ITERATION METHOD FOR APPROXIMATING SOLUTIONS OF PERTURBED ABSTRACT MEASURE DIFFERENTIAL EQUATIONS

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Abstract : In this paper we have proved the existence of the solution of perturbed abstract measure differential equation by using Dhage's iteration method. The main result is based on the iteration method included in the hybrid fixed point theorem in a partially ordered normed linear space. Also we have solved an example for the applicability of given results in the paper. Sharma [23] initiated the study of nonlinear abstract differential equations and some basic results concerning the existence of solutions for such equations. Later , such equations were studied by various authors for different aspects of the solutions under continuous and discontinuous nonlinearities. The study of fixed point theorem for contraction mappings in partial ordered metric space is initiated by different authors. The study of hybrid fixed point theorem in partially ordered metric space is initiated by Dhage[3,4,5] with applications to nonlinear differential and integral equations. The iteration method is also embodied in hybrid fixed point theorem in partially ordered spaces by Dhage[19,20]. In this paper we adopted this iteration method technique for abstract measure differential equations.

Key Words and Phrases: Abstract measure differential equation, Dhage iteration method, existence theorem, extremal solutions, approximation of solution, hybrid fixed point theorem.

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I. Introduction

The abstract measure differential equations involve the derivative of the unknown set-function with respect to the σ -finite complete measure. Some of the abstract measure differential equations have been studied in a series of papers by Joshi [1], Shendge and Joshi [2], Dhage [3–5], Dhage et al. [9] and Dhage and Bellale [10] for different aspects of the solutions.

The perturbed ordinary differential equations have been treated in Krasnoselskii [6] and it is mentioned that the inverse of such equations yields the sum of two operators in appropriate function spaces. The Krasnoselskii [6] fixed point theorem is useful for proving the existence results for such perturbed differential equations under mixed geometrical and topological conditions on the nonlinearities involved in them.

II. Preliminaries

A mapping $T : X \to X$ is called D-Lipschitz if there exists a continuous and nondecreasing function $\phi : R^+ \to R^+$ such that

$$\| Tx - Ty \| \le \phi(\| x - y \|)$$

for all $x, y \in X$, where $\phi(0) = 0$. In particular if $\varphi(r) = \alpha r, \alpha > 0$, T is called a Lipschitz function with a Lipschitz constant α. Further if $\alpha < 1$, then T is called a contraction on X with the contraction constant α.

Let X be a Banach space and let $T : X \to X$ T is called compact if $\overline{T(X)}$ is a compact subset of X . T is called totally bounded if for any bounded subsets S of X, $T(S)$ is a bounded subset of X . T is called completely continuous if T is continuous and bounded on X. Every compact operator is totally bounded, but the converse may not be true, however, two notions are equivalent on bounded subsets of X . The details of different types of nonlinear contraction, compact and completely continuous operators appear in Granas and Dugundji [7].

III. Statement of the Problem

Let X be a real Banach algebra with a convenient norm $\|.\|$. Let $x, y \in X$. Then the line segment $\overline{xy}$ in X is defined by

$$\overline{xy} = \{ z \in X \mid z = x + r(y - x), 0 \le r \le 1 \} \tag{3.1}$$

Let $x_0 \in X$ be a fixed point and $z \in X$. Then for any $x \in \overline{x_0 z}$, we define the sets S_x and $\overline{S}_x$ in X by

$$S_x = \{ rx \mid -\infty < r < 1 \}, \tag{3.2}$$

and $$\overline{S}_x = \{ rx \mid -\infty < r \le 1 \} \tag{3.3}$$

Let $x_1, x_2 \in \overline{xy}$ be arbitrary. We say $x_1 < x_2$ if $S_{x_1} \subset S_{x_2}$, or equivalently, $\overline{x_0 x_1} \subset \overline{x_0 x_2}$. In this case we also write $x_2 > x_1$.

Let M denote the σ -algebra of all subsets of X such that (X, M) is a measurable space. Let ca (X, M) be the space of all vector measures (real signed measures) and define a norm $\|.\|$ on ca (X, M) by

$$\|p\| = |p|(X), \tag{3.4}$$

where $|p|$ is a total variation measure of p and is given by

$$|p|(X)=\sup\sum_{i=1}^{\infty}|p(E_i)|,\quad E_i\subset X, \tag{3.5}$$

where the supremum is taken over all possible partitions $\{E_1 : i \in N\}$ of X. It is known that $ca(X,M)$ is a Banach space with respect to the norm ||.|| given by (3.4).

Let μ be a σ-finite positive measure on X, and let $p \in ca(X,M)$. We say p is absolutely continuous with respect to the measure μ if $\mu(E) = 0$ implies $p(E) = 0$ for some $E \in M$. In this case we also write $p << \mu$.

Let $x_0 \in X$ be fixed and let M_0 denote the σ - algebra on S_{x_0}. Let $z \in X$ be such that $z > x_0$ and let M_z denote the σ-algebra of all sets containing M_0 and the sets of the form $S_x, x \in \overline{x_0 z}$.

Throughout this paper, unless otherwise mentioned, let $(E, \preceq, \|\cdot\|)$ denote a partially ordered normed linear space. Two elements x and y in E are said to be **comparable** if either the relation $x \preceq y$ or $y \preceq x$ holds. A non-empty subset C of E is called a **chain** or **totally ordered** if all the elements of C are comparable. It is known that E is **regular** if $\{x_n\}$ is a non decreasing (resp. non increasing) sequence in E such that $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in N$. The conditions guaranteeing the regularity of E may be found in Heikkiländ and Lakshmikantham [21] and the references therein. We need the following definitions(see Dhage [17, 18, 19] and the references therein) in what follows.

Definition 3.1. A mapping $T : E \to E$ is called **isotone or non-decreasing** if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $Tx \preceq Ty$ for all $x, y \in E$. Similarly, T is called **nonincreasing** if $x \preceq y$ implies $Tx \succeq Ty$ for all $x, y \in E$. Finally, T is called **monotonic** or simply **monotone** if it is either non decreasing or non increasing on E.

Definition 3.2. A mapping $T : E \to E$ is called **partially continuous** at a point $a \in E$ if for $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|Tx - Ta\| < \varepsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. T called partially continuous on E if it is partially continuous at every point of it. It is clear that if T is partially continuous on E, then it is continuous on every chain C contained in E.

Definition 3.3. A non-empty subset S of the partially ordered Banach space E is called **partially bounded** if every chain C in S is bounded . An operator T on a partially normed linear space E into itself is called **partially bounded** if $T(E)$ is a partially bounded subset of E. T is called **uniformly partially bounded** if all chains C in $T(E)$ are bounded by a unique constant.

Definition 3.4. A non-empty subset S of the partially ordered Banach space E is called **partially compact** if every chain C in S is a relatively compact subset of E. A mapping $T : E \to E$ is called **partially compact** if $T(E)$ is a partially relatively compact subset of E. T is called **uniformly partially compact** if T is a uniformly partially bounded and partially compact operator on E. T is called **partially totally bounded** if for any bounded subset S of E,$T(S)$ is a partially relatively compact subset of E. If T is partially continuous and partially totally bounded, then it is called **partially completely continuous** on E.

Definition3.5. An upper semi-continuous and monotone non decreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a D-function provided $\psi(0) = 0$. An operator $T : E \to E$ is called partially nonlinear D-contraction if there exists a D-function ψ such that

$$\|Tx - Ty\| \le \psi(\|x - y\|) \tag{3.6}$$

For all comparable elements $x, y \in E$, where $0 < \psi(r) < r$ for $r > 0$. In particular, if $\psi(r) = kr$, $k > 0$, T is called a partial Lipschitz operator with a Lipschitz constant k and more over,if $0 < k < 1$, T is called a partial linear contraction on E with a contraction constant k.

The **Dhage iteration method** or Dhage iteration principle embodied in the following applicable hybrid fixed point theorem of Dhage[11] in a partially ordered normed linear space is used as a key tool for our work contained in this paper. The details of the Dhage iteration method or principle is given in Dhage [15,19,20], Dhage *et al.*[11, 14] and the references therein.

Theorem3.1 (Dhage[16]). *Let $(E, \preceq, \|\cdot\|)$ be a regular partially ordered complete normed linear space such that every compact chain C of E . Let A, $B : E \to E$ be two non decreasing operators such that*

(a) *A is partially bounded and partially nonlinear D-contraction,*

(b) *B is partially continuous and partially compact, and*

(c) *there exists an element $x_0 \in E$ such that $x_0 \preceq Ax_0 + Bx_0$ or $x_0 \succeq Ax_0 + Bx_0$.*

Then the operator equation $Ax + Bx = x$ has a solution x^* in E and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = Ax_n + Bx_n$, $n = 0, 1,\ldots$,converges monotonically to x^*.

IV. Main Result

In this section, we prove an existence and approximation result for the AMDE (4.3) on a closed and bounded interval $J = [a, b]$ under mixed partial Lipschitz and partial compactness type conditions on the nonlinearities involved in it. We place the AMDE (4.3) in the function space $C(J,\mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\le$ in $C(J,\mathbb{R})$ by

$$\|x\| = \sup_{t\in J}|x(t)| \tag{4.1}$$

and $x \le y \Leftrightarrow x(t) \le y(t)$ for all $t \in J$ (4.2)

Clearly, $C(J,\mathbb{R})$ is a Banach space with respect to above supremum norm and also partially ordered w. r. t. the above partially order relation $\le$. It is known that the partially ordered Banach space $C(J,\mathbb{R})$ is regular and lattice so that every pair of elements of E has a lower and an upper bound in it.

Given a $p \in ca(X,M)$ with $p << \mu$. Consider the following Abstract Measure Differential Equation (in short AMDE)

$$\left.\begin{aligned} &\frac{dp}{d\mu} = f(x, p(s_x)) + g\left(x, \max_{a\le\xi\le x} p(s_\xi)\right), \quad a.e\,[\mu], x \in x_0\overline{z} \\ &p(E) = q(E), E \in M_0, \end{aligned}\right\} \tag{4.3}$$

$\mu, f, g : s_z \times R \to R$ are continuous functions. where q is a given known vector measure, $\frac{dp}{d\mu}$ is a Randon-Nikodym derivate of p with respect to μ, $f, g : S_z \times R \to R$, and $f(x, p(S_x))$ and $K = \pi/2$, is μ-integrable for each $p \in ca(S_z, M_z)$.

By a solution of equation (4.3) we mean a differentiable function $x \in C(J, \mathbb{R})$ that satisfies equation (4.3), where $C(J, \mathbb{R})$ is the space of continuous real-valued functions defined on J.

Differential equations with maxima are often met in the applications, for instance in the theory of automatic control. Numerous results on existence and uniqueness, asymptotic stability as well as numerical solutions have been obtained. To name a few, we refer the reader to [16, 17, 18, 19] and the references there in. The AMDE (4.3) is a linear perturbation of first type of nonlinear differential equations. The details of different types of perturbation appears in Dhage [14]. The special cases of the AMDE (4.3) in the forms

$$\left.\begin{aligned} &\frac{dp}{d\mu} = f(x, p(S_x)),\ a.e\,[\mu], x \in x_0\overline{z} \\ &p(E) = q(E), E \in M_0 \end{aligned}\right\} \tag{4.4}$$

and

$$\left.\begin{aligned} &\frac{dp}{d\mu} = g\left(x, \max_{a\le\xi\le x} p(S_\xi)\right), \quad a.e\,[\mu], x \in x_0\overline{z} \\ &P(E) = q(E), E \in M_0 \end{aligned}\right\} \tag{4.5}$$

have already been discussed in the literature for different aspects of the solutions using usual Picard iteration method. See Bainov and Hristova [13] and the references there in for the details. In this paper we discuss the AMDE (4.3) for existence and approximation of solutions via a new approach based upon the Dhage iteration method. In consequence, we obtain the existence and approximation results for AMDEs (4.4) and (4.5) as special cases which are also new to the literature.

In the following section we give some preliminaries and the key tool that will be used for proving the main result of this paper.

Definition 4.1. A vector measure u ϵ ca(S_z, M_z) said to be a lower solution of the equation (4.3) if it satisfies

$$\left.\begin{aligned} &\frac{du}{d\mu} \le f(x, p(S_x)) + g\left(x, \max_{a\le\xi\le x} p(S_\xi)\right) \\ &u(E) = q(E), E \in M_0 \in R, \end{aligned}\right\} \tag{4.6}$$

$a.e\,[\mu], x \in \overline{x_0 z}$.. Similarly.A vector measure $v \in ca(S_z, M_z)$ is called an upper solution of the AMDE (4.3) if the above inequality is satisfied with reverse sign.

We consider the following set of assumptions in what follows:

(H_1) There exist constants $\lambda > 0$, $\mu > 0$ with $\lambda \ge \mu$ such that
$0 \le [f(x, p(S_x)) + \lambda x] - [f(x, p(S_y)) + \lambda y] \le \mu(x - y)$
$a.e\,[\mu], x \in \overline{x_0 z}$ and $x, y \in \mathbb{R}, x \ge y$.

(H_2) There exists a constant $M > 0$ such that $|g(x, \max p(S_\xi)| \le M$, for all $a.e\,[\mu], x \in \overline{x_0 z}$

(H_3) $g(x, \max p(s_\xi))$ is non decreasing in x for each $x \in \overline{x_0 z}$

(H_4) AMDE (4.3) has a lower solution $u \in C(S_z, M_z)$.

Now we consider the following AMDE

$$\left.\begin{aligned} &\frac{dp}{d\mu} + \lambda\, x\, p(S_x) = \tilde{f}(x, p(S_x)) + g\left(x, \max_{a\le\xi\le x} p(S_\xi)\right), \\ &p(E) \le q(E), E \in M_0 \end{aligned}\right\} \tag{4.7}$$

for all $x \in \overline{x_0 z}$ where $\tilde{f}, g : S_z \times \mathbb{R} \to \mathbb{R}$ and $\tilde{f}(x, p(S_x)) = f(x, p(S_x)) + \lambda p(S_x)$, $\lambda > 0$.

Remark 4.1. A vector measure $u \in ca(S_z, M_z)$ is a solution of the equation (4.7) if and only if it is a solution of the equation (4.3) defined on $\overline{x_0 z}$

We also consider the following condition in what follows.

(H_5) There exits a constant $K > 0$ such that $|\tilde{f}(x, p(S_x))| \le K$, for all $x \in \overline{x_0 z}$

Lemma 4.1. *Suppose that the hypotheses (H_2), (H_3) and (H_5) hold. Then a* vector measure $u \in ca(S_z, M_z)$ *is a solution of the AMDE (4.7) if and only if it is a solution of the nonlinear integral equation*

$$p(x) = a_0 e^{-\lambda x} + e^{-\lambda x}\int_a^x e^{\lambda x}\tilde{f}(x, p(S_x))\,dx$$
$$+ e^{-\lambda x}\int_a^x e^{\lambda x} g\left(x, \max_{a\le\xi\le x} p(S_\xi)\right) dx, \tag{4.8}$$

for all $x \in \overline{x_0 z}$

Theorem 4.1. *Suppose that hypotheses (H_1) − (H_5) hold. Then the AMDE (4.3) has a solution x* defined on* $\overline{x_0 z}$ *and the sequence* $\{p_n\}$ *of successive approximations defined by*

$x_0 = u,$

$$p_{n+1}(x) = a_0 e^{-\lambda x} + e^{-\lambda x}\int_a^x e^{\lambda x}\tilde{f}(x, p_n(S_x))\,dx + e^{-\lambda x}\int_a^x e^{\lambda x} g\left(x, \max_{a\le\xi\le x} p_n(S_\xi)\right)dx, \quad (4.9)$$

for all $x \in \overline{x_0 z}$ *converges monotonically to* x^*.

Proof. Set $E = C(J, \mathbb{R})$. Then, in view of Lemma 4.1, every compact chain C in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\le$ so that every compact chain C is in E.

Define two operators A and B on E by

$$Ap(x) = a_0 e^{-\lambda x} + e^{-\lambda x}\int_a^x e^{\lambda x}\tilde{f}(x, p(S_x))\,dx,\ x \in \overline{x_0 z}, \quad (4.10)$$

and $$Bp(x) = e^{-\lambda x}\int_a^x e^{\lambda x}\, g\left(x, \max_{a\le\xi\le x} p(S_\xi)\right)dx,\ x \in \overline{x_0 z}. \quad (4.11)$$

From the continuity of the integral, it follows that A and B define the operators $A, B : E \to E$. Applying Lemma 4.1, the AMDE (4.3) is equivalent to the operator equation

$Ap(x) + Bp(x) = p(x),\ x \in \overline{x_0 z}$.

Now, we show that the operators A and B satisfy all the conditions of Theorem 4.1 in a series of following steps.

Step I. *A and B are non decreasing on E.*

Let $x, y \in E$ be such that $x \ge y$. Then by hypothesis (H_1), we get

$x_1, x_2 \in \overline{x_0 z}$

$$\lim_{n\to\infty} Bp_n(x) = \lim_{n\to\infty} e^{-\lambda x}\int_a^x e^{\lambda x} g\left(x, \max_{a\le\xi\le x} p_n(S_\xi)\right)dx$$

$$= e^{-\lambda x}\int_a^x e^{\lambda x}\left[\lim_{n\to\infty} g\left(x, \max_{a\le\xi\le x} p_n(S_\xi)\right)\right]dx \qquad \because \lim_{n\to\infty} g\left(x, \max_{a\le\xi\le x} p_n(S_\xi)\right) = g\left(x, \max_{a\le\xi\le x} p_n(S_\xi)\right)$$

$$= e^{-\lambda x}\int_a^x e^{\lambda x} g\left(x, \max_{a\le\xi\le x} p_n(S_\xi)\right)dx$$

$$= Bp(x),$$

for all $x \in \overline{x_0 z}$

Next, we show that the operator B is also non decreasing on E. Let $x, y \in E$ be such that $x \ge y$. Then $p(S_x) \ge p(S_y)$ for all $x \in x_0 z$. Since y is continuous on $[a, x]$, there exists a $\xi^* \in [a, x]$ such that

$$p(S_\xi{}^*) = \max_{a\le\xi\le x} p(S_\xi).$$

By definition of $\le$, one has $p(S_x{}^*) \ge p(S_y{}^*)$. Consequently, we obtain

$$\max_{a\le\xi\le x} p(S_\xi) \ge p(S_x{}^*) \ge p(S_y{}^*) = \max_{a\le\xi\le x} p(S_\xi)$$

Now, using hypothesis (H_3), it can be shown that the operator B is also non decreasing on E.

Step II. *A is partially bounded and partially contraction on E.*

Let $x \in E$ be arbitrary. Then by (H_5) we have

$$|Ap(x)| \le |a_0 e^{-\lambda x}| + e^{-\lambda x}\int_a^x e^{-\lambda x}\,|\tilde{f}(x, p(S_x))|\,dx$$
$$\le |a_0| + K\int_a^x e^{\lambda x}\,dx$$
$$\le |a_0| + e^{\lambda a}K(x-a),$$

for all $x \in \overline{x_0 z}$ Taking the supremum over x, we obtain

$$\|Ap(x)\| \le |a_0| + e^{\lambda a}K(x-a),$$

So A is a bounded operator on E. This implies that A is partially bounded on E.

Let $x, y \in E$ be such that $x \ge y$. Then by (H_1) we have

$$Ap(x) - Ap(y) \le \left| a_0 e^{-\lambda x} + e^{-\lambda x}\int_a^x e^{\lambda x}\tilde{f}(x, p(S_x))\,dx - \left(a_0 e^{-\lambda x} + e^{-\lambda x}\int_a^x e^{\lambda x}\tilde{f}(x, p(S_y))\,dx\right)\right|$$

$$\le e^{-\lambda x}\int_a^x e^{\lambda x}[\tilde{f}(x, p(S_x)) - \tilde{f}(x, p(S_y))]\,dx$$

$$\le e^{-\lambda x}\int_a^x e^{\lambda x}\mu\,|p(S_x) - p(S_y)|\,dx$$

$$\le e^{-\lambda x}\int_a^x e^{\lambda x}\,\lambda\,|p(S_x) - p(S_y)|\,dx$$

$$\le e^{-\lambda x}\int_a^x \frac{d}{dx}e^{\lambda x}\,\|x-y\|\,dx$$

$$\le (1 - e^{-\lambda a})\,\|x-y\|$$

for all $x \in \overline{x_0 z}$ Taking the supremum over x, we obtain

$$\| Ap(x) - Ap(y) \| \le L \| x - y \|,$$

for all $x, y \in E$ with $x \ge y$. Hence A is a partially contraction on E and which also implies that A is partially continuous on E.

Step III. *B is partially continuous on E.*

Let $\{p_n\}_{n\in\mathbb{N}}$ be a sequence in a chain C such that $p_n \to p$, for all $n \in \mathbb{N}$. Then

$$\begin{aligned}\lim_{n\to\infty} Bp_n(x) &= \lim_{n\to\infty} e^{-\lambda x}\int_a^x e^{\lambda x} g\left(x, \max_{a\le\xi\le x} p(S_\xi)\right) dx \\ &= e^{-\lambda x}\int_a^x e^{\lambda x}\left[\lim_{n\to\infty} g\left(x, \max_{a\le\xi\le x} p(S_\xi)\right)\right] dx \\ &= e^{-\lambda x}\int_a^x e^{\lambda x} g\left(x, \max_{a\le\xi\le x} p(S_\xi)\right) dx \\ &= Bp(x),\end{aligned}$$

for all $x \in \overline{x_0 z}$ This shows that $B\, p_n(x)$ converges to $Bp(x)$ point wise on $\overline{x_0 z}$.

Now we show that $\{Bp_n\}_{n\in\mathbb{N}}$ is an equicontinuous sequence of functions in E. Let $x_1, x_2 \in \overline{x_0 z}$ with $x_1 < x_2$. We have $y \in B(C)$.

$$\begin{aligned}| y(x_2) - y(x_1) | &= | Bp(x_2) - Bp(x_1) | \\ &= \left| e^{-\lambda x_2}\int_a^{x_2} e^{\lambda x} g\left(x, \max_{a\le\xi\le x} p(S_\xi)\right) dx - e^{-\lambda x_1}\int_a^{x_1} e^{\lambda x} g\left(x, \max_{a\le\xi\le x} p(S_\xi)\right) dx\right| \\ &\le \left|(e^{-\lambda x_2} - e^{-\lambda x_1})\int_a^{x_1} e^{\lambda x} g\left(x, \max_{a\le\xi\le x} p(S_\xi)\right) dx\right| + \left| e^{-\lambda x_2}\int_{x_1}^{x_2} e^{\lambda x} g\left(x, \max_{a\le\xi\le x} p(S_\xi)\right) dx\right| \\ &\to 0 \quad \text{as } x_1 \to x_2\end{aligned}$$

Uniformly for all $n \in \mathbb{N}$. This shows that the convergence $Bp_n(x) \to Bp(x)$ is uniform and hence B is partially continuous on E.

Step IV. *B is partially compact operator on E*

Let C be an arbitrary chain in E. We show that $B(C)$ is uniformly bounded and equicontinuous set in E. First we show that $B(C)$ is uniformly bounded. Let $y \in B(C)$ be any element. Then there is an element $x \in E$ such that $y = Bx$. By hypothesis (H_2)

$$\begin{aligned}| y(x) | &= | Bp(x) | \\ &= \left| e^{-\lambda x}\int_a^x e^{\lambda x} g\left(x, \max_{a\le\xi\le x} p(S_\xi)\right) dx\right| \\ &\le \int_a^x e^{\lambda x} g\left(x, \max_{a\le\xi\le x} p(S_\xi)\right) dx \\ &\le \int_a^b e^{\lambda b} M\, dx \\ &\le e^{\lambda b} M(b-a) = r,\end{aligned}$$

for all $x \in \overline{x_0 z}$ Taking the supremum over x we obtain $\| y \| \le \| Bx \| \le x$, for all $y \in B(C)$. Hence $B(C)$ is uniformly bounded subset of E. Next we show that $B\,(C)$ is an equicontinuous set in E. Let $x_1, x_2 \in \overline{x_0 z}$, with $x_1 < x_2$. Then ,for any $y \in B(C)$, one has $y \in B(C)$.

$$\begin{aligned}| y(x_2) - y(x_1) | &= | Bp(x_2) - Bp(x_1) | \\ &= \left| e^{-\lambda x_2}\int_a^{x_2} e^{\lambda x} g\left(x, \max_{a\le\xi\le x} p(S_\xi)\right) dx - e^{-\lambda x_1}\int_a^{x_1} e^{\lambda x} g\left(x, \max_{a\le\xi\le x} p(S_\xi)\right) dx\right| \\ &\le \left|(e^{-\lambda x_2} - e^{-\lambda x_1})\int_a^{x_1} e^{\lambda x} g\left(x, \max_{a\le\xi\le x} p(S_\xi)\right) dx\right| + \left| e^{-\lambda x_2}\int_{x_1}^{x_2} e^{\lambda x} g\left(x, \max_{a\le\xi\le x} p(S_\xi)\right) dx\right| \\ &\to 0 \quad \text{as } x_1 \to x_2\end{aligned}$$

Uniformly for all $y \in B(C)$. This shows that $B(C)$ is an equicontinuous subset of E. So $B(C)$ is a uniformly bounded and equicontinuous set of functions in E and hence it is compact in view of Arzelá-Ascoli theorem. Consequently $B : E \to E$ is a partially compact operator of E into itself.

Step V. *u satisfies the inequality $u \le Au + Bu$.*

By hypothesis (H_4) the equation (4.3) has a lower solution u defined on E. Then we have

$$\left.\begin{aligned}&\frac{du}{d\mu} \le f(x, p(S_x)) + g\left(x, \max_{a\le\xi\le x} p(S_\xi)\right), \quad a.e\,[\mu], x \in x_0\overline{z} \\ &p(E) = q(E), E \in M_0,\end{aligned}\right\} \qquad (4.12)$$

Adding λ $u(x)$ on both sides of the first inequality in (4.12), we obtain

$$\left.\begin{array}{l}\frac{du}{d\mu}+\lambda\mu(x)\le f(x,p(s_x)+\lambda\mu(x)+g\left(x,\max_{a\le\xi\le x}p(S_\xi)\right),\quad a.e\,[\mu], x\in\overline{x_0z}\\ u(a)=\alpha_0\in R,\end{array}\right\}$$

Again, multiplying the above inequality by $e^{\lambda x}$,

$$\left(e^{\lambda x}\frac{du}{d\mu}\right)'\le e^{\lambda x}\left[f(x,p(S_x)+\lambda\mu(x)+g\left(x,\max_{a\le\xi\le x}p(S_\xi)\right)\right],\quad a.e\,[\mu], x\in\overline{x_0z} \tag{4.13}$$

A direct integration of (4.13) from a to x yields

$$u(x)\le a_0e^{-\lambda x}+e^{-\lambda x}\int_a^x e^{\lambda x}\tilde{f}(x,p(S_x))\,dx+e^{-\lambda x}\int_a^x e^{\lambda x}g\left(x,\max_{a\le\xi\le x}p(S_\xi)\right)dx, \tag{4.14}$$

for $x\in\overline{x_0z}$ From definitions of the operators A and B it follows that

$u(x)\le Au(x)+Bu(x)$,

for all $x\in\overline{x_0z}$ Hence $u\le Au+Bu$. Thus A and B satisfy all the conditions of Theorem 2.1 and we apply it to conclude that the operator equation $Ax+Bx=x$ has a solution. Consequently the integral equation and the equation (4.3) has a solution x^* defined on J. Furthermore, the sequence $\{p_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.1) converges monotonically to x^*. This completes the proof.

Remark4.2. The conclusion of Theorem 4.1 also remains true if we replace the hypothesis (H_4) with the following one.

$u\in ca(S_z,M_z)$

(H_4') The AMDE (4.3) has an upper solution $u\in ca(S_z,M_z)$

Example 4.1. We consider the following AMDE

$$\left.\begin{array}{l}\frac{dp}{d\mu}=\arctan xp(s_x)-p(s_x)+\tanh\left(\max_{0\le\xi\le x}p(S_\xi)\right),\ x\in\overline{x_0z}.\\ x(0)=1.\end{array}\right\} \tag{4.15}$$

Here $f(x,p(S_x))=\arctan xp(S_x)-p(S_x)$ and $g(x,p(S_x))=\tan hx$. The functions f and g are continuous on $J\times\mathbb{R}$. Next, we have

$$0\le\arctan xp(s_x)-\arctan yp(s_y)\le\frac{1}{\xi^2+1}(x-y),$$

for all $x,y\in\mathbb{R}$, $x>\xi>y$. Therefore $\lambda=1>\frac{1}{\xi^2}+1=\mu$. Hence the function f satisfies the hypothesis(H_1). Moreover, the function f $(x,p(S_x))=\arctan xp(S_x)$ is bounded on $\overline{x_0z}$ with bound $K=\pi/2$, so that the hypothesis (H_5) is satisfied. The function g is bounded on $J\times\mathbb{R}$ by $M=1$, so (H_2) holds. The function $g(x,p(S_x))$ is increasing in x for each $x\in\overline{x_0z}.$, so the hypothesis (H_3) is satisfied. The AMDE has a lower solution $u(x)=-2x+1$, $x\in\overline{x_0z}$ Thus all hypothesis of Theorem 4.1 are satisfied and hence the AMDE (4.15) has a solution x^*defined on $\overline{x_0z}.$ and the sequence $\{p_n\}_{n=0}^{\infty}$ defined by

$$x_0=-2x+1,$$

$$p_{n+1}(x)=e^{-x}+e^{-x}\int_0^x e^x\arctan x_np(s_x)\,dx+e^{-x}\int_0^x e^x\tanh\left(\max_{0\le\xi\le x}x_np(S_\xi)\right)dx$$

for each $x\in\overline{x_0z}.$, converges monotonically to x^*.

Remark 4.3. Finally while concluding, we mention that the study of this paper may be extended with appropriate modifications to the nonlinear abstract measure differential equation with maxima,

$$\left.\begin{array}{l}\frac{dp}{d\mu}=f\left(x,p(s_x),\max_{0\le\xi\le x}p(S_\xi)\right)+g\left(x,p(S_x),\max_{0\le\xi\le x}p(S_\xi)\right)\\ p(E)=q(E),E\in M_0,\end{array}\right\} \tag{4.16}$$

for all $x\in\overline{x_0z}$,where $f,g:S_z\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions. When $g\equiv 0$ the differential equation (4.16) reduces to the nonlinear differential equations with maxima,

$$\left.\begin{array}{l}\frac{dp}{d\mu}=f\left(x,p(S_x),\max_{0\le\xi\le x}p(S_\xi)\right),x\in\overline{x_0z}\\ p(E)=q(E),E\in M_0\end{array}\right\} \tag{4.17}$$

which is studied in Otrocol and Rus [22] for existence and uniqueness theorem via Picard iterations under strong Lipschitz condition. Therefore, the obtained results for differential equation (4.14) with maxima via Dhage iteration method will include the existence and approximation results for the differential equation with maxima (4.15) under weak partial Lipschitz condition.

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