INTEGRAL EQUATION INVOLVING HERMITE POLYNOMIAL AS ITS KERNEL..

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1. ABSTRACT: In this paper an integral equation involving Hermite polynomial as kernel is solved by the method of Laplace transform. Certain special cases are included.

Key words: Hermite polynomial, Laplace transforms

2. NOTATIONS AND RESULTS USED:

The Laplace Transform $f(p) = \int_0^\infty e^{-pt} f(t)dt, \quad \mathrm{Re}(p) > 0$ is represented as $F(p) \doteq f(t)$ (2.1)

Erdelyi { [1], p. 129,131,144,172,238}

If $F(p) \doteq f(t)$ then $F(p+a) \doteq e^{-at} f(t)$ (2.2).

If $F(p) \doteq f(t)$, $f(0) = f^{/}(0) = \ldots\ldots = f^{n-1}(0) = 0$ and $f^{n}(t)$ is continuous, then

$$p^n F(p) \doteq f^n(t) \tag{2.3}$$

If $F_1(p) \doteq f_1(t)$ and $F_2(p) \doteq f_2(t)$, then

$$\int f_1(u) f_2(t-u)du \doteq F_1(p)F_2(p) \tag{2.4}$$

$\Gamma m(p+b)^{-m} \doteq t^{m-1}e^{-bt}$, Re (m) > 0, Re (p) > - Re (b) (2.5)

$$e^{\beta t} He_{2n+1}[2^{\frac{1}{2}}(\alpha-p)^{\frac{1}{2}} t^{\frac{1}{2}} \doteq (-2)^{-n} (\frac{1}{2\pi})^{\frac{1}{2}} (\alpha-\beta)^{\frac{1}{2}} \frac{(2n+1)!}{n!} \frac{(p-\alpha)^n}{(p-\beta)^{n+\frac{3}{2}}}, \quad \mathrm{Re}(p) > \mathrm{Re}(\beta)$$

(2.6)

$$e^{\beta t} t^{-\frac{1}{2}} He_{2n}[2^{\frac{1}{2}}(\alpha-p)^{\frac{1}{2}} t^{\frac{1}{2}} \doteq (-2)^{-n} (\pi)^{\frac{1}{2}} \frac{(2n)!}{n!} \frac{(p-\alpha)^n}{(p-\beta)^{n+\frac{1}{2}}}, \quad \mathrm{Re}(p) > \mathrm{Re}(\beta) \tag{2.7}$$

$$t^{\alpha-1}{}_1F_1(n;\alpha;kt) \doteq \Gamma(\alpha)p^{n-2}(p-k)^{-n}, \quad \mathrm{Re}(p)>0, \mathrm{Re}(k)>0, \mathrm{Re}(\alpha)>0. \qquad (2.8)$$

$$t^{2\alpha+2n-1}{}_1F_2(n;\alpha+n;\alpha+n+\tfrac{1}{2};\tfrac{-1}{4}a^2t^2) \doteq \Gamma(2\alpha+2n)p^{-2\alpha}(p^2+a^2)^{-n}, \quad \mathrm{Re}(\alpha+n)>0. \quad (2.9).$$

3. MAIN RESULTS.

THEOREM-I

Each of the integral equation

$$f(t) = \int_0^t e^{b(t-u)} He_{2n+1}[(2a)^{\frac{1}{2}}(t-u)^{\frac{1}{2}}]g(u)du \qquad \text{and} \qquad (3.1)$$

$$g(t) = A\int_0^t e^{b(t-u)}(t-u)^{n-\frac{3}{2}}{}_1F_2\left[n, \frac{n}{2}-\frac{1}{4}, \frac{n}{2}+\frac{1}{4}, \frac{a^2}{4}(t-u)^2\right] \times \left[(D-b)(D+a-b)^n f(u)\right]du$$

(3.2) is the solution of the other, provided

i) n is a positive integer

ii) a and b are complex numbers

iii) $f^{n+1}(t)$ is sectionally continuous for $0 \le t \le x < \infty$ and $f(0) = f'(0) = \ldots\ldots = f^{n-1}(0) = 0$.

iv) $D \equiv \dfrac{d}{du}$, $\qquad A = \dfrac{(-2)^n n!(2\pi)^{\frac{1}{2}}}{a^{\frac{1}{2}}(2n+1)!\Gamma(n-\frac{1}{2})}$.

PROOF:

Let $F(p) \doteq f(t)$ and $G(p) \doteq g(t)$. Using (2.3) and (2.4) in (3.1), the integral equation (3.1) becomes;

$$F(p) = (-2)^{-n}\left(\frac{a}{2n}\right)^{\frac{1}{2}}\frac{(2n+1)!}{n!}(p-a-b)^n(p-b)^{-(n+\frac{3}{2})}G(p) \qquad (3.3)$$

Using (2.2), (2.3), (2.4) and (2.9)in (3.2), the integral equation (3.2) becomes,

$$G(p) = (-2)^{-n}\left(\frac{2\pi}{a}\right)^{\frac{1}{2}}\frac{n!}{(2n+1)!}(p-a-b)^{-n}(p-b)^{(n+\frac{3}{2})} \qquad (3.4).$$

The equations (3.3) and (3.4) can be obtained from each other. Hence by LERCHS theorem [3,P.5] it follows that each of the integral equation(3.1) and (3.2) is the solution of the other.

SPECIAL CASE:

Put a = 1, b = 0, in (3.1) and (3.2), to get the following result.

Each of the integral equations

$$f(t)=\int_0^t He_{2n+1}(2)^{\frac{1}{2}}(t-u)^{\frac{1}{2}}g(u)du \qquad (3.5) \quad \text{and}$$

$$g(t)=A\int_0^t (t-u)^{n-\frac{3}{2}}\,{}_1F_2\left[n,\frac{n}{2}-\frac{1}{4},\frac{n}{2}+\frac{1}{4},\frac{a^2}{4}(t-u)^2\right]\times\left[D(D+1)^n f(u)\right]du \qquad (3.6)$$

Is the solution of the other, provided,

i) n is a positive integer

ii) $f^{n+1}(t)$ is sectionally continuous for $0\le t\le x<\infty$ and $f(0)=f^{/}(0)=\ldots\ldots=f^{n-1}(0)=0$

iii) $D\equiv\frac{d}{du}$, $A=\frac{(-2)^n n!(2\pi)^{\frac{1}{2}}}{a^{\frac{1}{2}}(2n+1)!\Gamma(n-\frac{1}{2})}$.

THEOREM-II

Each of the integral equation

$$f(t)=\int_0^t e^{b(t-u)}(t-u)^{\frac{-1}{2}}He_{2n}[(2a)^{\frac{1}{2}}(t-u)^{\frac{1}{2}}]g(u)du \quad \text{and} \qquad (3.7)$$

$$g(t)=A\int_0^t e^{b(t-u)}(t-u)^{n-\frac{3}{2}}\,{}_1F_1\left[n,n+\frac{1}{2};a(t-u)\right]\times\left[(D-b)^{n+1}f(u)\right]du \qquad (3.8)$$

is the solution of the other, provided,

i) n is a positive integer

ii) a and b are complex numbers

iii) $f^{n+1}(t)$ is sectionally continuous for $0\le t\le x<\infty$ and $f(0)=f^{/}(0)=\ldots\ldots=f^{n-1}(0)=0$.

iv) $D\equiv\frac{d}{du}$, $A=\frac{(-2)^n n!}{\pi^{\frac{1}{2}}(2n)!\Gamma(n+\frac{1}{2})}$.

PROOF;

Let $f(t)\doteq F(p)\; and \quad g(t)\doteq G(p)$

Using (2.2), (2.4) and (2.7) in (3.7), the integral equation (3.7) becomes,

$$F(p)=(-2)^{-n}\pi^{\frac{1}{2}}\frac{(2n)!}{n!}(p-a-b)^n(p-b)^{-(n+\frac{1}{2})}G(p).$$

Using (2.2), (2.3), (2.4) and (2.8) in (3.8), the integral equation (3.8) becomes,

$$G(p) = A\Gamma(n+\frac{1}{2})(p-b)^{(n+\frac{1}{2})}(p-a-b)^{-n}F(p) \qquad (3.9)$$

$$= \frac{(-2)^n n!}{\pi^{\frac{1}{2}}(2n)!}(p-a-b)^{-n}(p-b)^{(n+\frac{1}{2})}F(p). \qquad (3.10)$$

The equations (3.9) and (3.10) can be obtained from each other. Hence by LERCHS theorem [3, P.5], it follows that each of the integral equation (3.7) and (3.8) is the solution of the other.

SPECIAL CASE:

In (3.7) and (3.8), put a =, and b = 0, to get:

Each of the integral equations

$$f(t) = \int_0^t (t-u)^{\frac{-1}{2}} He_{2n}[(2)^{\frac{1}{2}}(t-u)^{\frac{1}{2}}]g(u)du \qquad (3.10)$$

and

$$g(t) = A\int_0^t (t-u)^{n-\frac{3}{2}}\, {}_1F_1\left[n, n+\frac{1}{2}; (t-u)\right] \times \left[D^{n+1} f(u)\right]du \qquad (3.11).$$

Is the solution of the other, provided,

i) n is a positive integer

ii) $f^{n+1}(t)$ is sectionally continuous for $0 \le t \le x < \infty$ and $f(0) = f^{/}(0) = ... = f^{n-1}(0) = 0$

iii) $D \equiv \frac{d}{du}$, $A = \frac{(-2)^n n!}{\pi^{\frac{1}{2}}(2n)!\Gamma(n+\frac{1}{2})}$.

THEOREM-III.

Each of the integral equations

$$f(t) = \int_0^t e^{b(t-u)}(t-u)^{\frac{-1}{2}} He_{2n}[(2a)^{\frac{1}{2}}(t-u)^{\frac{1}{2}}]g(u)du \qquad (3.12)$$

and

$$g(t) = A\int_0^t e^{b(t-u)}(t-u)^{n-\frac{3}{2}} {}_1F_1\left[n, \frac{n}{2}-\frac{1}{4}, \frac{n}{2}+\frac{1}{4}; \frac{a^2}{4}(t-u)^2\right] \times \left[(D+a-b)^{n+1} f(u)\right] du. \quad (3.13)$$

Is the solution of the other, provided,

i) n is a positive integer

ii) a and b are complex numbers

iii) $f^{n+1}(t)$ is sectionally continuous for $0 \le t \le x < \infty$ and $f(0) = f^{/}(0) = ... = f^{n-1}(0) = 0$

iv) $D \equiv \frac{d}{du}$, $A = \frac{(-2)^n n!}{\pi^{\frac{1}{2}}(2n)!\Gamma(n-\frac{1}{2})}$.

PROOF: Similar to that of Theorem –II using (2.9).

SPECIAL CASE:

In (3.11) and (3.12), put a =, and b = 0, to get:

Each of the integral equations

$$f(t) = \int (t-u)^{\frac{-1}{2}} He_{2n}[2^{\frac{1}{2}}(t-u)^{\frac{1}{2}}] g(u) du \quad (3.14)$$

and

$$g(t) = A\int_0^t (t-u)^{n-\frac{3}{2}} {}_1F_2\left[n, \frac{n}{2}-\frac{1}{4}, \frac{n}{2}+\frac{1}{4}; \frac{1}{4}(t-u)^2\right] \times \left[(D+1)^{n+1} f(u)\right] du \quad (3.15)$$

Is the solution of the other, provided,

i) n is a positive integer

ii) $f^{n+1}(t)$ is sectionally continuous for $0 \le t \le x < \infty$ and $f(0) = f^{/}(0) = ... = f^{n-1}(0) = 0$

iii) $D \equiv \frac{d}{du}$, $A = \frac{(-2)^n n!}{\pi^{\frac{1}{2}}(2n)!\Gamma(n-\frac{1}{2})}$.

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