An Exact Static Spherical Solution of Einstein's Field Equations for Zeldovich Fluid Distribution

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Abstract :

The present paper provides an exact static spherical solution of Einstein's field equations for Zeldovich fluid distribution. Here we have also obtained and discussed various physical and geometrical features.

Key Words : Line element, energy momentum censor, pressure, density, rotation tensor.

1. INTRODUCTION :

Various researchers have tried to find exact solutions of Einstein's field equations for static perfect fluid sphere in general relativity by using different conditions and assumptions which are of much value [2-4-, 6-9,12]. If one takes e.g. poly tropic fluid sphere $p = ae^{1+\frac{1}{n}}$ (Klain [5], Tooper [12], Buchdahl [1], or a mixture of ideal gas and radiation (Suhonen [11]), one should has to use numerical method. Singh and Yadav [10] have also studied. The static fluid sphere with equation of state p = ρ. Some other workers in this line are Yadav and Saini [14]and Singh & Kumar [15]. In this paper I have obtained an exact Static spherical solution of Einstein's field equations for the Zeldovic fluid distribution. To overcome the difficulty of infinite density at the centre (i.e. singularity at ρ = 0), it is assumed that the distribution has a core of radius r_0 and constant density ε_0 which is surrounded by Zeldovich fluid.

2. The field Equations and Their Solutions

We use the static spherically symmetric metric given by

(2.1) $ds^2 = e^{\upsilon}dt^2 - e^{\lambda}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$

Where α and β are function of r only. The field equations

(2.2) $R^{\alpha}_{\beta} - \frac{1}{2} R^{\alpha}_{\beta} = -8\pi^{\alpha}_{\beta}$

for the metric (2.1) for the Zeldovich fluid which can be regarded as a perfect fluid having the energy momentum tensor.

(2.3) $T^{\alpha}_{j} = (p+p)^{\alpha} u_{\beta} - \delta^{\alpha}_{\beta} p$

Characterized by the equation of state p = ρ in comoving co-ordinates

(i.e. $u_1 = u_2 = u_3 = 0$ and $u_4 = e^{\frac{\upsilon}{2}}$) are

(2.4) $8\pi p = e^{-\lambda}\left(\frac{\upsilon'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2}$

(2.5) $8\pi p = e^{-\lambda}\left(\frac{\upsilon''}{r} - \frac{\lambda'\upsilon'}{4} + \frac{\upsilon'^2}{4} + \frac{\upsilon' - \lambda'}{2r}\right)$

(2.6) $8\pi\rho = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2}$

From equations (2.4), (2.6) and using p = ρ we have

(2.7) $e^{-\lambda}\left(\frac{\upsilon'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2}$

Equations (2.7) shows that if υ is know λ can be found, so we choose

(2.8) $\upsilon = Ar, \quad A = \text{constant}$

Using (2.8) equation (2.7) takes the form

(2.9) $\frac{e^{-\lambda}\lambda^{1}}{r} - \frac{3e^{-\lambda}}{r} + \frac{2}{r^2} = 0$

Putting $z = e^{-\alpha}$ the equation (2.9) is reduced to

(2.10) $\frac{dz}{dr} + \frac{3z}{r} = \frac{2}{r^2}$

Which is a linear differential equation whose solution is

(2.11) $z = \frac{2r^3}{3} + c.$

Therefore we get

(2.12) $e^{-\lambda} = \frac{2}{3} r^3 + c$

where C is an integration constant

Hence the metric (2.1) after suitable adjustment of constants takes the form

(2.13) $ds^2 = r\,dt^2 - r^{-3} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2)$.

The non zero component of Rhijk for the metric (2.13) are

(2.14) $R_{1212} = \frac{R_{1313}}{\sin^2\theta} = \frac{-R_{2323}}{2r\sin^2\theta} = -\frac{1}{2}$,

$$R_{1414} = -\frac{-3R_{2424}}{r} = \frac{-3R_{3434}}{r\sin^2\theta} = -\frac{9}{2r^5}$$

Choosing the orthonormal tetrad $\bar{\lambda}^i_{(j)}$ as

$$\lambda^{-i}_{(1)} = (r^{3/2}, 0, 0, 0), r^{-i}_{(2)} = \left(0, \frac{1}{r} 0, 0\right)$$

(2.15) $\lambda^{-i}_{(3)} = (0, 0, \frac{1}{r\sin\theta}, 0), \lambda^{-i}_{(4)} = (0, 0, 0, \frac{1}{\sqrt{r}})$

The physical component $R_{(abcd)}$ of the curvature tensor are

(2.16) $R_{(1212)} = R_{(1313)} = \frac{r^4 R_{(1414)}}{9} = \frac{-r^4 R_{(2323)}}{2} = -\frac{r}{2}$

$$R_{(2424)} = R_{(3434)} = \frac{3}{2r^7}$$

Also for the metric (2.13) the fluid velocity u^i is found to be

(2.17) $u^1 = u^2 = u^3 = u_1 = u_2 = u_3 = 0$ and $u^4 = \frac{1}{\sqrt{r}}, u_4 = \sqrt{r}$

The scalar of expansion $\theta = u^i$; i is identically zero.

The non-zero components of the tensor of rotation w_{ij} and shear tensor σ_{ij} are

(2.18) $w_{14} = -w_{41} = -\frac{1}{2\sqrt{r}}$,

(2.19) $\sigma_{14} = \sigma_{41} = -\frac{1}{2\sqrt{r}}$

3. Solution for the Perfect Fluid Core

Pressure and density for metric (2.13) are

(3.1) $$8\pi p = 8\pi\rho = -4r + \frac{1}{r^2}$$

It follows from (3.1) that the density of the distribution tends to infinity as r tends to zero. In order to get rid of the singularity at r = 0 in the density we visualize that the distribution has a core of radius r_0 and constant density ρ_0. The field inside the core is given by the Schwarzschild internal solution.

(3.2) $$e^{-\lambda} = 1 - \frac{r^2}{R^2}$$

$$e^{\upsilon} = [A - B\left(1 - \frac{r^2}{R^2}\right)^{1/2}]^{1/2}$$

$$8\pi p = \frac{1}{R^2} = \left[\frac{3B\left(1-\frac{r^2}{R^2}\right)^{1/2} - A}{A - B\left(1-\frac{r^2}{R^2}\right)^{1/2}}\right]$$

where A and B are constants and

$$R^2 = \frac{3}{8\pi\rho_o}$$

The constants appearing in the solution can be evaluated by the continuity conditions for the metric (2.13) and (3.2) at the boundary r = r_0.

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