SOME INVESTIGATION ON WEAKLY LINEAR SYSTEM OF FUZZY RELATION

Dharmendra Kumar Sharma
Research Scholar
Dept. of Math's, V.K.S. Univ. Ara.

Abstract: The present paper provides discussion on weakly linear systems. We have defined two basic type of homogeneous and heterogeneous weakly linear systems. Infact, an important feature of weakly linear systems is that the set of their solutions are closed under composition and arbitrary joins. Various related propositions and theorems are produced.

Key Words: Fuzzy relation, linear systems, weakly linear system, greatest solution, joins, dual pair.

INTRODUCTION:

Systems of fuzzy relation equations and inequalities were first studied by Sanchez, who used them in medical research (cf. [1-4]). Later they found a much wider field of application, and nowadays they are used in fuzzy control, discrete dynamic systems, knowledge engineering, identification of fuzzy systems, prediction of fuzzy system, decision-making, fuzzy information retrieval, fuzzy pattern recognition, image compression and reconstruction, and in many other areas (cf., e.g., [5, 6, 7, 8, 9, 10, 11]). Most frequently studied systems were the ones that consist of equations and inequalities with one side containing the composition of an unknown fuzzy relation and a given fuzzy relation or fuzzy set, while the other side contains only another or the same given fuzzy relation or fuzzy set. Such systems are called linear systems. Solvability and methods for computing the greatest solutions to linear systems of fuzzy relation equations and inequalities were first studied in the above mentioned papers by Sanchez, who discussed linear systems over the Godel structure. Later, linear systems over more general structures of truth values were investigated, including those over complete resituated lattices cf., e.g., [12, 13-15,]).

More complex non-linear systems of fuzzy relation inequalities and equations, called weakly linear, have been recently introduced and studied in [16-17]. Basically, weekly linear systems discussed in [16] consist of inequalities and equations of the form $A_i \circ X \otimes X \circ A_i$ and $X \leq M$, where A_i $(i \in I)$ and M are given fuzzy relation on a set U, X is an unknown fuzzy relation on U, o denotes the composition of fuzzy relations, and $\otimes$ is one of $\leq$, > and =. Besides, these systems can also include additional inequalities and equations of the from $A_i \circ X^{-1} \otimes X^{-1} \circ A_i$ and $X^{-1} \leq M$, where X^{-1} denotes the converse (inverse, transpose) relation of X.

Such weakly linear systems, which include only fuzzy relations on a single set, are called homogeneous. On the other hand, heterogeneous weakly linear systems, studied in [17], include fuzzy relations on two possible different sets and an unknown is a fuzzy relation between these two sets. Two basic types of heterogeneous weakly linear systems are systems of the form X^{-1} o $A_i \leq B_i$ o X^{-1} ($i \in I$), $X \leq N$, or A_i o $X \leq X$ o B_i ($i \in I$), $X \leq N$, where A_i and B_i ($i \in I$) are respectively given fuzzy relations on non-empty sets U and V, N is a given fuzzy relation between U and V, and X is an unknown fuzzy relation between obtained by combining the previous two types of systems (for X and X^{-1}).

Both homogeneous and heterogeneous weakly linear systems have their origins in the theory of fuzzy automata. Homogeneous weakly linear systems emerged from the research aimed at the reduction of the number of states of fuzzy finite automata, carried out in [18, 19], whereas heterogeneous weakly linear systems turned up from the study of simulation, bisimulation and equivalence of fuzzy automata, conducted in [20, 21]. In addition, weakly linear systems play an important role in the social network analysis.

WEAKLY LINEAR SYSTEMS :

Now we move to consideration of systems of weakly linear fuzzy relation inequalities and equations, that will be called simply weakly linear systems. As for linear systems, we can distinguish two basic types of weakly linear system: homogeneous and heterogeneous weakly linear systems.

Let U be a non-empty set (not necessarily finite), let $\{A_i\}$ $i \in I$ be a given family of fuzzy relations on U (where I is also not necessarily finite), let M be a given fuzzy relations on U, and let X be an unknown fuzzy relation on U. Two basic types of homogeneous weakly linear systems are systems of the from

X o $A_i \leq A_i$ o X ($i \in I$), $X \leq M$, (*wl*1.1)

A_i o $X \leq X$ o A_i ($i \in I$), $X \leq M$, (*wl*1.2)

and the third one is the conjunction of the first two:

X o $A_i = A_i$ o X ($i \in I$), $X \leq M$, (*wl*1.3)

Besides, in many situations we need that both a fuzzy relation R and its inverse R^{-1} are solutions to the above mentioned systems, and then we consider the following three systems:

X o $A_i \leq A_i$ o X, X^{-1} o $A_i \leq A_i$ o X^{-1}, ($i \in I$), $X \leq M \wedge M^{-1}$, (*wl*1.4)

A_i o $X \leq X$ o A_i, A_i o $X^{-1} \leq X^{-1}$ o A_i, ($i \in I$), $X \leq M \wedge M^{-1}$, (*wl*1.5)

X o $A_i = A_i$ o X, X^{-1} o $A_i = A_i$ o X^{-1}, ($i \in I$), $X \leq M \wedge M^{-1}$, (*wl*1.6)

Clearly, a symmetric fuzzy relation is a solution to (*wl*1.4) (resp. (*wl*1.5), (*wl*1.6) if and only if it is solution to (*wl*1.1) (resp. (*wl*1.2), (*wl*1.3). If $M(a_1, a_2)=1$, for all $a_1, a_2 \in U$, then the inequality $X \leq M$ becomes trivial, and it can be omitted. Systems (*wl*1.1) – (*wl*1.6) are called homogeneous weakly linear systems [16, 17]. For the sake of convenience, for each $t \in \{1, \ldots, 6\}$, system (*wl*1.t) will be denoted by $WL^{1.t}(U, I, A_i, M)$. If M $(a_1, a_2) = 1$, for all $a_1, a_2 \in U$, then system (*wl*1.t) is denoted simply by $WL^{1.t}(U, I, A_i)$. Note that the condition $X \leq M \;\; M^{-1}$ appearing in (*wl*1.4) – (*wl*1.6) is equivalent to $X \leq M$ and $X^{-1} \leq M$.

Another situation is when we work with two possibly different non-empty sets U and V (which are not necessarily finite). Let $\{A_i\}_{i \in I}$ be a given family of fuzzy relations on U and $\{B_i\}_{i \in I}$ a given family of fuzzy relations on V (where I is also not necessarily finite), let N be a given fuzzy relation between U and V, and let

X be an unknown fuzzy relation between U and V. Two basic types of heterogeneous weakly linear systems are systems of the from

$X^{-1} \circ A_i \leq B_i \circ X^{-1}$, $(i \in I)$, $X \leq N$ (*wl*2.1)

$A_i \circ X \leq X \circ B_i$, $(i \in I)$, $X \leq N$, (*wl*2.2)

Besides, there are four systems obtained by combinations of the previous two systems (for X and X^{-1})

$X^{-1} \circ A_i \leq B_i \circ X^{-1}$ $X \circ B_i \leq A_i \circ X$, $(i \in I)$, $X \leq N$ (*wl*2.3)

$A_i \circ X \leq X \circ B_i$ $B_i \circ X^{-1} \leq X^{-1} \circ A_i$ $(i \in I)$, $X \leq N$ (*wl*2.4)

$A_i \circ X = X \circ B_i$ $(i \in I)$, $X \leq N$ (*wl*2.5)

$X^{-1} \circ A_i = B_i \circ X^{-1}$ $(i \in I)$, $X \leq N$ (*wl*2.6)

Systems (*wl*2.1)–(*wl*2.6) are be called heterogeneous weakly linear systems. For the sake of convenience, for each t ∈ {1, …., 6}, system (*wl*2.t) is denote by $WL^{2.t}$ (U, V, I, A_i, B_i, N). If N (a, b) = 1 for all a ∈ U and b ∈ V, then system (*wl*2.t) is denoted simply by $WL^{2.t}$ (U, V, I, A_i, B_i).

Note first that some of the above introduced systems are mutually dual. For instance, (*wl*1.1) and (*wl*1.2) are dual, in the sense that a fuzzy relation R is a solution to $WL^{1.1}$(U, I, Ai, M) if and only if its inverse R^{-1} is a solution to $WL^{1.2}$(U, I, A^{-1}, M^{-1}). Similarly, dual pairs are systems (*wl*1.4) and(*wl*1.5), (*wl*2.1) and (*wl*2.2), (*wl*2.3) and (*wl*2.4), as well as (*wl*2.5) and (*wl*2.6). The duality means that for any universally valid statement concerning one of the systems in a dual pair there is the corresponding universally valid statement concerning another system in this dual pair, and vice versa. For this reason, we deal mainly with one of the systems which from a dual pair.

An important feature of weakly linear systems is that the sets of their solutions are closed under composition and arbitrary joints. More formally, the following is true.

Proposition 1. [16] : For arbitrary fuzzy relations R_1, R_2 R_a ∈ R(U) (α ∈ Y) and t ∈ {1, …, 6} we have

(a) If R_1 and R_2 are respectively solutions to systems $WL^{1.t}$ (U, I, A_i, M1) and $WL^{1.t}$ (U, I, A_i, M2) then $R_1 \circ R_2$ is a solution to the system $WL^{1.t}$ (U, I, A_i, $M_1 \circ M_2$).

(b) If R_α is a solution to the system $WL^{1.t}$ (U, I, A_i, M) for each α ∈ Y, then $\bigvee_{\alpha \in Y}$ Ra is also a solution to the system $WL^{1.t}$ (U, I, A_i, M)

Proposition 2. [17] : For arbitrary fuzzy relations and R_1, R_α ∈ R (U, V) (α ∈ Y) and (R_2 ∈ R) (V, W), an arbitrary t ∈ {1, …, 6} we have

(a) If R_1 and R_2 are respectively solutions to systems $WL^{1.t}$ (U, V, I, A_i, B_i, N_1) and $WL^{1.t}$ (V.W, I, B_i, C_i, N_2) then $R_1 \circ R_2$ is a solution to the system $WL^{1.t}$ (U, W, I, A_i, C_i, $N_1 \circ N_2$).

(b) If Ra is a solution to the system $WL^{1.t}$ (U, V, I, A_i, B_i, M) for each α ∈ Y, $\bigvee_{\alpha \in Y} R\,\alpha$ then Ra is also a solution to the system $WL^{1.t}$ (U, V, I, A_i, B_i, M)

According to the second statements in Propositions 1 and 2 every weakly linear system has the greatest solution, the join of all its solutions. Evidently, the greatest solution to a weakly linear system may be the empty relation, but if the given fuzzy relation M in a homogeneous weakly linear system is reflexive, then the greatest solution to this system must contain the equality relation, and hence, it must be reflexive. Besides, by the first statements in Propositions 1 and 2 we obtain the following.

Theorem 1 [16] : All homogeneous weakly linear systems (*wl*1.1)-(*wl*1.6) have the greatest solutions. In addition, the following is true:

- **(a)** If M is a reflexive fuzzy relation, then the greatest solutions to (*wl*1.1)-(*wl*1.6) are also reflexive fuzzy relations.
- **(b)** If M is a fuzzy quasi-order, then the greatest solutions to (*wl*1.1)-(*wl*1.3) are also fuzzy quasi-orders.
- **(c)** If M is a fuzzy equivalence, then the greatest solution to (*wl*1.4)-(*wl*1.6) are also fuzzy equivalences.

Theorem 2 [17] : All heterogeneous weakly linear systems (*wl*2.1)-(*wl*2.6) have the greatest solutions.

In addition, if N is a partial fuzzy function, then the greatest solution to systems (*wl*2.3) and (*wl*2.4) are also partial fuzzy functions.

As we have proved that weakly linear systems have the greatest solutions, we naturally come to the question how to compute these greatest solutions. For this purpose, it is convenient to represent these systems another equivalent from, which will be done in the sequel.

We start with the homogeneous case. Let U be a non-empty set, let $\{A_i\}_{i \in I}$ be a family of fuzzy relations on U, and let M be a fuzzy relation on U. Define functions $\phi^{(1.t)}$: R(U) → R(U), for $1 \leq t \leq 6$, as follows:

$$\phi^{(1.1)}(R) = \bigwedge_{i \in I} (A_i \text{ o } R)/A_i \quad (1)$$

$$\phi^{(1.2)}(R) = \bigwedge_{i \in I} A_i/(R \text{ o } A_i) \quad (2)$$

$$\phi^{(1.3)}(R) = \bigwedge_{i \in I} [(A_i \text{ o } R)/A_i] \wedge [A_i/(R \text{ o } A_i)] = \phi^{(1.1)}(R) \wedge \phi^{(1.2)}(R) \quad (3)$$

$$\phi^{(1.4)}(R) = \bigwedge_{i \in I} [(A_i \text{ o } R)/A_i] \wedge [A_i/(R \text{ o } A_i)]^{-1} = \phi^{(1.1)}(R) \wedge [\phi^{(1.1)}(R^{-1})]^{-1} \quad (4)$$

$$\phi^{(1.5)}(R) = \bigwedge_{i \in I} [A_i/(R \text{ o } A_i)] \wedge [A_i/(R^{-1} \text{ o } A_i)]^{-1} = \phi^{(1.2)}(R) \wedge [\phi^{(1.2)}(R^{-1})]^{-1} \quad (5)$$

$$\phi^{(1.6)}(R) = \bigwedge_{i \in I} [(A_i \text{ o } R)/A_i] \wedge [(A_i \text{ o } R^{-1})/A_i]^{-1} \wedge [A_i/(R \text{ o } A_i)] \wedge [A_i/(R^{-1} \text{ o } A_i)]^{-1}$$

$$= \phi^{(1.4)}(R) \wedge [\phi^{(1.5)}(R)] = \phi^{(1.1)}(R) \wedge [\phi^{(1.1)}(R^{-1})]^{-1} \wedge$$

$$\phi^{(1.2)}(R) \wedge [\phi^{(1.2)}(R^{-1})]^{-1} \quad (6)$$

For each R ∈ R(U).

Using these functions we can represent system ((*wl*1.1)-(*wl*1.6) in the following way.

Theorem 3 [16].: For every t ∈ {1, …., 6}, system (*wl*.t) is equivalent to system

$$X \leq = \phi^{(1.1)}(X), \qquad X \leq M^{(t)}, \quad (7)$$

Where $M^{(t)}$= M, for t ∈ {1, 2, 3}, and $M^{(t)} = M \wedge M^{-1}$, for t ∈ {4, 5, 6}.

In the heterogeneous case we consider two possible different non-empty sets U and V, a family $\{A_i\}_{i \in I}$ of fuzzy relations on U, a family $\{B_i\}_{i \in I}$ of fuzzy relations on V, and a fuzzy relation N between U and V. Define functions $\phi^{(2.t)}$: R(U, V) → R(U, V), for $1 \le t \le 6$, as follows:

$$\phi^{(2.1)}(R) = \bigwedge_{i \in I} [(B_i \text{ o } R^{-1})/A_i]^{-1} \quad (8)$$

$$\phi^{(2.2)}(R) = \bigwedge_{i \in I} A_i/(R \text{ o } B_i) \quad (9)$$

$$\phi^{(2.3)}(R) = \bigwedge_{i \in I} [(B_i \text{ o } R^{-1})/A_i]^{-1} \wedge [(A_i \text{ o } R)/B_i] = \phi^{(2.1)}(R) \wedge [\phi^{(2.1)}(R^{-1})]^{-1} \quad (10)$$

$$\phi^{(2.4)}(R) = \bigwedge_{i \in I} [(A_i/(R \text{ o } B_i)] \wedge [B_i/(R^{-1} \text{ o } A_i)]^{-1} = \phi^{(2.2)}(R) \wedge [\phi^{(2.2)}(R^{-1})]^{-1} \quad (11)$$

$$\phi^{(2.5)}(R) = \bigwedge_{i \in I} [A_i/(R \text{ o } B_i)] \wedge [(A_i \text{ o } R)B_i)] = \phi^{(2.2)}(R) \wedge [\phi^{(2.1)}(R^{-1})]^{-1} \quad (12)$$

$$\phi^{(2.6)}(R) = \bigwedge_{i \in I} [(B_i \text{ o } R^{-1})/A_i]^{-1} \wedge [(B_i/(R^{-1} \text{ o } A_i)]^{-1} = \phi^{(2.1)}(R) \wedge [\phi^{(2.2)}(R^{-1})]^{-1} \quad (13)$$

For each R ∈ R(U, V). Notice that in the expression "$\phi^{(2.1)}(R^{-1})$" (t ∈ {1, 2}), we denote by $\phi^{(2.t)}$ a function from R(V, U) into itself.

Now we represent systems (*wl*2.1)-(*wl*2.6) in the following equivalent form.

Theorem 4 [17].: For every t ∈ {1, …, 6}, system (w.t) is equivalent to system

$$X \le = \phi^{(2.t)}(X), \qquad X \le N. \quad (14)$$

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