A STUDY OF THE STRUCTURE OF LIE ALGEBRAS

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Abstract :

The Aim of this paper is to develop the structure of Lie algebras so as of getting the full classification for semi simple Lie algebra and their representations. Lie algebras and their representations in the field of ground theory having canonical structure.

Key Words : Lie algebras, structure, semi simple, group theory, canonical

Introduction

Since a subalgebra of g is a vector subspace closed under the commutator, and an ideal is a vector subspace h such that $[x, y] \in h$ for any $x \in g$, $y \in h$. This definition is the natural analog of an ideal in an associative algebra. It is because of skew-symmetry of the commutator there is no difference between left and right ideals: every right ideal is also automatically a left ideal.

As in the theory of associative algebras, if h is an ideal of g then the quotient space g/h has a canonical structure of a Lie algebra.

If $f : g_1 \to g_2$ is a morphism of Lie algebras, then Ker f is an ideal in g_1, Im f is a subalgebra in g_2, and f gives rise to an isomorphism of Lie algebras g/Ker f $\simeq$ Im f.

Again Let I, I_2 be ideals in g. Define

$$I_1 + I_2 = \{x + y \mid x \in I_1, y \in I_2\}$$

$$[I_1, I_2] = \text{Subspace spanned by } [x, y], x \in I_1, y \in I_2.$$

Then $I_1 \cap I_2$, $I_1 + I_2$, $[I_1, I_2]$ are ideals in g.

We study Lie algebras is by analyzing how close the Lie algebra is to a commutative Lie algebra. There are several ways of making it simple.

First, the center $z(g) = \{x \in g \mid [x, y] = 0 \text{ for all } y \in g\}$ is. It is more effective to study commutative quotients of g.

Since The commutant of a Lie algebra g is the ideal [g, g].

The quotient g/[g, g] is an abelian Lie algebra. Moreover, [g, g] is the smallest ideal with this property: if g/I is abelian for some ideal $I \subset g$, then $I \supset [g, g]$.

Commutant gives us another way of measuring how far a Lie algebra is from being commutative: the smaller [g, g] (and the larger g/[g, g]), the closer g is to being commutative. For example, for commutative g, we have [g, g] = 0.

For a Lie algebra g, define the series of ideals $D^i g$ (called derived series) by $D^0 g = g$ and

$$D^{i+1}g = [D^i g, D^i g].$$

Them the following conditions are equivalent:

(1) $D^n g = 0$ for large enough n.

(2) There exists a sequence of subalgebras $a^0 = g \supset a^1 \supset \cdot \cdot \cdot \supset a^k = \{0\}$ such that a^{i+1} is an ideal in a^i and the quotient a^i/a^{i+1} is abelian.

(3) For large enough n, every commutator of the form

$$[\dots[[x_1, x_2], [x_3, x_4]]\dots]$$

(2^n terms, arranged in a binary tree of length n) is zero.

Proof. Equivalence of (1) and (3) are easy when. Implication (1) $\Rightarrow$ (2) is also clear: we may write $a^i = D^i g$. To prove (2) $\Rightarrow$ (1), if a^i satisfies the conditions of the proposition, then we have $a^{i+1} \supset [a^i, a^i]$. So, we see that $a^i \supset D^i g$.

Lie algebra g is called solvable if it satisfies any of the equivalent conditions of above a solvable Lie algebra is an "almost commutative" Lie algebra: it is an algebra that can be obtained by successive extensions of commutative algebras.

For a Lie algebra g, define a series of ideals $D^i g \subset g$ by $D^0 g = g$ and $D^i{+}1g = [g, D_i g]$.

The following conditions are equivalent:

(1) $D_n g = 0$ for large enough n.

(2) There exists a sequence of ideals $a_0 = g \supset a_1 \supset \cdot \cdot \cdot \supset a_k = \{0\}$ such that $[g, a_i] \subset a_{i+1}$.

(3) For large enough n, every commutator of the form

$$[\dots[[x_1, x_2], x_3], x_4]\dots x_n]$$

(n terms) is zero.

Proof. Equivalence of (1) and (3) are the Implication (1) $\Rightarrow$ (2) is also clear: we can take $a_i = D_i g$. To prove (2) $\Rightarrow$ (1), note that if a_i satisfies the conditions of the proposition, then by induction, we see that $a_i \supset D_i g$.

Let $b \subset gl(n, K)$ be the subalgebra of upper triangular matrices, and n be the subalgebra of all strictly upper triangular matrices. Then b is solvable, and n is nilpotent.

If F is a flag in a finite-dimensional vector space V :

$F = (\{0\} \subset V_1 \subset V_2 \subset . . . V_n = V)$

with $\dim V_i < \dim V_{i+1}$ (we do not require that $\dim V_i = i$), then define

$b(F) = \{x \in gl(V) \mid xV_i \subset V_i \text{ for all } i\}$,

$n(F) = \{x \in gl(V) \mid xV_i \subset V_{i-1} \text{ for all } i\}$.

By taking F to be the standard flag in K^n then the Lie algebras b, n defined above.

We find that b(F) is solvable and n(F) is nilpotent. We may define $a_k(F) = \{x \in gl(V) \mid xV_i \subset V_{i-k} \text{ for all } i\}$ so that $b(F) = a_0$, $n(F) = a_1$. for $x \in a_k$, $y \in a_l$, we have $xy \in a_{k+l}$ so $[a_k, a_l] \subset a_{k+l}$, so $D_i n \subset a_{i+1}$. This proves nilpotency of n(F).

In over to show solvability of b, note that even though for $x, y \in b$ we can only say that $xy \in b$, for the commutator we have a stronger condition: $[x, y] \in n = a_1$. Diagonal entries of xy and yx coincide. Thus, $D^1 b \subset n = a_1$. It easily follows by induction that $D^{i+1} b \subset a_2 i$.

As b is not nilpotent: $D_2 b = [b,D_1 b] = D_1 b = n$, which can be easily deduced from $[x, E_{ij}] = (\lambda_i - \lambda_j)E_{ij}$ if x is a diagonal matrix with entries λ_i.

Them

(1) A real Lie algebra g is solvable (respectively, nilpotent) iff its complexification g_C is solvable (respectively, nilpotent).

(2) If g is solvable, then any subalgebra, quotient of g is also solvable. If g is nilpotent, then any subalgebra, quotient of g is also nilpotent.

(3) If g is nilpotent, then g is solvable.

(4) If $I \subset g$ is an ideal such that both I, g/I are solvable, then g is solvable.

Proof. Parts (1), (2) are obvious if we use definition of solvable algebra in the form "any commutator of the form . . . is zero", and similarly for nilpotent. Part (3) follows from inclusion $D^i g \subset D_i g$, which can be easily proved by induction.

Finally, to prove part (4), denote by φ the canonical projection $g \to g/I$. Then $\varphi(D^n g) = D^n(g/I) = 0$ for some n. Thus, $D^n g \subset I$. Therefore, $D^{n+k} g \subset D^k I$, so $D^{n+k} g = 0$.

References :

[1] Introduction to Lie Algebras and Representation theory, James E. Humphreys, Graduate texts in Mathematics, Springer,1972.

[2] Introduction to Lie algebras, Karin Erdmann and Mark. J. Wildon, Springer international edition, Springer, 2009.

[3] A. Vinogradov and M. Vinogradov. On multiple generalizations of Lie algebras and Poisson manifolds. Contemporary Mathematics, 1998.

[4] V. Filippov. n-Lie algebras. Sibirskii Matematicheskii Zhurnal, 2008.