BI-IDEALS IN $C_1$ AND $C_2$ NEAR SUBTRACTION SEMIGROUPS

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Abstract:

In this paper we introduce the notation of bi-ideals in $C_1$ and $C_2$ near-subtraction semigroup and study some of their properties.

Key words:

Near subtraction semigroups, IFP, strong IFP, GNF, bi-ideals in $C_1$ and $C_2$ near-subtraction semigroup.

1. Introduction

B.M.Schein [7] considered systems of the form $(X; o;/)$, where $X$ is a set of functions closed under the composition “$o$” of functions (and hence $(X; o)$ is a function semigroup) and the set theoretic subtraction “/” (and hence $(X;/)$ is a subtraction algebra in the sense of [1]). Y.B.Jun et al[4] introduced the notation of ideals in subtraction algebras and discussed the characterization of ideals. For basic definition one may refer to Pilz[6]. Mahalakshmi et al. [5] studied the notation of bi-ideals in near subtraction semigroups. The purpose of this paper is to introduce the notation of bi-ideals in $C_1$ and $C_2$ near-subtraction semigroups. We investigate some basic results, examples and properties.

2. Preliminaries

Definition:2.1. A nonempty set $X$ together with binary operations “$−$” is said to be subtraction algebra if it satisfies the following conditions

(i) $x−(y−x)=x$.

(ii) $x−(x − y ) = y −(y − x)$.

(iii) $(x − y) – z = (x − z) − y$, for every $x, y, z \in X$.

Definition:2.2. A nonempty set $X$ together with two binary operations ““$−$” and “$•$” is said to be a subtraction semigroup if it satisfies the following conditions

(i) $(X, −)$ is a subtraction algebra.

(ii) $(X, •)$ is a semigroup.

(iii) $x (y − z) = xy − xz$ and $(x−y)z= xz − yz$, for every $x, y, z \in X$. 
Definition: 2.3. A non empty set X together with two binary operations ‘−’ and ‘•’ is said to be a right near subtraction semigroup if it satisfies the following conditions:

(i) \((X, −)\) is a subtraction algebra.
(ii) \((X, •)\) is a semigroup.
(iii) \((x − y) z = xz − yz\), for every \(x, y, z \in X\).

It is clear that \(0x = 0\), for all \(x \in X\). Similarly we can define a left near subtraction semigroup. Hereafter a near subtraction semigroup means only a right near subtraction semigroup.

Definition: 2.4. A nonempty subset \(S\) of a subtraction semigroup \(X\) is said to be a subalgebra of \(X\), if \(x−y \in S\), for every \(x, y \in X\).

Definition: 2.5. Let \((X, −, •)\) be a near − subtraction semigroup. A nonempty subset \(I\) of \(X\) is called

(i) A left ideal if \(I\) is a subalgebra of \((X, −)\) and \(xi − x (y − i) \in I\) for all \(x, y \in X\) and \(i \in I\).
(ii) A right ideal \(I\) is a subalgebra of \((X, −)\) and \(IX \subseteq I\).
(iii) If \(I\) is both a left and right ideal then, it is called a two-sided ideal (simply, ideal) of \(X\).

Definition: 2.6. A near subtraction semigroup \(X\) is said to be Zero − symmetric if \(x0 = 0\), for every \(x \in X\).

Definition: 2.7. An element \(e \in X\) is said to be idempotent if for each \(e \in X\), \(e^2 = e\).

Definition: 2.8. A subalgebra \(Q\) of \((X, −)\) is said to be a quasi-ideal of zero-symmetric near subtraction semigroup of \(X\) if \(QX \cap XQ \subseteq Q\).

Definition: 2.9. A subalgebra \(B\) of \((X, −)\) is said to be a bi-ideal of zero-symmetric near subtraction semigroup of \(X\) if \(BXB \subseteq B\).

Definition: 2.10. We say that \(X\) is an s (s') near subtraction semigroup if \(a \in X(a(aX))\), for all \(a \in X\).

Definition: 2.11. A s-near subtraction semigroup \(X\) is said to be a \(s\)-near subtraction semigroup if \(x \in xX\), for all \(a \in X\).

Definition: 2.12. A near subtraction semigroup \(X\) is said to be sub commutative if \(aX = Xa\), for every \(a \in X\).

Definition: 2.13. A near subtraction semigroup \(X\) is said to be left-bipotent if \(Xa = X a^2\), for every \(a \in X\).

Definition: 2.14. An element \(a \in X\) is said to be regular if for each \(a \in X\), \(a = aba\), for some \(b \in X\).

Definition: 2.15. An element \(a \in X\) is said to be strongly regular if for each \(a \in X\), \(a = b a^2\), for some \(b \in X\).

Definition: 2.16. \(X\) is said to fulfill the insertion-of factors property ( IFP ) provided for all \(a, b, n \in X\), \(ab = 0 \Rightarrow anb = 0\).
Definition: 2.17. \( X \) has **strong IFP**, if for all ideals \( I \) of \( X \) and for all \( a, b \in X \), \( ab \in I \Rightarrow anb \in I \).

3. **Bi-ideals in \( C_1 \) and \( C_2 \) near-subtraction semigroup**

In this section we define bi-ideals in \( C_1 \) and \( C_2 \) near-subtraction semigroups and give some examples of these new concepts.

**Definition: 3.1.** Let \( X \) be a right near subtraction semigroup. If for all \( x \in X \), \( xX = xXx \) then we say \( X \) is a \( C_1 \) near subtraction semigroup.

**Definition: 3.2.** Let \( X \) be a right near subtraction semigroup. If for all \( x \in X \), \( Xx = xXx \) then we say \( X \) is a \( C_2 \) near subtraction semigroup.

**Example 3.3.** Let \( X = \{0, a, b, c\} \) be the Klein’s four group. Define subtraction and multiplication in \( X \) as follows:

\[
\begin{array}{ccc|c}
- & 0 & a & b & c \\
\hline
0 & 0 & 0 & 0 & 0 \\
 a & a & 0 & c & b \\
 b & b & 0 & 0 & 0 \\
 c & c & 0 & c & 0 \\
\end{array}
\quad
\begin{array}{ccc|c}
. & 0 & a & b & c \\
\hline
0 & 0 & 0 & 0 & 0 \\
 a & 0 & 0 & a & a \\
 b & 0 & a & c & b \\
 c & 0 & a & b & c \\
\end{array}
\]

Here \((X, -, \cdot)\) is a near subtraction semigroup (see [6], pg.407, scheme 4 (0, 14, 2, 1)).

Then \( X \) is a \( C_1 \) near subtraction semigroup. But \( X \) is not a \( C_2 \) near subtraction semigroup. Since \( Xa \neq aXa \).

**Example 3.4.** Let \( X = \{0, a, b, c\} \) be the Klein’s four group. Define subtraction and multiplication in \( X \) as follows:

\[
\begin{array}{ccc|c}
- & 0 & a & b & c \\
\hline
0 & 0 & 0 & 0 & 0 \\
 a & a & 0 & a & a \\
 b & b & 0 & 0 & 0 \\
 c & c & b & a & 0 \\
\end{array}
\quad
\begin{array}{ccc|c}
. & 0 & a & b & c \\
\hline
0 & 0 & 0 & 0 & 0 \\
 a & 0 & a & 0 & c \\
 b & 0 & 0 & 0 & 0 \\
 c & 0 & a & 0 & c \\
\end{array}
\]

Here \((X, -, \cdot)\) is a near subtraction semigroup (see [6], pg.407, scheme 14 (0, 7, 0, 9)).

Then \( X \) is a \( C_2 \) near subtraction semigroup. But \( X \) is not a \( C_1 \) near subtraction semigroup. Since \( cX \neq cXc \).

**Remark: 3.5.** Neither \( C_1 \implies C_2 \) near subtraction semigroup.
Lemma: 3.6. Let $X$ is a $C_1$ and $C_2$ near subtraction semigroup $\iff X$ is $C_1$ near subtraction semigroup and every idempotent is central.

Proof: To prove that $X$ is $C_1$ near subtraction semigroup and every idempotent is central.

Since $e \in E$ and $X$ is $C_1$ near subtraction semigroup. $\therefore eX = eXe$ for some $x \in X$.

$\Rightarrow exe = eu$ and $ex = eve$ for some $u, v$ in $X$.

Now $exe = (ex)e = (eve)e = exe \quad \ldots \ldots \ldots (1)$

Since $X$ is a $C_2$ near subtraction semigroup. $\therefore Xe = eXe$ for some $x \in X$.

$\Rightarrow exe = ue$ and $xe = eve$ for some $u, v$ in $X$.

Now $exe = e(xe) = e(evbe) = eve = xe \quad \ldots \ldots \ldots (2)$

From (1) and (2) we get $ex = exe = xe$. Hence $E \subseteq C(X)$.

Conversely, assume that $X$ is $C_1$ near subtraction semigroup and every idempotent is central. (ie) $ex = xe$.

Let $u \in eXe \Rightarrow u = e x' e = x' e e = x' e \in Xe$, for all $x' \in X$. $\therefore eXe \subseteq Xe \quad \ldots \ldots \ldots (3)$

Let $u \in Xe \Rightarrow u = xe = ex \in eX = eXe$, for all $x \in X$. $\therefore Xe \subseteq eXe \quad \ldots \ldots \ldots (4)$

From (1) and (2) we get $Xe = eXe$. Hence $X$ is a $C_2$ near subtraction semigroup.

Theorem: 3.7. Let $X$ be a $C_1$ near subtraction semigroup then $X$ has strong IFP.

Proof: Let $I$ be an ideal of $C_1$ near subtraction semigroup of $X$.

Assume that $ab = 0$, for some $a, b \in X$.

For $n \in X$, $a \in aX = aXa \Rightarrow an = a n' a$, for all $n' \in X$.

$anb = a n' ab \in XI \subseteq I \Rightarrow anb \in I$. Hence $X$ has strong IFP.

Theorem: 3.8. If $X$ is a zero symmetric $C_1$ near subtraction semigroup then $X$ has an IFP.

Proof: Let $X$ is a zero symmetric $C_1$ near subtraction semigroup.

If $ab = 0$, for some $a, b \in X$ and $an \in aX = aXa$, for some $n \in X$.

$\Rightarrow an = a n' a$, for some $n' \in X$.

$\Rightarrow anb = a n' a b = a n' . 0 = 0 \Rightarrow anb = 0$. Hence $X$ has an IFP.

Theorem: 3.9. If $X$ is a $C_2$ near subtraction semigroup then $X$ has an IFP.

Proof: Let $X$ is a $C_2$ near subtraction semigroup.
If \( ab = 0 \), for some \( a, b \in X \) and \( nb \in Xb = bXb \), for some \( n \in X \).

\[ \Rightarrow nb = bn' b \text{, for some } n' \in X \]

\[ \Rightarrow anb = ab n' b = 0. \text{ Hence } X \text{ has an IFP.} \]

**Theorem: 3.10.** Let \( X \) be a \( C_1 \) be a near subtraction semigroup, if \( xX = yX \) then \( Xx = Xy \), for all \( a, b \in X \).

**Proof:** Let \( X \) be a \( C_1 \) be a near subtraction semigroup.

If \( xX = yX \), for some \( x, y \in X \).

Let \( x \in xX = Yx = yXy \Rightarrow x = y n' y \), for some \( n' \in X \) [\( \because C_1 \) is near subtraction semigroup]

\[ \Rightarrow nx = n y n' y \Rightarrow Xx \subseteq Xy \; \ldots \ldots \ldots (1) \]

Similiarly, let \( y \in yX = xX = xXx \Rightarrow y = x n' x \), for some \( n' \in X \)

[\( \because C_1 \) is near subtraction semigroup]

\[ \Rightarrow ny = n x n' x \Rightarrow Xy \subseteq Xx \; \ldots \ldots \ldots (2) \]

From (1) and (2), hence \( Xx = Xy \).

**Theorem: 3.11.** Let \( X \) be a \( C_2 \) be a near subtraction semigroup, if \( Xx = Xy \) then \( xX = yX \), for all \( a, b \in X \).

**Proof:** Let \( X \) be a \( C_2 \) be a near subtraction semigroup.

Assume \( Xx = Xy \), for some \( x, y \in X \).

Let \( x \in xX = xY = yXy \Rightarrow x = y n' y \), for some \( n' \in X \) [\( \because C_2 \) is near subtraction semigroup]

\[ \Rightarrow xn = y n' yn \Rightarrow xX \subseteq yX \; \ldots \ldots \ldots (1) \]

Similiarly, let \( y \in yX = Xx = xXx \Rightarrow y = x n' x \), for some \( n' \in X \)

[\( \because C_2 \) is near subtraction semigroup]

\[ \Rightarrow yn = x n' xn \Rightarrow yX \subseteq xX \; \ldots \ldots \ldots (2) \]

From (1) and (2), hence \( xX = yX \).

**Corollary: 3.11.1** If \( X \) is a \( C_1 \) and \( C_2 \) near subtraction semigroup then \( Xx = Xy \iff Xx = Yx \), for all \( x, y \in X \).
Theorem: 3.12. Let \( X \) is a \( \bar{s} \), \( C_1 \) and \( C_2 \) near subtraction semigroup then \( M_1 \cap M_2 = M_1 M_2 \) for any two left \( N \)-subgroup \( M_1 \) and \( M_2 \) of \( X \).

Proof: Let \( X \) is a \( \bar{s} \), \( C_1 \) and \( C_2 \) near subtraction semigroup and let \( x \in M_1 \cap M_2 \), then
\[ x \in M_1 \text{ and } x \in M_2. \]

Let \( x \in Xx = xXx \) [since \( C_2 \) is near subtraction semigroup]
\[ = xxXx \in XM_1 XM_2 \subseteq M_1 M_2. \]

Therefore \( M_1 \cap M_2 \subseteq M_1 M_2 \) ...........(1)

If \( x \in M_1 M_2 \) then \( x = yz \), for some \( y \in M_1 \) and \( z \in M_2 \).

Now \( x = yz \in yX = yXy \in XM_1 \subseteq M_1 \) [since \( C_1 \) is near subtraction semigroup]

Also \( x = yz \in Xz = zXz \in XM_2 \subseteq M_2 \) [since \( C_2 \) is near subtraction semigroup]

Therefore \( M_1 M_2 \subseteq M_1 \cap M_2 \) ...........(2)

From (1) and (2), hence \( M_1 \cap M_2 = M_1 M_2 \).

Remark: 3.13. Let \( X \) be a \( \bar{s} \) either \( C_1 \) or \( C_2 \) near subtraction semigroup then \( X \) is regular.

Proof: For every \( x \in xX = x X x \) [\( \because C_1 \) is near subtraction semigroup]

For every \( x \in Xx = x X x \) [\( \because C_2 \) is near subtraction semigroup]

Hence \( X \) is regular.

Remark: 3.14. Let \( s \) is a \( C_1 \) near subtraction semigroup then \( X \) is strongly regular.

Proof: For all \( x \in X \), let \( x \in xX = x X x = xXx \in Xx^2 \Rightarrow x \in Xx^2 \)

Hence \( X \) is strongly regular.

Theorem: 3.15. Let \( X \) be a \( s, C_1 \) near subtraction semigroup. Then \( X \) is strongly regular iff \( B = BXB \), for every bi-ideal \( B \) of \( X \).

Proof: Let \( X \) is a \( s, C_1 \) - near subtraction semigroup and \( X \) is strongly regular. Then \( X \) is regular. Let \( B \) be bi-ideal of \( X \). Now for \( b \in B, b = bab \in BXB, \) for some \( a \in X \).

Thus \( B \subseteq BXB \), for every bi-ideal \( B \) of \( X \).
Conversely, let \( a \in X \), \( Xa \) is a bi-ideal of \( X \). Since \( X \) is a s-near subtraction semigroup.

Let \( a \in Xa = XaXXa \subseteq XaXa = XaXaa \subseteq Xa^2 \). Therefore \( a \in Xa^2 \).

Hence \( X \) is strongly regular.

**Theorem: 3.16.** Let \( X \) be a s, \( C_1 \) near subtraction semigroup. Then \( B = BXB \), for every bi-ideal \( B \) of \( X \) iff \( X \) is a left bi-potent near subtraction semigroup.

**Proof:** Let \( X \) be a left bi-potent near subtraction semigroup. Let \( B \) be a bi-ideal of \( X \). Since \( X \) is a s-near subtraction semigroup. Let \( b \in bX = bXb = bXbb \subseteq Xb^2 \). Thus \( X \) is strongly regular. By theorem 3.15, \( B = BXB \) for every bi-ideal \( B \) of \( X \).

Conversely let \( a \in X \). Then \( Xa \) is a bi-ideal of \( X \).

Let \( y \in Xa = XaXXa \subseteq XaXa = XaXaa \subseteq Xa^2 \). Therefore \( Xa \subseteq Xa^2 \).

But \( Xa^2 \subseteq Xa \), for all \( a \in X \). Therefore \( Xa^2 = Xa \).

Hence \( X \) is a left bi-potent near subtraction semigroup.

**Lemma: 3.17.** [Refer V. Mahalakshmi [5]]

Let \( X \) be a zero–symmetric near subtraction semigroup. If \( L = \{ 0 \} \), then \( en = ene \), for \( 0 \neq e \in E \) and \( n \in X \).

**Lemma: 3.18.** [Refer V. Mahalakshmi [5]]

If \( X \) has the condition, \( eX = eXe = Xe \), for all \( e \in E \), then \( E \subseteq C(X) \).

**Remark: 3.19.** [Refer V. Mahalakshmi [5]]. The following are equivalent

(i) \( X \) is a GNF.

(ii) \( X \) is regular and each idempotent is central.

(iii) \( X \) is regular and subcommutative.

**Theorem: 3.20.** Let \( X \) be a left permutable s- near subtraction semigroup then the following are equivalent.

(i) \( X \) is \( C_1 \) - near subtraction semigroup.

(ii) \( B = BXB \) for every bi-ideal \( B \) of \( X \).

(iii) \( Q = QXQ \) for every quasi-ideal \( Q \) of \( X \).

(iv) \( X \) is left bi-potent near subtraction semigroup.
(v) X is regular.

(vi) aXa = Xa = Xa^2 for all a ∈ X.

(vii) X is a C_2 near subtraction semigroup and X is strongly regular.

(viii) X is GNF.

**Proof:** (i) ⇒ (ii) Assume that X is C_1 - near subtraction semigroup.

By remark:3.14, X is strongly regular.

By theorem:3.16, B=BXB for every bi-ideal B of X.

(ii) ⇒ (iii) Since every quasi-ideal is also a bi-ideal. We have Q = QXQ for every quasi-ideal Q of X.

(iii) ⇒ (iv) Let a ∈ X then Xa is a quasi-ideal of X.

If x ∈ Xa = XaXXa ⊆ XaXa then x = x_1 a x_2 a, for some x_1, x_2 ∈ X. Since X be a left permutatable s- near subtraction semigroup. Therefore x = x_1 a x_2 a = x_1 x_2 a^2 ∈ X a^2.

That is Xa ⊆ X a^2. But X a^2 ⊆ Xa, for all a ∈ X. Therfore Xa^2 = Xa .

Hence X is left bi-potent near subtraction semigroup.

(iv) ⇒ (v) Let a ∈ Xa = X a^2. This implies that X is strongly regular. Then X is regular.

(v) ⇒ (vi) Let x ∈ aXa. Since X is left permutatable s- near subtraction semigroup.

Now x = a x_1 a = x_1 a^2 ∈ Xa^2 ⇒ aXa ⊆ X a^2 ..........(1)

Similarly x ∈ X a^2 = Xaa then x = x_1 a a = ax_1 a ∈ aXa ⇒ X a^2 ⊆ aXa ..........(2)

From (1) and (2), we get aXa = X a^2. Since X is regular for every a ∈ X, there exists x ∈ X such that a = axa. Thus Xa = Xaxa ∈ Xaxa . That is for every y ∈ Xa,

y = x_1 a x_2 a = x_1 x_2 a^2 ∈ X a^2. Therefore Xa ⊆ X a^2. But X a^2 ⊆ Xa, for all a ∈ X.

Hence aXa = X a^2 = Xa, for every a ∈ X.

(vi) ⇒ (vii) Obviously true. Therefore X is a C_2 near subtraction semigroup and

X is strongly regular. Since x is strongly regular. Let a ∈ X a^2 then L = { 0 }.

Then by Lemma:3.17, eXe = eX, for every e ∈ E. Since X is C_2 near subtraction semigroup. Therefore Xe = eXe and so eX = eXe = Xe for every e ∈ E.
Again by the Lemma: 3.18, E ⊆ C(X). Hence X is GNF.

(viii) ⇒ (i) Let X is a GNF, by Remark:3.19, X is regular and sub commutative for a ∈ X.

Now aX = axaX = axXa ∈ aXXa ⊆ aXa. This implies aX ⊆ aXa ………..(1)

Obviously, aXa ⊆ aX ………(2).

From (1) and (2), we get aX = aXa. Hence X is $C_1$ near subtraction semigroup.

REFERENCES