Exploring the Algebraic and Geometric Nature of π.

[1*]Archisman Roy, [2]Tushar Banerjee

[1]Author (Student), AOSHS, Higher Secondary School of Science; [2]Research Scholar, SSSUTMS
[1]Department of Science, [2]Department of Zoology
[1]Asansol, India. * denotes correspondence

Abstract: This bodacious study has been endeavored to probe different natures of π. Π comes from the Greek script of the word 'periphery' which means circumference. In connection to that π has a birthright for claiming geometry as its origin. Metaphorically, it is a real and definite number which is algebraically inexpressible. Our piece of research is thus willing to persuade that every geometric relation can be expressed with some equations of algebra and each algebraic expansion has its geometric interpretation. In addition to that, the article produces numerously validated as well as cited works of famous mathematicians to cajole about the article's plausibility. It approaches a good deal of mathematical history for congenially maintaining its engagement with the content. Sufficient statistical data have been produced to sway every bit of the mathematical perspective. The piece of writing consists of suitable graphs and images adhering to its requirement. Furthermore, it expresses enthusiastically about its new substantial contribution regarding the cognizance of π. Finally, the article suffers a lenient way of producing π subsequently through three stages of visualizing a mathematical paradox viz geometric background, augmentation of algebra, and the real connection between them both.

***Index Terms* - Irrational, constant, transcendental, infinite-fraction, exact.**

I. INTRODUCTION

The world of algebraical mathematics especially the number theorems are gigantically fascinated by a few irrational numbers such as Euler's constant 'e' and the 'π'. From the very beginning of discussing mathematics, the number π has been entangled to numerous controversies and paradoxes. Following it, some of the few legendary mathematicians like, Zeno, Poincare, Reimann, David Hilbert have invested their genius beyond exploring the exactness of π. Needlessly, a simple math adorer, not even knowing a great deal of mathematics, would be blown out with paradoxes of π at its most basic level.

The 'π' can be expressed both geometrically and algebraically. As aforestated, π in its basic definition relates to the geometry of circles. Considering a circle of radius unity, the area of it is exactly π. For a tenderfoot, π seems to be an exact number quite naturally because whatever you may do to compute the exact area of a circle with a radius of 1 will result in the same number. On the other hand, π is a well-known irrational number which nor is an integer itself not be a ratio of two integers (of course, the denominator is not zero). That's why π is hard enough to be expressed as an exact number. Since two simultaneous views of π contradict each other, it is quite evident to have paradoxes in mind. In this write-up, we are intending to explore that ever-contradicting nature of it.

While there is no exact value of π, many mathematician and math fans are keen on calculating π to as many digits as possible. The genius of the world record for reciting the most decimals of π belongs to Rajveer Meena of India who blindfolded recited to seventy thousand decimals after the point in 2015 [1]. Meanwhile, some computer programmers (of supercomputers) evaluated π up to more than thirty trillion decimals [2-5].

For any circle the ratio of the circumference to its diameter is π. Since this definite ratio can certainly be transformed into a ratio of two integers, π can be stated as a rational number. Although this conjecture is not right. We are to convince ourselves about the hidden problem. If the history of mathematics is concerned, Zeno's paradox of the Achilles and the Turtle [6, 7] first invoked a fuzzy concept of π in Archimedes [8]. One of the leading mathematicians of the Eighteenth century, Euler, for the first time expressed any infinite fraction with numerators 1 and denominators of any integer inputs to be an irrational number. Using this theorem one of his colleague Polymath Johann Lambert first successfully shown that π is an irrational number [9]. Following that the godfather of calculus, Gottfried Wilhelm Leibniz claimed π as an infinite sum of wonderfully rhythmic odd fractions [10]. His theorem was greatly based on and supported by another superstar of mathematical physics prof. Fermat's Christmas theorem. These formulations have pertained wildly in higher-order physics through the hands of prof. Huygens.

In due course of time, the nature of π and its geometrical interpretation was profoundly elucidated by David Hilbert and his colleague S. Cohn Vossen in their book 'Anschauliche Geometrie' [11]. The whole history, we just narrated, has started its journey successfully from Lambert in 1761. Although, an Indian Mathematician (sage) Madhava of Sangamagrama one-handedly invented the irrationality of π and its expression of infinite fraction series in the fourteenth century, almost 350 years preceding Lambert. The world has recognized his contribution later and named the delineating odd integer infinite fraction series of π as the Madhava-Leibniz theorem. Our Article is thus pointing towards describing the real nature of π and its exact correlation with geometric interpretation as well as an algebraic expression.

II. DATA AND SOURCES OF DATA

This study conceives various mathematical expansions, their history and, analytical studies. We are exploring different natures of π with established works of numerous mathematical genius. The variables of the study contain dependent and independent variables that have been cited at each place of its textual endorsement. The study uses a post-specified method of selecting variables (of course they are standard ones). For this study, secondary data has been collected using library consultation and telephonic surveys. Few analytics and data enlisted in the tables were reckoned from the German mathematicians, Prof. Burkard Polster. We availed a huge amount of offprint books in library consultation along with online journal exploration for a validated and apodictic study over the subject. All such primary information has been cited thoroughly both in text and in references. Neither field study nor lab experimentations were approached for the article.

III. THEORETICAL FRAMEWORK AND ANALYSIS

3.1 Π is a definite number

Although π being tagged as the constant [12-14] of a circle, it is an irrational number thus, impossible to be expressed specifically on the number line. But if anyone considers π to be the ratio of circumference to diameter viz geometric relation then it is indeed a definite number yet cannot be expressed exactly. π can be related in a non-numerical sense as follows, $\pi = \frac{C}{d}$ [13, 15] where C is the circumference and d being the diameter of a circle.

The certitude that π is an absolute constant of a circle in geometry [12-13] can be comprehended easily with logic. If the diameter of a circle is changed by any amount, the circumference gets calibrated accordingly. Say $\pi = 3.14$, implies the circumference expands by 3.14 units due to increment of unity in its diameter. As an instance, let the diameter gets amplified to 2 units from 1 unit. Similarly does the circumference hence, it turns out to be 6.28 from 3.14 unit. Quite evidently the ration $\frac{6.28}{2} = \frac{3.14}{1} = k$ remains the constant. If the relation is not definite then the ratio would have changed whenever there is a change in the diameter. Thus, the circle would naturally have turned to an ellipse or, hyperbola based on the eccentricity. Since in ideal sense, a circle remains as it is on changing its diameter, the ratio is certainly an exact number.

3.2 Why can't the perfect geometric ratio of C/d be known precisely?

The problem with converting geometric quantities to numerical quantities can be understood in terms of Zeno's paradox [7]. Zeno was a Greek philosopher who posed the riddle of how swift Achilles was not able to catch a slow tortoise. Zeno argued that each time Achilles gets to the place where the turtle was, the turtle has moved to a new position. With each step, Achilles gets a fraction closer to the turtle but never catches it mathematically. Zeno's paradox is the problem of translating continuous phenomena into numbers which are by nature, digital [7]. It is the problem of calculating infinity and zero. The paradox has been phrased in various ways. Another way is the riddle of a jumping frog, to wit, a frog located 10 feet (or any distance) from a wall jumps half-way to the wall with each jump. The question is: How many jumps will it take for the frog to reach the wall? The mathematical answer is that if the frog could jump half-way each time, it would jump infinitely but never reach the wall. The same problem can be seen with a ruler – say a foot ruler based on the English system of halving measurements. The ruler is continuous but the number markings on it are digital. Now, start with an inch and cut it in half with each measurement and see if you can get to zero. With the first cut, you get ½ inch, then ¼, 1/8, 1/16, 1/32, 1/64….ad infinitum and eventually, you never reach zero numerically. As we shall see, the methods for calculating π are based on the Zeno problem.

3.3 The Archimedean Method of Calculating Π and the Zeno Problem

Now the relevance of Zeno's paradox to π can be seen in Archimedes' method of calculating it. Archimedes began by inscribing a hexagon in a circle and kept on doubling the sides. He Used the Pythagorean theorem over and over to figure out the side lengths of the left-over little triangles formed [8, 16, 17]. Despite inscribing the 96-sided polygon, Archimedes was unable to delineate a rectilinear figure into a smooth curve and measure the circumference precisely. He could come closer and closer to the approximation as like as Achilles gets closer to the tortoise but never catches it in Zeno's paradox. The idea that an infinite number of polygons would convert straight lines to curve ones is, of course, imaginary because, there is no limitless infinite number. A concept of infinity may exist, but a number is bound to be within limits as a number is finite by definition.

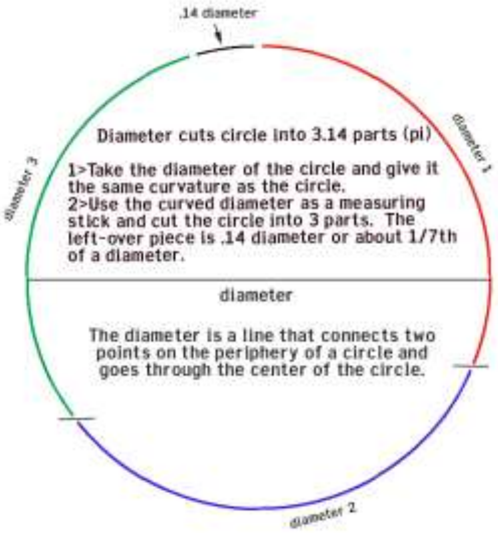
Therefore, we see that it is difficult, to convert a continuously curving line to a straight line (or vice versa) and to measure it numerically with great precision. There are other methods for deriving π, but they also run into the Zeno-problem of getting closer and closer but never quite getting there. So, again, an ideal circle has a definite circumference, a definite diameter, and a definite area, but so far, all the methods for deriving π numerically with millions of decimal places have failed to get there using pure mathematics. We see that it is not feasible to directly divide the circumference by the diameter because of the difficulty of converting a curved line to a straight line. Instead, Archimedes tried to convert rectilinear figures (hexagons) to the curved line of the circumference [18-19]. Archimedes assumed that if one can know the circumference, then one can know π since π = C/d [15]. Π is also the area of the circle if the radius is considered to be 1. So, Gaddy concludes that π as a geometric ratio is perfect, exact, rational, and not transcendental even though the numerical calculations of it are approximate, irrational, and transcendental [13]. This is the key distinction in understanding the contradictory claims about π. It all depends on whether one is referring to perfect geometric π or approximate numerical π.

3.4 Π and Tau

Some mathematical purists insist that since the radius is used in calculations of the circle (A = πr^2, C=2πr), the radius rather than the diameter should be used to determine the constant of the circle. Thus, the formula C/r = tau should be used, rather than C/d = π. Considering that the radius is exactly half the diameter if the circumference is divided by the radius, the circle constant, tau is numerically calculated to be 6.28…ad Infinitum or, two times π. The point to be noted about an ideal circle is that π is in exact one-half of tau if geometrically speaking. However, 3.14…ad Infinitum divided by 6.28…ad Infinitum cannot be calculated to exactly ½. Again, Gaddy's insight illuminates this conundrum, that is, π and tau are exact geometric ratios but cannot be calculated specifically in the numerical sense.

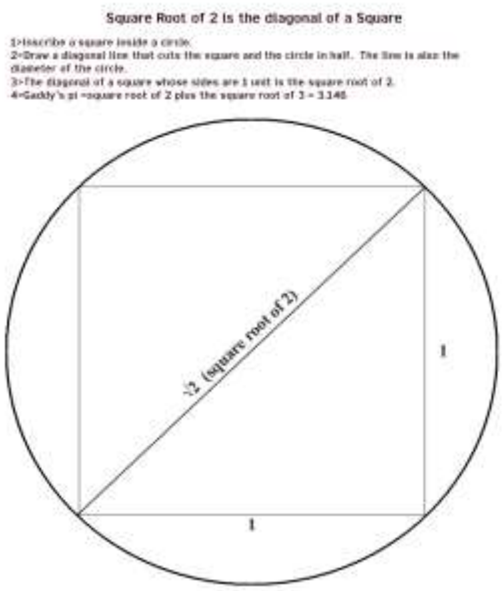
3.5 Converting a Curved Line to a Straight line

The following graphic shows an attempt to convert the circumference of a circle to units of diameter. This graphical representation is not drawn to scale, but it illustrates the point. We can see that the diameter divides the circle into three parts plus a remainder of about 0.14d. That remainder cannot be divided by the diameter to obtain a number with a finite number of decimal places. However, the numerical value of π can be trapped between limits, so it is not an infinite number, but it does involve an infinity of decimal places. Archimedes found that π lies in between $3\frac{10}{71}$ and $3\frac{1}{7}$. However, trying to get to the dot number between those limits is like Achilles trying to catch the tortoise by going a fraction of the way with each step.

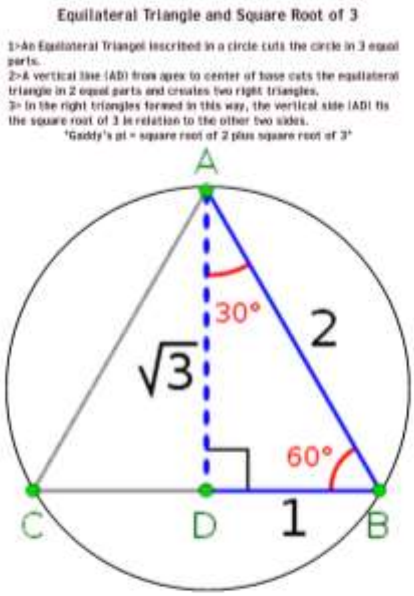


3.6 Gaddy's solution of converting curved line to a straight line

Dan Gaddy's proposal to convert a curved line into a straight line is to consider π as the sum of two exact geometric ratios – the square root of 2 plus the square root of 3 which equals 3.146….ad infinitum. We know that the square root of 2 is the hypotenuse of an isosceles right-angled triangle of side length unity and therefore can be represented as a finite, straight line. Therefore, if we inscribe a square within a circle, and cut the square in half with a diagonal line, that line is the square root of two and also the diameter of the circle.



Likewise, if we inscribe an equilateral triangle into a circle dividing the circle into three equal parts. the square root of 3 can be represented as a finite straight line that cuts an equilateral triangle from apex to the midpoint of the base, thus creating two right triangles.



The attractive feature of what has become known as Gaddy's π is that it makes π, in a geometric sense, rational. Both the √2 and the √3 can be represented as finite straight lines and thus converts a curved line of the circumference to a straight line. Again, the great insight that can be gleaned from Gaddy's work is that the geometric π, represented as C/d, is a perfect ratio but numerical π, represented as 3.14…ad Infinitum is an imperfect representation of perfect geometric π. Geometric π is Platonic after Plato's dictum, the forms in the mind are perfect, but the representations of those ideal forms in the empirical world are imperfect reflections of the perfect world of the mind.

3.7 Proving the irrationality and infinitive expressions of Π

Till now we have somewhat dealt with the geometrical point of view of π. But, assigning the real sense of it as a number comes algebraically. The first extrapolation that appears in every keen mind is exploring the inexactness and irrationality of π. It is quite evident that any irrational number is naturally an inexact number, that's why understanding π as an irrational number solves the rest of the problems quite simply. Now, following the path of history of mathematics the mathematical giant Euler's contemporary prof. Lambert first introduced proof of the irrationality of π [9]. Johan Lambert was a colleague of Euler at the Prussian Academy of Sciences. During that time the infinite nested fraction series was very much of in things. In 1761, he deduced the proof of π's irrationality based on Euler's theorem. The conjecture of its irrationality was at first proposed by Legendre [20] and Euler [21] who were confirmed mathematically through the pen of Lambert. Euler inculcated an infinite fraction series as a general solution to irrational numbers (refer to equation 3.1).

$$I = A + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots}}}} \qquad (3.1)$$

where 'I' is the irrational number and 'A' is the box function of I.

Any infinite fraction series with numerator 1 and denominators of integer inputs $[a_i \in I^+, \forall i \in \{1,2,3, \dots\}]$ always denotes an irrational number. We will be returning to the works of Lambert but before that, we should explore some infinite series. One of the forgotten but finest mathematicians of all times, Madhava of Sangamagrama took birth in the Kerala State. The mathematical achievements of Madhava were perceived in detail later in Rajagopal and Rangachari's texts [22]. Another mathematical report of Kerala like Nilakantha registers the brilliant but lost works of Madhava. Around the latter half of 14th century (some mathematical historian claims period around 1400 AD.) Madhava discovered the infinite series expressions of $\sin x$, $\cos x$, $\tan^{-1} x$, and many more, preceding 300 years to its validated rediscovery in Europe. We are going to use the expression of $\sin x$, $\cos x$, $\tan^{-1} x$, for improvising the irrationality of π. A notable mathematical work 'Mahajyanayana Prakara' takes note of the previously stated infinite nested sin function [23]. All the mathematical history discussed here is in accordance with a detailed report of [24]. Jyesthadeva wrote Yukti-Bhasa which individually describes Madhava's series [25]. Later some of the infinite expressions of number theorems of Madhava were rediscovered by the Swiss Mathematician James Gregory. The expressions of the sin and cos function were named Maclaurin expression after the name of its re-discoverer mathematician. Let us produce some analogy. As known to all, the natural logarithm of 5 is an irrational number. The procedure of proving it is by following the contradiction method of mathematical reasoning.

$$Let, \log_{10} 5 \;\; is\; a\; rationa\; number.$$
$$\therefore \log_{10} 5 = \frac{u}{v}$$
$$\therefore 5 = 10^{\frac{u}{v}}$$
$$\therefore 5^v = 10^u$$
$$\therefore odd = even.$$

Which is never possible. So, our mathematical statement is contradicting. That's why the natural logarithm of 5 is an irrational number. With the help of Euler's expression, Lambert proved that π is an irrational quantity using the contradiction method [9]. We are going to use the Sin and Cos series to evaluate the Tan function.

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} +$$

$$\cos x = \frac{1}{1!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\therefore \tan x = x.\frac{\frac{1}{1!}-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\cdots}{\frac{1}{1!}-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots} = x.\frac{B}{A}\ (say)$$

$$\therefore \tan x = \frac{x}{\frac{\frac{1}{1!}-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots}{\frac{1}{1!}-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\cdots}}$$

$$\therefore \tan x = \frac{x}{\frac{A+B-B}{B}} = \frac{x}{\frac{B+(A-B)}{B}}$$

$$\therefore \tan x = \frac{x}{\frac{B-\frac{2x^2}{3!}+\frac{4x^4}{5!}-\frac{6x^6}{7!}+\cdots}{B}} = \frac{x}{1-\frac{\frac{2x^2}{3!}-\frac{4x^4}{5!}+\frac{6x^6}{7!}-\cdots}{B}}$$

$$= \frac{x}{1-x^2\frac{\left[\frac{1}{3}-\frac{x^2}{3!}\frac{1}{5}+\frac{x^4}{5!}\frac{1}{7}-\cdots\right]}{B}}$$

$$= \frac{x}{1-\frac{x^2}{\frac{B}{\left[\frac{1}{3}-\frac{x^2}{3!}\frac{1}{5}+\frac{x^4}{5!}\frac{1}{7}-\cdots\right]}}} \qquad \left(Let\ C = \left[\frac{1}{3}-\frac{x^2}{3!}\frac{1}{5}+\frac{x^4}{5!}\frac{1}{7}-\cdots\right]\right)$$

$$= \frac{x}{1-\frac{x^2}{\frac{3C+(B-3C)}{C}}}$$

$$= \frac{x}{1-\frac{x^2}{\frac{3-x^2.\left(\frac{1}{3.5}-\frac{x^2}{3!}\frac{1}{5.7}+\frac{x^4}{5!}\frac{1}{7.9}-\frac{x^6}{7!}\frac{1}{9.11}+\cdots\right)}{C}}}$$

Evaluating this infinite series expression times and again in the same manner, we can have,

$$\tan x = \frac{x}{1-\frac{x^2}{3-\frac{x^2}{5-\frac{x^2}{7-\frac{x^2}{9-\cdots}}}}} \qquad (3.2)$$

Now let us assume x to be a rational number. Therefore, $x = \frac{u}{v}$ for u and v be integers and v be a nonzero number.

$$\therefore \tan\frac{u}{v} = \frac{\frac{u}{v}}{1-\frac{\frac{u^2}{v}}{3-\frac{\frac{u^2}{v}}{5-\frac{\frac{u^2}{v}}{7-\frac{\frac{u^2}{v}}{9-\cdots}}}}}$$

$$\therefore \tan\frac{u}{v} = \frac{u}{v-\frac{u^2}{3v-\frac{u^2}{5v-\frac{u^2}{7v-\frac{u^2}{9v-\cdots}}}}} \qquad (3.3)$$

Now, notice that the numerators are squared u which is the same for the rest of all the terms excluding the first one. On the other hand, the denominators are consistently increasing with the multiplicity of odd numbers. So, it is quite evident that with constant numerator there must a point onwards where the denominator is greater than the numerator. For better visualization let us suppose that u and, v is chosen in such a manner that $5v>u^2$.

$$\therefore\ \frac{u^2}{5v-\frac{u^2}{7v-\frac{u^2}{9v-\cdots}}} < 1$$

$$Let\ us\ assume\ the\ fraction\ to\ be\ rational,\ \frac{u^2}{5v-\frac{u^2}{7v-\frac{u^2}{9v-\cdots}}} = \frac{B}{A} < 1 \qquad \Rightarrow B < A$$

If $\frac{u^2}{5v-\frac{u^2}{7v-\frac{u^2}{9v-\cdots}}}$ is less than 1 then, $\frac{u^2}{7v-\frac{u^2}{9v-\frac{u^2}{11v-\cdots}}}$ is definitely less than 1.

$$\therefore\ \frac{u^2}{7v-\frac{u^2}{9v-\frac{u^2}{11v-\cdots}}} = \frac{5Bv-Au^2}{B} < 1\ ,$$

$$Let,\ C = 5Bv - Au^2$$

$\Rightarrow C < B$ and onwards.

Therefore, we can assign infinitely many positive values in between 1 and 0 followingly, $1 > A > B > C > D > E \dots\dots > 0$. It is to be noted that B<A Implies, $C = 5Bv - Au^2$ is less than the difference of $5v-u^2$. A general theorem says such kind of ever decreasing numbers can never be ended till zero, which can also be assigned with logic. If the numbers are ever decreasing with relation in their difference, not in their ratio, then they will go beyond negation. Thus, the punch line is, 'It is not possible to assign

infinitely many such ever decreasing numbers within any range of positive numbers. Hence, our consideration that the fraction to be rational is wrong.'.

$$\frac{u^2}{5v-\frac{u^2}{7v-\frac{u^2}{9v-\cdots}}} = irrational\ (I)$$

Since any integer subtracted to or, divided by an irrational number is irrational, we can conclude that $\tan\frac{u}{v} = \frac{u}{v-\frac{u^2}{3v-\frac{u^2}{5v-\frac{u^2}{7v-\frac{u^2}{9v-\cdots}}}}}$ is an irrational number. That's why the tan function of a rational number gives an output of an irrational one. For better visualization, some graphs have been plot to convince that the infinite fraction meets the tan function. Referred to that it is definite that for any rational input in tan function, the output is an irrational one. Go for the following figures. The blue-lined function in the figures is the plot of the Tan function and the red one plots equation 3.2 in steps. As the plot of the referred equation tends towards infinity (up to figure 4, thirteen fractions are plotted) the red curve and the blue one gradually coalesces over each other.

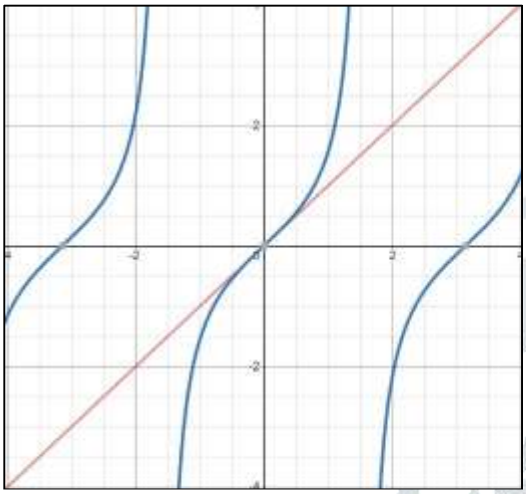


Figure 1

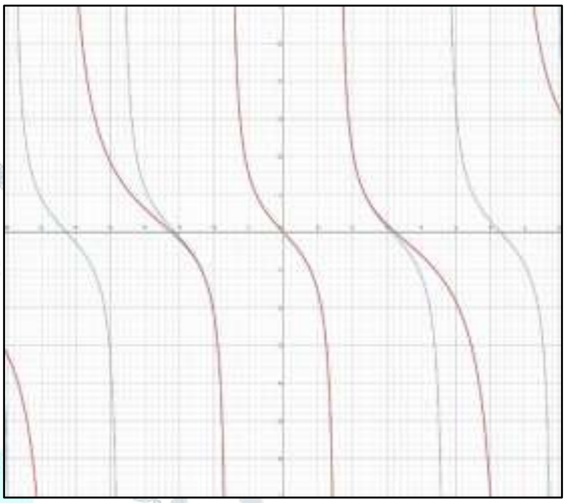


Figure 3

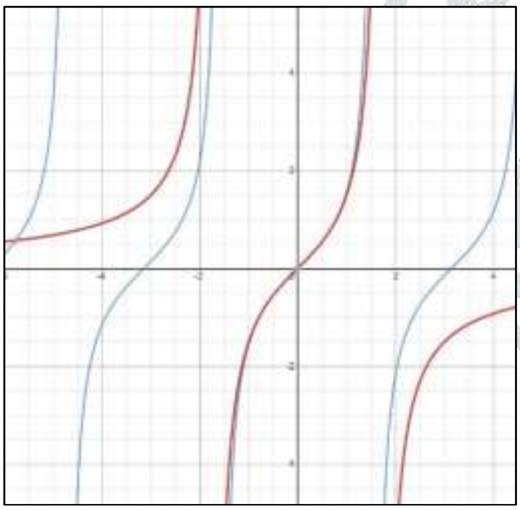


Figure 2

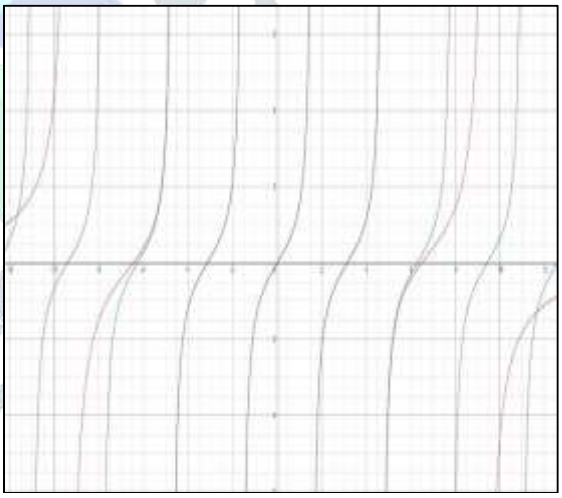


Figure 4

Now, being convinced enough it is to be noted that $\tan\frac{\pi}{4} = 1$ is a well-known value, where 1 is a rational number. But we just proved that tan of any rational value is irrational so, assuming π as a rational quantity, $\tan\frac{\pi}{4}$ should have been an irrational number, which being 1 is certainly a false statement. So, our supposition of π to be rational is wrong, which insinuates that π is an irrational number. Till now we can infer that π is a definite irrational number [13] which is a geometric circular constant that cannot be expressed exactly. Yet, the prime intention of this article is to convince ourselves about the geometric interpretation beyond all these infinite nested fraction series. Before entering that section, it is obvious to mention that for a long journey of mathematics there was no periodic algebraic expression to demonstrate π. Later mathematicians were able to express Π as several periodic and generalized infinite fraction series. Refer to some of them in the following augmentations [26].

$$\pi = \frac{4}{1+\frac{1^2}{2+\frac{3^2}{2+\frac{5^2}{2+\cdots}}}} = 3+\frac{1^2}{6+\frac{3^2}{6+\frac{5^2}{6+\frac{7^2}{6+\cdots}}}} = \frac{4}{1+\frac{1^2}{3+\frac{2^2}{5+\frac{3^2}{7+\cdots}}}} \qquad (3.4)$$

3.8 Exploring a massive geometry behind infinity

We can start with the Leibniz expansion of π. One of the most celebrated brains of mathematics, the godfather of calculus, Gottfried Wilhelm Leibniz has been credited with the discovery of the expansion of π [10], although preceding to 300 years Madhava originally invented this expansion. James Gregory also rediscovered the expansions of Madhava by his means [27]. The Old mathematicians like Archimedes of 250 BC were aggrieved on the naming of π as Feynman's constant after the famous physicist Richard Feynman even without a bit of connectivity. However, our esteemed expansion says,

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots \quad (3.5)$$

This wonderfully rhythmic expression of π consists of consecutive odd numbers with alternative negation. But I don't think anyone can visualize even the smallest of any geometric touch. It is quite impressive to have such an algebraic expression to describe the number but, simultaneously there is some gap of understanding it without some geometry because after all, π is the legal son of circular geometry. Sarcastically, the whole infinite fraction sum deals with circular kinds of stuff without even consisting of any hints buried rather only the sums of some weird odd numeral patterns. Originally, the expansion has a huge circle hidden in it. This is wonderfully explained in the book "Anschauliche Geometrie" (German) [In English, 'Geometry and The Imagination'] by the mathematical megastar David Hilbert and his colleague Stefan Cohn Vossen (1932) [11]. We are going to stabilize that geometry of π within its algebraic expression following Fermat's Christmas Theorem.

Fermat, a genius of mathematics and physics proposed that the set of all consecutive positive odd numbers consists of some good numbers and some bad numbers. The positive odd numbers like 1, 5, 9, 13… are good numbers whereas, 3, 7, 11, 15… are said to be bad numbers. Now using these, Fermat discovered a theorem called Christmas theorem. Any number can be factorized into prime numbers. Among them there would be some good numbers and some bad numbers which can be called good factors and bad factors accordingly, the theorem says, the number of ways [W(N)] for expressing a number [N] as the sum of two squared integers [I_1, I_2] is equal to 4 times the bad factors subtracted to good ones. Mathematically the statement can be given as follows,

$$W\left(N = {I_1}^2 + {I_2}^2\right) = 4.(Good - Bad)factors. \quad (3.6)$$

Let us explore some concepts of calculating the area under a circle. Plotting a circle on a graph paper, you can count the number of lattice points of the unit squares embedded in the circle drawn. Since are of each of the squares is unity, approximately the number of lattice points [N(r)] in the circle is the area of it. Refer to the figure… for exploring the evaluating process of a circular area.

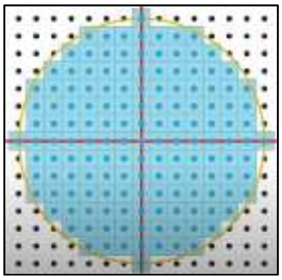


Figure 5

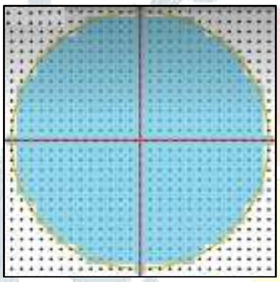


Figure 6

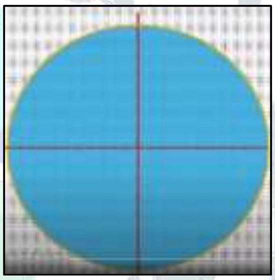


Figure 7

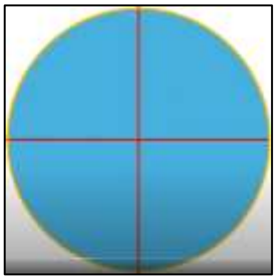


Figure 8

Now, the mathematical formula of the area is πr^2.

$$\therefore \pi r^2 = N(r)$$

$$\therefore \pi = \frac{N(r)}{r^2} \quad (3.7)$$

Now we can evaluate the value of π by putting the number of lattice points and its squared radius. Assuming different radii I am trying to give a statistical review of different observed values of π, refer to Table 1. $\pi(n)$ denotes the measured value of π when the radius of the circle is n.

Table 1: List of the values of Π corresponding to different values of radius of a circle.

Slot No.	Value of radius (r)	Value of Π (π)
1.	7	3.0408…
2.	10	3.17
3.	13	3.1301…
4.	20	3.1425…
5.	30	3.134…
6.	54	3.1361…
7.	100	3.1417…
8.	200	3.140725…
9.	300	3.14107…
10.	1000	3.1415…

The values corresponding to the slot number 2, 4, 5, 7, 8, 9 have been taken directly from the book, Geometry and The Imagination, by Hilbert. They may have been estimated following different paths of geometry. Refer to the book for detail [28]. The further conclusions drawn are based on the rest of the values.

$\therefore \pi(7) = \frac{N(7)}{49} = 3.0408$ ... (not a good approximation but not a bad either) [N (7) =149]

$\therefore \pi(13) = \frac{N(13)}{169} = 3.1301$ ...

$\therefore \pi(54) = \frac{N(54)}{2916} = 3.1361$ ...

$\therefore \pi(10000) = \frac{N(1000)}{100000000} = 3.1415$ ... (wonderfully approximated)

Therefore, it can be noted that the value of π is getting more correctly approximated as the circle gets bigger in radius. Finally, π can be expressed as,

$$\pi = \lim_{r\to\infty} \frac{N(r)}{r^2} \qquad (3.8)$$

In an alternative manner, using basic geometry and trigonometry, we can obtain a wonderful approximation of π's expansion as,

$$\pi = \lim_{n\to\infty} n \sin(\frac{180°}{n}) \qquad (3.9)$$

We can verify this statement very simply in the domain of calculus.

$$\pi \ (in\ radians) \ = 180° \ (in\ degrees)$$

$$\therefore \lim_{n\to\infty} n \sin\left(\frac{180°}{n}\right) = \lim_{n\to\infty} n \sin\left(\frac{\pi}{n}\right)$$

$$= \lim_{n\to\infty} n \times \frac{\pi}{n} \times \sin\left(\frac{\pi}{n}\right)$$

$$= \lim_{n\to\infty} n \frac{\sin\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}}$$

$$= \lim_{h\to 0} \frac{\pi \sin h}{h} \quad \left[h = \frac{\pi}{n}\right]$$

$$= \pi. \lim_{h\to 0} \frac{\sin h}{h}$$

$$= \pi$$

So, the expression is verified and can further be justified sensibly with some statistical data. Plug into Table 2 [29] to get convinced that π has a consanguineal connection with the infinite series. We are further going to invest our brains in developing such convincing statistics for equation 3.9.

Table 2: Listing the values of Π with estimated accurate digits vide equation 3.9.

Serial Number	**Value of n**	**Value of π**	**Number of accurate digits**
1.	3	2.598076211353316	0
2.	4	2.8284271274619	0
3.	5	2.938926261462366...	0
4.	6	2.9999999999999996...	0
5.	7	3.037186173822907...	1
6.	8	3.0614674589207183...	1
7.	10	3.090169943749474...	1
8.	12	3.105828541230249...	2
9.	20	3.1286893008046173...	2
10.	25	3.1333308391076065...	2
11.	50	3.1395259764656687	2
12.	75	3.1406740296899716...	3
13.	96	3.1410319508905093...	4
14.	100	3.141075907812829...	4
15.	500	3.14171827794755...	5
16.	1000	3.1415874858795636...	5
17.	10000	3.141592601912665...	7
18.	100000	3.1415926530730216...	9
19.	1000000	3.1415926535846257...	12
20.	10000000	3.141592653589741...	14

By the by, let us proceed with equation 3.8. But putting values of lattice points via counting is a very rigorous and limited process for approaching to our inference. In case a generalized expression is needed for evaluating the numbers of lattice points. For that sake, Fermat's Christmas theorem can be employed. Let us assume that the distance of a lattice point be 'd'. So, d^2 is always less than

r^2. If we can form a right-angled triangle with two lattice points and the origin as the three vertices then, the squared hypotonus, d^2 is the sum of two integer's squares. Thus, the number of ways of expressing $d^2 < r^2$ as the sum of two squared integers can be found using Fermat's theorem. The expression of a number as the sum of two squared integers was first observed and noted as well as described by prof. Girard is known as Girard's Theorem [30, 31]. Mathematician Fermat did not use to prove the mathematical theorems he proposed. In most cases, the proofs were lost in due course of time. The Christmas Theorem is no exception. The proof of it was first invented by prof. Euler [32, 33]. Exactly the total number of lattice points is the possible ways of expressing d^2as the aforestated sum.

$$\therefore N(r) = \sum 4(good\ factros - bad\ factors)+1 \text{ [because the origin is to be included]} \quad (3.10)$$

$$\therefore N(r) = 4.[\sum good - \sum bad]+1$$

$$\therefore N(r) = 4.\left[\left(\left[\frac{r^2}{1}\right]+\left[\frac{r^2}{5}\right]+\left[\frac{r^2}{9}\right]+\left[\frac{r^2}{13}\right]+\cdots\right)-\left(\left[\frac{r^2}{3}\right]+\left[\frac{r^2}{7}\right]+\left[\frac{r^2}{11}\right]+\left[\frac{r^2}{15}\right]+\cdots\right)\right]+1 \text{ where [.] denotes GIF function.}$$

$$\therefore \lim_{r\to\infty}\frac{N(r)}{r^2} = 4.\lim_{r\to\infty}\left[\frac{\left(\left[\frac{r^2}{1}\right]+\left[\frac{r^2}{5}\right]+\left[\frac{r^2}{9}\right]+\left[\frac{r^2}{13}\right]+\cdots\right)}{r^2}-\frac{\left(\left[\frac{r^2}{3}\right]+\left[\frac{r^2}{7}\right]+\left[\frac{r^2}{11}\right]+\left[\frac{r^2}{15}\right]+\cdots\right)}{r^2}\right]+\frac{1}{r^2}$$

Without making the mathematics more critical it can be assigned very logically that the box function tends to an output of integers as r tends to infinity. So, simply we can remove the boxes while evaluating the limits, and applying La-Hospital's rule the limits normally get reduced to each corresponding fractions.

$$\therefore \lim_{r\to\infty}\frac{N(r)}{r^2} = 4.\left[\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots\right]+0$$

$$\therefore \pi = 4.\left(\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots\right)$$

Quite evidently it is enough visualizable that the algebraic expansion is an alternative form of a huge circle hidden in it. Using Fermat's Christmas theorem, we discussed the derivation of Madhava-Leibniz expansion. Finally, it is to be noted that π can therefore be inferred to be a definite irrational number which is not expressible exactly although the geometric sense of it is in to-to adherence to its algebraic expression. The geometric π and the algebraic ones are the same rather the paradox can be nullified.

IV. Inference

This article explores every pros and con of π. Inspecting each inch of geometry, we concluded that π can be said as a rational and definite number although, it is not true when algebraic expansions are consulted. This article points towards the sanguine relation of π with infinite fraction series. Using Euler's theorem and Lambert's identity we arrived after π's irrationality. It is again to be certified that the irrational number π is derived from a rhythmic odd infinite fraction series. Going beyond the numbers this write-up successfully establishes the existence of a huge circle embeds secretly within those infinite series. Finally, it is to be concluded that π is not a rational one rather it is a definite irrational quantity that is not expressible exactly. The paradoxes regarding π's irrationality thus can be disregarded. From any sense of mathematics whether it is geometry or, algebra the expression of π and its value is constant. Geometrically, the formula of the circumference with π is an approximation. The ratio of area to the squared radius of an infinitesimally large circle is the real tone of π. At last, the inference is that the geometric articulation of utterance of π is in absolute accordance with its algebraic pronouncement.

V. Acknowledgment

The entire funding of the research project was lugged by personal investment. This work was appreciably supported by Sanghamitra Roy, with her supervision and guidance in editing. We would like to extend our gratitude to those people who succored us to bring this Paper to fruition. First, we would like to thank Tushar Banerjee for giving us the opportunity and admirable guidance in paving up to fulfillment as well as showing us the process of this paper's publication. Finally, author Archisman Roy is eagerly expressing his due regards with the warmest and hearty gratefulness for his parents to aid a great deal in the paper's publication, for his father's lucrative advice, and especially thanks to his mother who helped throughout the whole process by all means. From all the co-authors an exuberant salutation is being delivered to dear, Prof. Asim Kumar Mukherjee, for his aid in the formulation of mathematics.

VI. References

[1] .

[2] 2016. π^e trillion digits of π. pi2e.ch. Archived from the original.

[3] Iwao, H. Emma. 2019. Pi in the sky: Calculating a record-breaking 31.4 trillion digits of Archimedes' constant on Google Cloud. Google Cloud Platform. Retrieved 12 April 2019.

[4] Arndt, J. and Haenel, C. 2006, Pi Unleashed. Springer-Verlag: 17. ISBN 978-3-540-66572-4.

[5] Bailey, David H.; Plouffe, Simon M.; Borwein, Peter B.; Borwein, Jonathan M. 1997. The quest for PI. The Mathematical Intelligencer, 19 (1): 50–56.

[6] Huggett, N. 2010. Zeno's Paradoxes: 3.2 Achilles and the Tortoise. Stanford Encyclopedia of Philosophy. Retrieved 2011.

[7] 2010. Zeno's Paradoxes. Stanford University. Retrieved April 3, 2017.
[8] Arndt, J. and Haenel, C. 2006, Pi Unleashed. Springer-Verlag: 170. ISBN 978-3-540-66572-4.
[9] Lambert, J. Reprinted in 1997. Mémoire sur quelques propriétés remarquables des quantités transcendantes circulaires et logarithmiques. Berggren, Borwein & Borwein: 129–140.
[10] Eymard, P. and Lafon, Jean P. 1999. The Number Pi. American Mathematical Society: 53-54. ISBN 978-0-8218-3246-2.
[11] Hilbert, D. and Vossen, Cohn S. (translated by Nemeyi P.). 1952. Geometry and The Imagination (Anschauliche Geometrie). Chelsea Publishing Company: 32-35. ISBN 0-8284-0087-3.
[12] Retrieved Document. 2018. Compendium of Mathematical Symbols. Math Vault.
[13] Weisstein, Eric W. Retrieved Document. 2018. Pi. mathworld.wolfram.com.
[14] Bogart, S. Retrieved Document. 2018. What Is Pi, and How Did It Originate? Scientific American.
[15] Arndt, J. and Haenel, C. 2006, Pi Unleashed. Springer-Verlag: 8. ISBN 978-3-540-66572-4.
[16] Arndt, J. and Haenel, C. 2006, Pi Unleashed. Springer-Verlag: 177, 205. ISBN 978-3-540-66572-4.
[17] Retrieved Document. 2013. The Computation of Pi by Archimedes: The Computation of Pi by Archimedes – File Exchange – MATLAB Central. Mathswork.com.
[18] Archimedes. http://itech.fgcu.edu/faculty/clindsey/mhf4404/archimedes/archimedes.html.
[19] Archimedes. Heath L. and Thomas. 2002. The Works of Archimedes. Courier Corporation. ISBN - 0-486-42084-1.
[20] Arndt, J. and Haenel, C. 2006, Pi Unleashed. Springer-Verlag: 196. ISBN 978-3-540-66572-4.
[21] Hardy and Wright 1938 and 2000: 177 footnotes. 1882. § 11.13–14 references Lindemann's proof as appearing at Math. Ann. 20: 213–225.
[22] Hayashi, T.; Kusuba, T. and Yano, M. 1990. The Correction of The Madhava Series for the circumference of a Circle. Centaurus 33 (2-3): 149-174.
[23] Sarma, k. V. 1972. A History of the Kerala School of Hindu Astronomy. Hoshiarpur.
[24] Gold D. and Pingree D. A hitherto unknown Sanskrit work concerning Madhava's derivation of the power series for sine and cosine. Historia Sci. 42: 49-65.
[25] Gupta R. C. 1973. The Madhava-Gregory series. Math. Education. 7. B67-B70.
[26] Lange, L.J. 1999. An Elegant Continued Fraction for π. The American Mathematical Monthly. 106 (5): 456–458.
[27] Arndt, J. and Haenel, C. 2006, Pi Unleashed. Springer-Verlag: 188-189. ISBN 978-3-540-66572-4.
[28] Hilbert, D. and Vossen, Cohn S. (translated by Nemeyi P.). 1952. Geometry and The Imagination (Anschauliche Geometrie). Chelsea Publishing Company: 34. ISBN 0-8284-0087-3.
[29] Srimani, S. 2015. A Geometrical Derivation of Pi. IOSR Journal of Mathematics, 11(6): 21.
[30] Stevin, S. 1625. l'Arithmétique de Simon Stevin de Bruges, annotated by Albert Girard. Leyde, 622.
[31] Dickson L. E. History of the Theory of Numbers. 2(6). 227-228.
[32] Retrieved Document. 1758. De numerus qui sunt aggregata quorum quadratorum. Novi commentarii academiae scientiarum Petropolitanae, 4(1752/3): 3-40
[33] Retrieved Document. 1760. Demonstratio theorematis FERMATIANI omnem numerum primum formae 4n+1 esse summam duorum quadratorum. Novi commentarii academiae scientiarum Petropolitanae, 5 (1754/5): 3-13.