UNSTEADY PLANE COUETTE FLOW BETWEEN TWO PARALLEL POROUS PLATES

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ABSTRACT

In this paper we discuss the unsteady flow of viscous incompressible fluid between two parallel plates in which the lower wall be suddenly accelerated from rest to move with uniform velocity U in a direction parallel to another flat wall (upper) at rest and at a distance h from it with uniform suction at the lower wall an equal injection at the upper wall. This is the problem of flow formation in couette motion in plane case. The complete solutions for the longitudinal velocity distribution and shearing stress have been obtained in closed forms which depend on the suction parameter σ *and time* $\bar{t}$ *.*

Keyword: *Incompressible fluid, Laplace transformation, Porous wall, suction, longitudinal velocity, shearing stress.*

Introduction

An exact solution of the Navier-stokes equations for steady laminar flow of viscous incompressible fluid between two parallel flat plates, one is uniform motion and other at rest, without suction is well known. This represents the solution of the well known problem of plane couette flow. In this case the velocity distribution is linear and the maximum velocity is at the plate in motion. Sinha and Choudhary obtain the solution of the Navier-Stokes equation in case of the steady plane couette flow between two parallel plate with small uniform suction at the stationary plate under the assumption that the pressure between plates is uniform.

Verma and Bansal discussed the same problem without neglecting the pressure gradient normal to the plate. Gupta and Goyal investigated the problem of plane couette flow between two parallel plate with uniform suction at stationary plate.

Governing Equation

Let us consider the unsteady laminar flow of a viscous incompressible fluid between two parallel porous flat plates in which the lower plate be suddenly accelerated from rest and moves with constant velocity U in a direction parallel to the upper wall at rest and at a distance h from it. It is assumed that the fluid is being sucked from the lower wall with a constant velocity V_0 and the fluid is being injected into the upper wall at the same rate. Let u and v denote the components of the velocity in the direction of x and y increasing. Let the axis of x taken in the direction of flow along the lower wall and y axis at right angle to it. Then the Navier-Stokes equations for the two dimensional unsteady laminar flow of a viscous incompressible fluid in absence of body forces assume the form

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right] \tag{1}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right] \tag{2}$$

Equation of continuity

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{3}$$

The initial and boundary conditions are :

$$\left.\begin{array}{lll} t\le 0, & u=0 & \text{for}\, all\, y, if\ \ 0\le y\le h \\ t>0, & u=U & \text{for}\, y=0, \quad V=V_0<0 \text{ (Suction)} \\ & u=0 & \text{for}\, y=h, \quad V=V_0<0 \text{ (injection)} \end{array}\right\} \tag{4}$$

Assuming the longitudinal velocity as independent of x,

we have

$$\left.\begin{aligned}\frac{\partial u}{\partial x}&=0\\ \frac{\partial^2 u}{\partial x^2}&=0\end{aligned}\right\} \tag{5}$$

Also as solution is uniform, Vis independent of *t* and *x*

$$\left.\begin{aligned}\frac{\partial v}{\partial t}&=0\\ \frac{\partial v}{\partial x}&=0\\ \frac{\partial^2 v}{\partial x^2}&=0\end{aligned}\right\} \tag{6}$$

Using (5) the equation (3) reduces to

$$\frac{\partial v}{\partial y}=0 \qquad \text{and} \quad \frac{\partial^2 v}{\partial y^2}=0 \tag{7}$$

Using (6) and (7) the equation (2) reduces to

$$-\frac{1}{\rho}\frac{\partial p}{\partial y}=0 \qquad \text{or} \qquad \frac{\partial p}{\partial y}=0 \tag{8}$$

Now using (5) the equation (3) takes the form

$$\frac{\partial u}{\partial t}+V_0\frac{\partial u}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2} \tag{9}$$

As the present problem deals with plane couette flow, so the pressure gradient $\frac{\partial p}{\partial x}$ may be neglected.

The equation (9) becomes

$$\frac{\partial u}{\partial t}+V_0\frac{\partial u}{\partial x}=\nu\frac{\partial^2 u}{\partial y^2} \tag{10}$$

Introducing the following dimension less quantities

$$\left.\begin{aligned} \bar{x} &= \frac{x}{h} \\ \eta &= \frac{y}{h} \\ \bar{u} &= \frac{u}{U} \\ \bar{t} &= \frac{\nu t}{h^2} \end{aligned}\right\} \tag{11}$$

where U represents the characteristic velocity.

Thus,

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial \bar{u}}.\frac{\partial \bar{u}}{\partial \bar{t}}.\frac{\partial \bar{t}}{\partial t} = \frac{U\nu}{h^2}\frac{\partial \bar{u}}{\partial \bar{t}} \\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial \bar{u}}.\frac{\partial \bar{u}}{\partial \eta}.\frac{\partial \eta}{\partial y} = \frac{U}{h}\frac{\partial \bar{u}}{\partial \eta} \\ \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial \eta}\left[\frac{U}{h}\frac{\partial \bar{u}}{\partial \eta}\right]\frac{\partial \eta}{\partial y} = \frac{U}{h^2}\frac{\partial^2 \bar{u}}{\partial \eta^2} \end{aligned}\right\} \tag{12}$$

The equation (10) with the help of (12) assume the form

$$\frac{U\nu}{h^2}\frac{\partial \bar{u}}{\partial \bar{t}} + V_0\frac{U}{h}\frac{\partial \bar{u}}{\partial \eta} = \nu\frac{U}{h^2}\frac{\partial^2 \bar{u}}{\partial \eta^2}$$

i.e. $$\frac{\partial \bar{u}}{\partial \bar{t}} + V_0\frac{h}{\nu}.\frac{\partial \bar{u}}{\partial \eta} = \frac{\partial^2 \bar{u}}{\partial \eta^2}$$

$$\therefore \quad \frac{\partial \bar{u}}{\partial \bar{t}} + \sigma\frac{\partial \bar{u}}{\partial \eta} = \frac{\partial^2 \bar{u}}{\partial \eta^2} \tag{13}$$

where σ define the suction parameter and is denoted by

$$\sigma = \frac{V_0 h}{\nu} \tag{14}$$

The initial and boundary condition are

$$\left.\begin{array}{lll} \bar{t} \le 0, & \bar{u}(\eta,0)=0 & \text{for } all\ \eta,\ 0\le\eta\le 1 \\ \bar{t} > 0, & \bar{u}(0,\bar{t})=1 & \text{for } \eta=0 \\ & \bar{u}(1,\bar{t})=0 & \text{for } \eta=1 \end{array}\right\} \tag{15}$$

Present Method of Solution

The present method of the solution of the differential equation (13) deals with the use of Laplace transformation which is introduced below :

Let

$$u'(\eta,s)=\int_0^\infty \bar{u}(\eta,\bar{t})e^{-S\bar{t}}\,d\bar{t}, S>0 \tag{16}$$

Now multiplying both sides of (13) by $e^{-S\bar{t}}$ and then integrating w.r. to $\bar{t}$ from 0 to ∞, we get

$$\int_0^\infty \frac{\partial\bar{u}}{\partial\bar{t}}e^{-S\bar{t}}\,d\bar{t}+\sigma\int_0^\infty \frac{\partial\bar{u}}{\partial\eta}.e^{-S\bar{t}}\,d\bar{t}=\int_0^\infty \frac{\partial^2\bar{u}}{\partial\eta^2}e^{-S\bar{t}}\,d\bar{t}$$

$$Su'+\sigma\frac{\partial u'}{\partial\eta}=\frac{\partial^2 u'}{\partial\eta^2}$$

Thus the required differential equation is

$$\frac{\partial^2 u'}{\partial\eta^2}-\sigma\frac{\partial u'}{\partial\eta}-Su'=0 \tag{17}$$

The boundary conditions are $\bar{t}>0$

$$\left.\begin{array}{ll} u'(0,S)=\frac{1}{S} & \text{for}=\eta=0 \\ u'(1,S)=0 & \text{for}=\eta=1 \end{array}\right\} \tag{18}$$

The general solution of the differential equation (17) is given by

$$u' = P.e^{\frac{\sigma+\sqrt{\sigma^2+4S}}{2}\eta} + Qe^{\frac{\sigma-\sqrt{\sigma^2+4S}}{2}\eta} \tag{19}$$

$$u(\eta, S) = \frac{1}{S}e^{\frac{\sigma-\sqrt{\sigma^2+4S}}{2}\eta} + \frac{e^{\frac{\sigma-\sqrt{\sigma^2+4S}}{2}\left[e^{\frac{\sigma-\sqrt{\sigma^2+4S}}{2}\eta} - e^{\frac{\sigma+\sqrt{\sigma^2+4S}}{2}\eta}\right]}}{S\left[e^{\frac{\sigma+\sqrt{\sigma^2+4S}}{2}} - e^{\frac{\sigma-\sqrt{\sigma^2+4S}}{2}}\right]} \tag{20}$$

Now applying complex inversion formula to (20).

We get

$$\bar{u}(\eta, \bar{t}) = \frac{1}{2\pi i}\int_{y-i\infty}^{y+i\infty} e^{-S\bar{t}}\left[f(s) + F(s)\right]ds$$

$$= \quad \frac{1}{2\pi i}\left[2\pi i \text{ Sum of the residues of } \{f(s) + F(S)\}.e^{S\bar{t}} \text{ at the poles}\right] \tag{21}$$

Where $f(s) = \frac{1}{S}e^{\frac{\sigma-\sqrt{\sigma^2+4S}}{2}\eta}$ (22)

$$F(s) = \frac{e^{\frac{\sigma-\sqrt{\sigma^2+4S}}{2}}\left[e^{\frac{\sigma-\sqrt{\sigma^2+4S}}{2}\eta} - e^{\frac{\sigma+\sqrt{\sigma^2+4S}}{2}\eta}\right]}{S\left[e^{\frac{\sigma+\sqrt{\sigma^2+4S}}{2}\eta} - e^{\frac{\sigma-\sqrt{\sigma^2+4S}}{2}\eta}\right]} \tag{23}$$

Now two cases arise

Case I : Let us take the function defined in (22) in case of function *f*(*s*), pole is given by *S* = 0

Thus the residue of $f(s)e^{-S\bar{t}}$ at *S* = 0 is

$$= \quad \lim_{S\to 0}\frac{S-0}{S}e^{S\bar{t}}.e^{\frac{\sigma-\sqrt{\sigma^2+4S}}{2}\eta}$$

$$= \quad 1.e^{\frac{\sigma-\sigma}{2}\eta} = 1 \tag{24}$$

Case II : In case function $F(S)$, the pole are given by $S = 0$ (Once)

and $e^{\frac{\sigma+\sqrt{\sigma^2+4S}}{2}} - e^{\frac{\sigma-\sqrt{\sigma^2+4S}}{2}} = 0$

The second gives

$$e^{\sqrt{\sigma^2+4S}} = 1 = e^{2K\pi i}, K \text{ being any integer}$$

$$\sigma^2 + 4S = -4K^2\pi^2$$

The residue of $F(S)e^{S\bar{t}}$ at S = 0 is

$$\lim_{S\to 0}\frac{S-0}{S}e^{S\bar{t}}.\frac{e^{\frac{\sigma-\sqrt{\sigma^2+4S}}{2}}\left[e^{\frac{\sigma-\sqrt{\sigma^2+4S}}{2}\eta} - e^{\frac{\sigma+\sqrt{\sigma^2+4S}}{2}\eta}\right]}{\left[e^{\frac{\sigma+\sqrt{\sigma^2+4S}}{2}} - e^{\frac{\sigma-\sqrt{\sigma^2+4S}}{2}}\right]}$$

$$\frac{1-e^{\sigma\eta}}{e^{\sigma}-1} = -\frac{1-e^{\sigma\eta}}{1-e^{\sigma}} \qquad (25)$$

Again the residue of $F(s)e^{S\bar{t}}$ at $S = \frac{-4K^2\pi^2+\sigma^2}{4}$

$$= \frac{-4e^{-\frac{(4K^2\pi^2+\sigma^2)}{4}\bar{t}}.e^{\frac{\sigma-2K\pi i}{2}}}{(4K^2\pi^2+\sigma^2)}\left[e^{\frac{\sigma-2K\pi i}{2}\eta} - e^{\frac{\sigma+2K\pi i}{2}\eta}\right]\times S$$

$$\to \frac{-4K^2\pi^2+\sigma^2}{4}\frac{S+\frac{4K^2\pi^2+\sigma^2}{4}}{\left[e^{\frac{\sigma+\sqrt{\sigma^2+4S}}{2}} - e^{\frac{\sigma-\sqrt{\sigma^2+4S}}{2}}\right]}$$

$$= -\frac{8K\pi}{4\pi^2K^2+\sigma^2}e^{\frac{-(\sigma^2+4K^2\pi^2)}{4}.\bar{t}}.e^{\frac{\sigma\eta}{2}}.\sin K\pi\eta \qquad (26)$$

Using the results of (24), (25) and (26) in (21), we get

$$\bar{u}(\eta,\bar{t}) = 1 - \frac{1-e^{\sigma\eta}}{1-e^{\sigma}} - \sum_{K=1}^{\infty} \frac{8K\pi\, e^{-\frac{(\sigma^2+4K^2\pi^2).\bar{t}}{4}} . e^{\frac{\sigma\eta}{2}} . \sin K\pi\eta}{\left(4\pi^2K^2+\sigma^2\right)} \qquad (27)$$

Shearing stress at the lower wall is given by

$$\frac{Th}{\mu U_0} = \left(\frac{\partial\bar{u}}{\partial\eta}\right)_{\eta=0}$$

$$= \frac{\sigma}{1-e^{\sigma}} - \sum_{K=1}^{\infty} \frac{8K^2\pi^2 e^{-\frac{(\sigma^2+4\pi^2K^2).\bar{t}}{4}}}{\left(4\pi^2K^2+\sigma^2\right)} \qquad (28)$$

The equation (27) and (28) represent the required a longitudinal velocity distribution and shearing stress at the lower plate respectively.

PARTICULAR RESULT

The velocity distribution and the shearing stress in case of unsteady plane couette flow between two parallel impermeable plates $(\sigma = 0)$ from (27) and (28).

The longitudinal velocity distribution is

$$\bar{u} = 1 - \eta - \sum_{K=1}^{\infty} \frac{2\sin K\pi\eta}{K\pi}\, e^{-\pi^2K^2\bar{t}} \qquad (29)$$

$$\left(\text{Since } \lim_{\sigma\to 0} \frac{1-e^{\sigma\eta}}{1-e^{\sigma}} = \eta\right)$$

Shearing stress at the lower wall is given by

$$\frac{T_0 h}{\mu U_0} = 1 - 2\sum_{K=1}^{\infty} e^{-K^2\pi^2\bar{t}} \qquad (30)$$

$$\left(\text{Since } \lim_{\sigma\to 0} \frac{\sigma}{1-e^{\sigma}} = 1\right)$$

When $\bar{t}$ is very large,

i.e. $\bar{t} = \infty$

Equation (29) reduces to

$$\bar{u} = 1 - \eta$$

This represents the longitudinal velocity distribution for steady plane couette flow without suction.

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