PERFECT MATCHING NEIGHBORHOOD IN GRAPHS

[1] Yogeesha K M, [2] RajathaGiri D T
[1] Assistant professor of mathematics,[2] Assistant professor of mathematics
[1] Department of mathematics.
[1] Government first grade collage, Davanagere, Karnataka, India
[2] Government first grade collage, B H Road,Tumkur , Karnataka, India

ABSTRACT: A set S $\subseteq$ V is a neighborhood set of G, if $G = \bigcup_{v \in S} \langle N[v] \rangle$, Here $\langle N[v] \rangle$ is the subgraph of G induced by v by all vertices adjacent to v. The neighborhood number η(G) of G is the minimum cardinality of a neighborhood set of G. A neighborhood set S $\subseteq$ V is a perfect matching neighborhood set if the induced subgraph $\langle S \rangle$ contains at least one perfect matching. The perfect matching neighborhood number of G, denoted by η_{pm} (G), is the smallest cardinality of a perfect matching neighborhood set of G. In this chapter, we determine a best possible upper and lower bound for η_{pm} (G), and its exact values for some particular classes of graphs are found. Also investigate its relationship with other parameters.

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1. INTRODUCTION

The graphs considered here are undirected without loops or multiple edges, finite and without isolated vertices. Let G = (V, E) be a graph with |V| = p and |E| = q. For the open neighborhood of a vertex v $\in$ V is N(v)={u$\in$V/uv$\in$E}, the set of vertices adjacent to v. The closed neighborhood is N[v]=N(v)$\cup${v}, We use $\langle S \rangle$ to denote the subgraph induced by the set of vertices S. The degree of a vertex v in a graph G is the number of edges of G incident with v and is denoted by deg v. the minimum and maximum degree of vertices of G are denoted by δ(G) and Δ(G), respectively.

If M is a matching in a graph G with the property that every vertex of G is incident with an edge of M, then M is perfect matching in G. Clearly, if G has a perfect matching M, the G has even order and <M> is a 1-regular spanning sub graph of G, see[2]

A set S$\subseteq$V is a neighborhood set of G, if $G = \bigcup_{v \in S} \langle N[v] \rangle$, where $\langle N[v] \rangle$ is the subgraph of G induced by v and all vertices adjacent to v. The neighborhood number $\eta(G)$ of G is the minimum cardinality of a neighborhood set of G, see [6].

A neighborhood set S $\subseteq$ V is an independent neighborhood set of G, if S is independent. The independent neighborhood number $\eta_i(G)$ of G is the minimum cardinality of an independent neighborhood set of G, see [5].

A neighborhood set S $\subseteq$ V is a connected neighborhood set, if the induced subgraph of $\langle S \rangle$ is connected. The connected neighborhood number $\eta_c(G)$ of G is the minimum cardinality of a connected neighborhood set of G, see [7].

A set D of vertices in a graph G is a dominating set if every vertex in V-D is adjacent to some vertex in D. Further, it is total dominating set if the induced subgraph $\langle D \rangle$ has no isolates. The domination number (total domination number) $\gamma(G)(\gamma_t(G))$ is the minimum cardinality of a dominating (total dominating) set of G, see [3].

A dominating set S $\subseteq$ V is a paired-dominating set, if the induced subgraph of $\langle S \rangle$ has a perfect matching. The paired-domination number $\gamma_P(G)$ of G is the minimum cardinality of a paired-dominating set of G. This concept was first introduced by Haynes and Slater, see [4].

Analogously, we now define The perfect matching neighborhood number as follows. A neighborhood set $S \subseteq V$ is a The perfect matching neighborhood set, if the induced subgraph of $\langle S \rangle$ contains at least one perfect matching. The perfect matching neighborhood number $\eta_{pm}(G)$of G is the minimum cardinality of a The perfect matching neighborhood set of G.

A dominating set D of a graph G with $|D| = \gamma(G)$ is called a γ-set and neighborhood set S of graph G with $|S|=n(G)$ is called an η-set. Similarly, the other sets can be expected.

For example, we consider the subdivided star $G = K^*_{l,k}$.

$G = K^*_{l,k}$

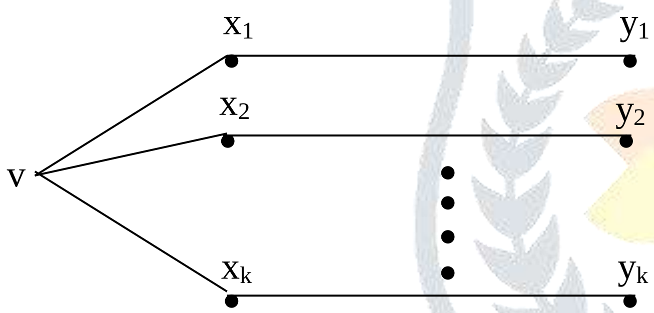


Figure – 1

Here, the set $S = \{x_1, x_2, x_3, ..., x_k\}$ is an $\eta(K^*_{1,k})$ –set as well as a $\gamma(K^*_{1,k})$-set and $S \cup \{v\}$ is an $\eta_{c}(K^*_{1,k})$-set as well as $\gamma_{t}(K^*_{1,k})$-set

For $\eta_{pm}(K^*_{1,k})$, note that if S is any perfect matching neighborhood set and $y_i \in S$, then $x_i \in S$ with x_i and y_i paired, and if $y_i \notin S$, then $x_i \in S$ to dominate y_i and then x_i and v are paired, since v can be at most once, it follows that

$$\eta(K^*_{1,k}) = \gamma(K^*_{1,k}) = k < \eta_c(K^*_{1,k})$$

$= \gamma_t (K^*_{1,k}) = k + 1 < \eta_{Pair} (K^*_{1,k}) = \gamma_P (K^*_{1,k}) = 2k.$

Many application of domination in graphs can be extended to perfect matching neighborhood. For example, if we think in neighborhood set which has a fileserver for a computer network, then each computer in the network has direct access to a fileserver. It is sometimes reasonable to assume that the access is available even when one of the fileservers goes down. A perfect matching neighborhood set provides the desired fault tolerance for such cases because each computer has access to at least two fileservers and each of the fileservers has direct access to at least one backup fileserver.

In the guard application, we can think of a perfect matching neighborhood set as a set of guards able to secure each location and satisfy the requirement that each guard be assigned an adjacent guard as a designated backup.

2. FUNDAMENTAL PROPERTIES OF PERFECT MATCHING NEIGHBOURHOOD NUMBER

Property 1. In a graph G without isolated vertices, the vertices of any maximal independent set of edges form a perfect matching neighborhood set. Thus G has a perfect matching neighborhood set if and only if the minimum vertex degree

$\delta(G) \geq 1$.

Property 2. If u is adjacent to an end vertex of G, then u is in every perfect matching neighborhood set and $\eta_{pm}(G)$ is even.

Property 3. Every perfect matching neighborhood set is a paired-dominating set of G. Hence, $\gamma_P(G) \leq \eta_{pm}(G)$.

For example, we consider G_1.

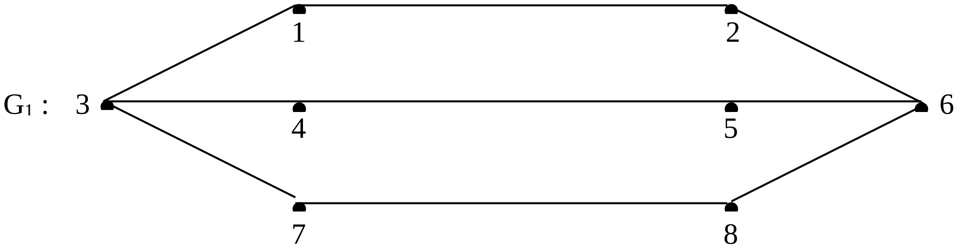


Figure – 2

Here {3, 4, 5, 6} is a γ_P-set and {1, 2, 4, 5, 7, 8} is an η_{pm}-set. Hence, $\gamma_P(G_1) < \eta_{pm}(G_1)$.

Property 4 : Every perfect matching neighborhood set is a neighborhood set of G. Hence, $\eta(G) \leq \eta_{pm}(G)$.

Property 5 : These easily computed values of $\eta_{pm}(G)$ are stated without proof.

i) For any path graph P_p with p vertices

$$\eta_{pm}(P_p) = \begin{cases} \lfloor 2p/3 \rfloor & \text{if } p \equiv -1,0 \pmod 3 \\ \lfloor 2p/3 \rfloor - 1 & \text{if } p \equiv 1 \pmod 3. \end{cases}$$

ii) For any cycle graph C_p with p vertices

$$\eta_{pm}(C_p) = \begin{cases} \lfloor 2p/3 \rfloor + 1 & \text{if } p \equiv 1 \pmod 3 \\ \lfloor 2p/3 \rfloor & \text{otherwise.} \end{cases}$$

iii) For any complete bipartite graph $K_{r,s}$ with $1 \leq r \leq s$,

$$\eta_{pm}(K_{r,s}) = 2\,r$$

iv) For any complete graph K_p with p vertices $\eta_{pm}(K_p)=2$

v) For any wheel graph W_p with p vertices

$$\eta_{pm}(W_p)=2$$

3 RESULTS

Theorem 1. If G has a vertex of degree p-1, then $\eta_{pm}(G) = 2$. (1)

Proof : Let deg v = Δ. Then there exists a vertex $u \in V$ such that u and v are adjacent. Thus the set {u, v} is a perfect matching neighborhood set of G. Hence (1) holds.

Now, we characterize graphs G for which $\eta_{pm}(G)$ is the order of G.

Theorem 2 : A graph G with no isolated vertices had $\eta_{pm}(G)=p$ if and only if $G=mK_2$ or $K_{2,2}$.

Proof : Suppose $\eta_{pm}(G)=p$ holds. On contrary suppose and $G\neq mK_2$ or $K_{2,2}$. Then there exist three vertices u,v and w such that u and v are adjacent and w is adjacent to any one of u and v .assume that w is adjacent to v, this implies that V-{W} is a perfect matching neighborhood. Necessity is easy to check

Next, we present sharp lower bound based on the maximum degree Δ(G).

Theorem 3 : If G has no isolated vertices, then $\eta_{pm}(G)\geq p/\Delta(G)$(2) Further, the bound is attained if $G = mK_2$.

Proof : Let $S = \{v_1, v_2, ..., v_k\}$ be an η_{pm}-set of G with matching $S'=\{e_1, e_2, ..., e_k\}$, where each edge e_i connects two elements of S. Furthermore, let q be the number of edges in G having one vertex in S and the other in V-S. Since $\deg(v) \leq \Delta(G)$ for all $v \in S$ and each vertex in S has at least one unique neighbor in S,

$$q \leq (\Delta(G)-1)|S| = (\Delta(G) - 1)\, \eta_{pm}(G)$$

Also, $q \geq |V - S| = p - \eta_{pm}(G)$

Hence, $|V - S| \leq \sum_{j=1}^{k} d(v_j) \leq (\Delta(G) - 1)|S|$,

$p - \eta_{pm}(G) \leq (\Delta(G) - 1)\, \eta_{pm}(G) \text{ or } p \leq \Delta(G)\, \eta_{pm}(G)$

Thus, (2) holds.

Further, the bound is attained if $G = mK_2$.

Now, we present an upper bound for $\eta_{pm}(G)$.

Theorem 4 : For any graph G with no isolated vertices then

$$\eta_{pm}(G) \leq 2\beta_1(G) \qquad (3)$$

where $\beta_1(G)$ is the maximum number of independent edges.

Proof : Let G be a graph with no isolated vertices. Then any maximal independent set of edges form a perfect matching neighborhood set. Thus, (3) holds.

We use the following result to prove our next result.

Theorem A [1]. For any graph G of order $p \geq 3$,

i) $\beta_1(G) + \beta_1(\overline{G}) \leq 2\lfloor p/2 \rfloor$,

ii) $\beta_1(G) + \beta_1(\overline{G}) \leq \lfloor p/2 \rfloor^2$, where $\overline{G}$ is the complement of a graph G, if G and $\overline{G}$ have the same vertex set and two vertices are adjacent in G if and only if they are not adjacent in $\overline{G}$.

Theorem 5 : Let G be a graph such that both G and $\overline{G}$ have no isolated vertices. Then

i) $\eta_{pm}(G) + \eta_{pm}(\overline{G}) \leq 4\lfloor p/2 \rfloor$

ii) $\eta_{pm}(G)\eta_{pm}(\overline{G}) \leq 4\lfloor p/2 \rfloor^2$

Proof : These follow from Theorem 4 and Theorem A.

To prove our next result, we make use of the following definitions.

The Private neighbor set of a vertex v with respect to a set S is denoted by PN[v,S] = N[v] – N[S – {v}]. If PN{v, S] ≠ ∅ for some vertex v and some S $\subseteq$ V, then every vertex of PN[v, S] is called a private neighbor of v with respect to S [4].

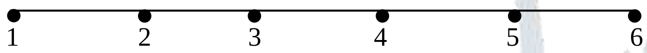
A neighborhood set S is a private neighborhood set, if u∈S has a private neighbor. Hence every graph without isolated vertices has a minimum neighborhood set which is also a private neighborhood.

Theorem 6 : If G has no isolated vertices and $\eta_{pm}(G) = 2\eta(G)$, then every $\eta(G)$-set is an $\eta_i(G)$-set, where $\eta_i(G)$ is the independent neighborhood number of G.

Proof : Let G be a graph having $\eta_{pm}(G) = 2\eta(G)$, and consider an $\eta(G)$-set S. Suppose that S is not independent. Then, there is an adjacent pair of vertices in S, say u and v, we form a perfect matching neighborhood set for G by pairing u and v and pairing each vertex in S – {u, v} with a neighbor in V – S. This is possible since the minimality of S implies that for each x ∈ S, either x has a private neighbor or x is isolated in $\langle S \rangle$. Let I be a set of isolates in S without private neighbors. Now each vertex in I must have at least one neighbor in V – S, since G has a no isolates. The minimality of S implies that no two vertices in I have a common neighbor. Hence, each vertex in V – {u, v} can be paired with a neighbor forming a perfect matching neighborhood set of order $\eta(G)+ \eta(G)–2<2\eta(G)$, contrary to the hypothesis. Hence $\eta(G) = \eta_i(G)$ and since S was arbitrary, every minimum neighborhood set of G is independent.

Remark : The converse of the above theorem is not true.

For example, consider G :



Here {1, 3, 5} is an η_i-set, {2, 4, 6} is an η-set and {2, 3, 4, 5} is an η_{pm} -set.

Now, we obtain a relationship between $\gamma_t(G)$ and $\eta_{pm}(G)$.

Theorem 7 : For any graph G without isolated vertices, which is not a C_4, then

$$\eta_{pm}(G) \leq 2\,\gamma_t(G)-2 \qquad (4)$$

Proof : Let S be a γ_t-set of G. Any vertex of S which does not have a private neighbor in S must have a private neighbor with respect to S in V – S, that is, any vertex of S that cannot be paired with a neighbor in S has a private neighbor in V – S. Hence a perfect matching neighborhood set may be formed by pairing the vertices of maximal matching M of the induced subgraph $\langle S \rangle$ and adding a private neighbor for V – S to S for each unpaired vertex in S.

Since S is a total dominating set, $|M| \geq 1$, and so at most $\gamma_t(G) – 2$ vertices in S must be paired with private neighbors from V – S. Hence $\eta_{pm}(G) \leq 2 + 2(\gamma_t(G) – 2)$.

Thus (4) holds.

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REFERENCES :

1). R.P Gupta, Independence and covering of line graphs and total graphs, proof techniques in graph theory (ed. F. Harary) Academic press, New York (1969) 61-62

2). F.Harary Graph theory, Addision-Wesley, Reading Mass(1969)

3). T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamental Domination in Graphs, Marcel Dekker, Inc., New York (1998).

4). T.W. Haynes, P.J. Slater, Paired domination in graphs, Networks 32 (1998).199-206

5). V.R. Kulli and N.D. Soner, The independent neighborhood number of a graph. Nat. Acad. Sci. Lett. 19 (1996), 159-161.

6). E. Sampathkumar and P.S. Neeralagi, The neighborhood number of a graph. Indian J. Pure Appl. Math., (16) (1985) 126-132.

7). E. Sampathkumar and P.S. Neeralagi, Independent, Perfect and connected neighborhood numbers of a graph. J. Combin. Inform. System Sci. (19) (1994) 139-145.

8). N.D. Soner, B. Chaluvaraju and B. Janakiram, The Maximal neighborhood number of a graph. Far East J. Appl. Math., 5(3) (2001) 301-307.