that R0>1 and that :=. Then the unique endemic equilibrium P* is globally asymptotically stable in 11 when 0$min[#, =,(*&:) a2c2a1], where a1=(#+:)(#&$)>0, a2=(*&:)*>0.

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Proof. It suffices to show that each positive semiorbit in 11 converges to an equilibrium. Let f(x) denote the vector field defined by system (7.1) and x=(s, e, i, r). Then the system (5.1) for (7.1) is

x v=f(x), z v=

f x

[3]

(x) z, (7.5)

where z=(z1, z2, z3, z4)#R4$R(4 3). Using the Appendix, the third additive compound f x [3] for (7.1) can be written as f x [3] =(&3b&*i+3:i) I+9

and 9 is the following matrix _&=&#&:+:i #+:r 0 0 0 &=&$ = 0

0 *s+:e &#&$&:+:i *i

$ *s&:s 0 *i&=&#&$&:+:i&. (7.6)

Let

V(x, z)=max{a1 |z1|+|z2|,e i

(|z3|+a2 |z4|)=, (7.7) where a1, a2 are as stated in the theorem. Then V(x, z)-a |z| for (x, z)#K_R4"[0] for some constant a>0, since e-c and i-c for x in the compact absorbing set K/11 . Thus, the function V satisfies the condition (5.2). Let (x(t), z(t)) be a solution to (7.5) and set V(t)=V(x(t), z(t)). Then V v(t)=V$(7.5)(x(t), z(t)) for almost all t. The following differential inequalities follow from (7.5) and (7.6).

D+a1 |z1(t)|&(3b+*i+=+#+:&4:i) a1 |z1(t)|+$a 1 |z4(t)|

&(3b+*i+=+#&3:i) a1 |z1(t)|+

a1$i a2e

e i

a2 |z4(t)| (7.8)

D+ |z2(t)|(#+:r)| z1(t)|&(3b+*i+=+$&3:i)| z2(t)| +(*s+:e)| z3(t)|+(*&:) s |z4(t)| (#&$) a1 |z1(t)|&(3b+*i+=+$&3:i)| z2(t)| +*is e +:i+e i |z3(t)|+ *is e e i a2 |z4(t)| (7.9)

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D+ |z3(t)|= |z2(t)|&(3b+*i+#+$+:&4:i)| z3(t)| = |z2(t)|&(3b+#+$+:&3:i)| z3(t)| (7.10)

D+a2 |z4(t)|*a2 i |z3(t)|&(3b+=+#+$+:&4:i) a2 |z4(t)|

(*i&:i)| z3(t)|&(3b+#+$+:&3:i) a2 |z4(t)|, (7.11)

since i<1, :i<:=, and *>:. Set

v1(t)=a1 |z1(t)|+|z2(t)| and v2(t)= e(t) i(t)

(|z3(t)|+a2 |z4(t)|). (7.12)

Then using (7.8), (7.9) we have D+v1(t)&(3b+*i+=+$&3:i) v1(t)+*is e

+:i+

$a1 a2e+v2(t). (7.13)

From (7.10), (7.11) we derive D+v2(t)=\e$ e & i$ i+v2(t)+e i

D+(|z3(t)|+a2 |z4(t)|)

=e i

|z2(t)|+\e$ e

&

i$ i

&3b&#&:&$+3:i+v2(t)

=e i

v1(t)+\e$ e

&

i$ i

&3b&#&:&$+3:i+v2(t). (7.14)

Using (7.13) and (7.14) we can show

D+V(t)+~ (t) V(t), (7.15)

where +~ (t)=max[g1(t), g2(t)] and g1(t)=&3b&*i&=&$+4:i+*is e

+

$a1 a2e+ (7.16)

g2(t)=

=e i

+

e$ e

&

i$ i

&3b&#&:&$+3:i. (7.17)

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Rewriting (7.1) we find

*si e

+:i=

e$ e

+=+b, (7.18)

=e i

+:i=

i$ i

+#+:+b (7.19)

r$ r

&

#i r

=&b&$+:i. (7.20)

Recall that &(t)=:i(t)&b. We thus have from (7.16)(7.20),

+~ (t)&&(t)e$(t) e(t)

&b&$+:i(t)+max{0, &(*&:) i(t)+ $a1 a2e(t)=

e$(t) e(t)

+

r$(t) r(t)

&

#i(t) r(t)

+max{0, &(*&:) c+$a1 a2c=, for all t>T=T(K1) and solutions x=x(t) such that x(0)#K1, by (7.4). Set $=min[#, =,(*&:) a2c2a1]>0. Then, if $$, |t 0 (+~ ({)&&({))d{loge(t)+log r(t)&|t 0 #i({) r({) d{2 |logc|&#ct for t>T. Thus V(x, z) also satisfies conditions (5.4), and Theorem 7.1 follows from Theorem 6.1. K

APPENDIX

The third additive compound matrix A[3] for a 4_4 matrix A=(aij) is A[3]=_a11+a22+a33 a43 &a42 a41 a34 a11+a22+a44 a32 &a31 &a24 a23 a11+a33+a44 a21 a14 &a13 a12 a22+a33+a44&.

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