# FLOW OF VISCO-ELASTIC FLUID BETWEEN TWO INFINITE ROTATING PARALLEL PLATES

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**Abstract**

Consideration is given to the flow of visco-elastic fluid contained between two infinite parallel plates which rotates with the same angular velocity Ω about axis normal to the plates but not coincident. Certain formulas are derived in the fourth order approximation of the Rivilin-Ericksen [6] fluid.

## Introduction

An exact solution of Navier-stokes equations including interial effects was given by Abbott and Walters [1]. These authors also gave an approximate solution to this problem for a general linear visco-elastic fluid assuming the distance between axes of rotation small [2, 3]. Wineman and Pipkin [4] had analysed the flow of essentially a third fluid flowing by gravity down a tilted through. Rajagopal [5] and Goddard [6] independently showed that the flow field considered was also dynamically compatible for an isotropic fluid of constant density. Rajagopal [5] also derived a set of similar differential equation, but in addition, showed that for the type of motion considered in [1], the Cauchy stress in an incompressible fluid must depend only upon the first two Riylin-Ericksen teresors $A_1$ and $A_2$. P.N. Kaloni, A.M. Siddiqui [7] and Singh [10] studied the flow problem of Visco-Elastic between eccentric disks and found some interesting results for third order fluid. Evelyne *et al*. [9] determine the linear visco elastic behavior.

In this paper we propose to study the flow of Visco-Elastic fluid contained between two infinite parallel plates which rotate with same angular velocity Ω about axes normal to the plate but not coincident. For this the fourth order approximation of Rivlin-Erickson [8] model of fluid is considered. By expanding the variable in ascending powers of suitable parameters, we determine the force on the upper plate. Soundalgebar *et al*. [12] and Subha yd[11] have studied the viscous dissipation effect in non-Newtonian studies.

**Mathematical Analysis:**

The fourth order approximation of the general constitutive equation given by Rivlin-Erickson [6] is

$$S = -pI + \phi_1 A_1 + \phi_2 A_2 + \phi_3 A_1^2 + \phi_4 (A_1 A_2 + A_2 A_1) +$$
$$+ \phi_5 (\text{tr } A_1^2) A_1 + \phi_6 A_2^2 + \phi_7 (A_2 A_1^2 + A_1 A_2^2) +$$
$$+ \phi_8 (\text{tr } A_2) A_2 + \phi_9 (\text{tr } A_2) A_1^2 \quad \ldots\ldots\ldots\ldots\ (1)$$

Where p is an arbitrary hydrostatic pressure and $\phi_1$'s are constant and $A_1$ and $A_2$ are Rivlin-Erickson tensors.

We consider steady flow of a fourth order fluid, rotating between two disks, each rotating with angular velocity Ω and separated by a gap h. The disks are eccentric to each other, the upper disk has its centre of rotation displaced from lower by a distance h.

Following [1], [3] and [4] we assume the velocity field of the form

$$V_x = g(z) - y, \qquad V_y = K(z) + \Omega x, \qquad V_z = 0 \quad \ldots\ldots\ldots\ (2)$$

and boundary condition of the form:

$$V_x = -\Omega a/2 - \Omega y, \qquad V_y = \Omega x, \qquad V_z = 0 \text{ at } z = 0$$
$$V_x = \Omega a/2 - \Omega y, \qquad V_y = \Omega x, \qquad V_z = 0 \text{ at } z = h \quad \ldots\ldots\ (3)$$

For above these assumption the flow field of the fourth order fluid equation (1) reduced to :

$$S = -pI + (\mu - \beta_1\Omega^2)A_1 + (\alpha_1 - \gamma_1\Omega^2)A_2 + (\alpha_2 - 2\gamma_2\Omega^2)A_1^2 +$$
$$\beta_2 (A_1 A_2 + A_2 A_1) + \beta_3 (\text{tr}.A_1^2) A_1 + \gamma_3 A_2^2 +$$
$$\gamma_4 (A_2 A_1 2 + A_1^2 A_2) + \gamma_5 (\text{tr}.A_2) A_2 + \gamma_6 (\text{tr}.A_2) A_1^2 \quad \ldots\ldots\ (4)$$

where

$\mu, \alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6$

all are material constants.

Now introducing the non-dimensional quantities:

$$\alpha = \frac{z}{h}, \quad G(\alpha) = \frac{g(\tau)}{a\Omega}, \quad K(\alpha) = \frac{k(\tau)}{a\Omega} \qquad \text{.......... (5)}$$

and defining :

$$F = G - iK, \quad T = S_{xz} - i\,S_{yz}, \quad T = \eta^* F \qquad \text{.......... (6)}$$

where η* is the complex shear modules and dash denotes differentiation with respect to α.

The equations of motion are reduced to :

$$\frac{d}{d\alpha}\eta^* F' + i\rho\, ah\Omega^2 F = (q - is)h \qquad \text{............. (7)}$$

or

$$F'' - \alpha^2 F + \in \frac{d}{d\alpha}\left[F|F'|^2\right] = (q - is) \qquad \text{................ (8)}$$

where

$$\eta^* = \left[\frac{\mu\partial\Omega}{h}(\lambda_1 + i\lambda_2)\left\{1 + \in |F'|^2\right\}\right] L = \frac{h^2}{\mu\partial\Omega(\lambda_1 + i\lambda_2)}$$

$$\lambda_0 = \frac{\rho\Omega h^2}{\mu}, \lambda_1 = \left(1 - \frac{\beta_1}{\mu}\Omega^2\right), \lambda_2 = (\alpha_1 - \gamma_1\Omega^2)\frac{\Omega}{\mu} \qquad \text{................ (9)}$$

$$\lambda_3 = \frac{2(\beta_2 + \beta_3)}{\left(\frac{a^2}{h^2}\frac{\Omega^2}{\mu}\right)}, \quad \lambda_4 = 2\left(\gamma_3 + \frac{1}{2}\gamma_4 + \gamma_5\right)\frac{a^2}{h^2}\frac{\Omega^3}{\mu}$$

The boundary condition in terms of F are

F = –1/2 on α = 0, F = ½ on α = 1 ............. (10)

After integrating once the pressure is given

$$P = \frac{\rho\Omega^2}{2}(x^2 + y^2) + (qx + sy) + \left\{2(\alpha_1 - 2\gamma_1\Omega^2)\right\}$$

$$\left\{\frac{a^2}{h^2}\Omega^2|F'|^2\right\} + \left(\gamma_3 + \gamma_4 + \gamma_5 + \frac{1}{2}\gamma_6\right)\frac{a^4}{h^4}\Omega^2|F'|^4 + C_0 \qquad \text{............. (11)}$$

where $C_0$ is an arbitrary constant. We see from (1) that non-zero values of q and s would give rise to the pressure gradient between the plate. In order to remove the possibility of this component we take q = s = 0. F(α) is expanded in powers of ∈.

On equating terms of zero$^{th}$ order in ∈ in (8) we get:

$$F_0'' - \beta^2 F_0 = 0 \qquad \text{……………..} (12)$$

The boundary condition reduced to :

$$F_0(0) = -\frac{1}{2}, \quad F_0\left(\frac{1}{2}\right) = 1 \qquad \text{…………….} (13)$$

The solution of equ. (12) is same as [1], [5].

$$F_0 = \frac{1}{2}\left[\frac{\sinh\beta\left(\alpha - \frac{1}{2}\right)}{\sinh\frac{\beta}{2}}\right] \qquad \text{………………….} (14)$$

where

$$\beta = \xi + i\zeta, \xi^2 = \frac{\lambda_0}{2}\left\{\frac{\sqrt{\lambda_1^2 + \lambda_2^2}}{\left(\lambda_1^2 + \lambda_2^2\right)} - \lambda_2\right\} \qquad \text{……,,,,,,,………} (15)$$

$$\zeta^2 = \frac{\lambda_0}{2}\left\{\frac{\sqrt{\lambda_1^2 + \lambda_2^2}}{\left(\lambda_1^2 + \lambda_2^2\right)} + \lambda_2\right\}$$

The first order of the equation (8) to be solved as :

$$F_1'' = \beta F_1 = H\,\Psi'(\alpha) \qquad \text{………………….} (16)$$

where

$$H = \frac{\beta\left|\beta^2\right|}{8\sinh\left(\frac{\beta}{2}\right)\left|\sinh\left(\frac{\beta}{2}\right)\right|^2}$$

and

$$\Psi(\alpha) = \sinh\beta\left(\alpha - \frac{1}{2}\right)\left|\cosh\beta\left(\alpha - \frac{1}{2}\right)\right|^2 \qquad \text{……………….} (17)$$

The boundary condition is :

$$F_1(0) = 0, \quad F_1(1) = 0 \qquad \text{………………. (18)}$$

The solution of the equation (16) subjected to (18) is :

$$F_1(\alpha)=H\left[\frac{(2\beta+\bar{\beta})\sinh(2\beta+\bar{\beta})}{4\left\{\beta^2-(2\beta+\bar{\beta})^2\right\}}+\frac{(2\beta-\bar{\beta})\sinh(2\beta-\bar{\beta})\left(\alpha-\frac{1}{2}\right)}{4\left\{\beta^2-(2\beta-\bar{\beta})^2\right\}}+\right.$$

$$\frac{\bar{\beta}\sinh\bar{\beta}\left(\alpha-\frac{1}{2}\right)}{(2\beta-\bar{\beta})^2}-\frac{\sinh\beta\left(\alpha-\frac{1}{2}\right)}{\sinh\frac{\beta}{2}}\left\{\frac{(2\beta+\bar{\beta})\sinh\frac{(2\beta+\bar{\beta})}{2}}{4\left\{\beta^2-(2\beta+\bar{\beta})^2\right\}}\right.$$

$$\left.\left.+\frac{(2\beta-\bar{\beta})\sinh\frac{(2\beta-\bar{\beta})}{2}}{4\left\{\beta^2-(2\beta-\bar{\beta})^2\right\}}+\frac{\bar{\beta}\sinh\left(\frac{\bar{\beta}}{2}\right)}{2(\beta^2-\bar{\beta}^2)}\right\}\right] \qquad \text{………………. (19)}$$

where $\bar{\beta}$ is the complex conjugate of β.

The first order expression for the velocity field can now be given as:

$$G - iK = G_0 - iK_0 + \in (G_1 - ik_1) \qquad \text{………………. (20)}$$

By separating the real and imaginary parts in (20) with the help of (14) and (19) we determine the velocity components in horizontal plane. If X and Y are components of forces exerted by the fluid on one of plate then

$$X-iY=-\frac{\mu\partial\Omega\pi R^2}{h}\left[(\lambda_1+i\lambda_2)F'+\in(\lambda_1+\lambda_2)F'|F'|^2\right] \qquad \text{……………... (21)}$$

On substituting $F = F_0 + \in F_1$ and neglecting higher powers of ∈, the force field at upper plate at α = 1 is given by with (20) a,

$$X-iY=-\frac{\mu\partial\Omega\pi R^2}{h}(\lambda_2+i\lambda_1)\left[F_0'(1)+\in\left\{F_1'(1)+F_0'(1)|F_0'(1)|^2\right\}\right] \qquad \text{………… (22)}$$

And putting the values of $F_0$ and $F_1$ from equations (14) and (19) in (22) we get

$$X - iY = -\frac{\mu\pi\partial\Omega R^2}{h}(\lambda_1+\lambda_2)\left[\frac{\beta\cosh\frac{\beta}{2}}{2\sinh\frac{\beta}{2}} + \frac{\in H}{\sinh\frac{\beta}{2}}\right.$$

$$\left\{-\frac{\beta^2\sinh\left(\beta+\frac{\bar\beta}{2}\right)}{4\left\{\beta^2-(2\beta+\bar\beta)^2\right\}} - \frac{\beta(\beta+\bar\beta)\sinh\frac{1}{2}(2\beta+\bar\beta)\cosh\frac{\beta}{2}}{4\left\{\beta^2-(2\beta-\bar\beta)^2\right\}}\right.$$

$$-\frac{\beta^2\sinh\left(\frac{\beta-\bar\beta}{2}\right)}{4\left\{\beta^2-(2\beta+\bar\beta)^2\right\}} - \frac{\beta(\beta+\bar\beta)\sinh\left(\frac{2\beta-\bar\beta}{2}\right)\cosh\frac{\beta}{2}}{4\left\{\beta^2-(2\beta-\bar\beta)^2\right\}}$$

$$\left.\left.+\frac{\beta^2\cosh\frac{\beta}{2}\sinh\frac{\beta}{2}}{2(\beta^2-\bar\beta)^2} - \frac{\beta\bar\beta\sinh\frac{\bar\beta}{2}\cosh\frac{\beta}{2}}{2(\beta^2-\bar\beta^2)}\right\}\right] \quad \ldots\ldots\ldots. (23)$$

Now if we put $\beta_i = \gamma_i = 0, \quad \in = 0, \quad \lambda_1 = \lambda_2 = \alpha_1\frac{\Omega}{\mu}$

in equation (21), the force at upper plate is given by

$$X - iY = -\pi R^2(\mu + i\alpha_1\Omega)\left[a\Omega\left\{\frac{\beta(1+\cosh\beta)}{\sinh\beta}\right\}\right]$$

$$= -\pi R^2\partial\Omega\mu^*\left[a\Omega\left\{\frac{\beta(1+\cosh\beta)}{\sinh\beta}\right\}\right] \quad \ldots\ldots\ldots\ldots\ldots.. (24)$$

where

$$\zeta^* = \mu + i\alpha_1\Omega$$

Again if we consider all the tens with the penish order and neglecling other higher terms, we get the dynamical viscosity

$$\mu^{**} = (\mu - \beta_1\Omega^2) + i\Omega(\alpha_1 - v_1\Omega^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots (25)$$

Since dynamical rigidity is always positive so the dynamical viscosity is also positive. Hence

$$\mu > \beta_1\Omega^2 \quad and \quad \alpha_1 < v_1\Omega^2$$

For $B = \alpha^2 h^2$

and $\mu^{*} = \eta^{*}$

The equation (24) is reduced to equation (49) of Abboot and Walter [1] for second order fluid.

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