MATHEMATICAL ANALYSIS OF UNSTEADY BLOOD FLOW IN A BRANCH OF ARTERY

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Abstract: A mathematical model has been developed to study the unsteady blood flow of a suspension of small uniformly sized spherical particles in a branch in presence of a uniform transverse magnetic field. A simple bifurcation model has been analyzed to study the effect of magnetic field on the nature of local disturbances and high shear forces which cause certain cardiovascular lesions. Laplace transform has been used to solve the problem.

Keywords: unsteady, transverse magnetic field, bifurcation model ,Laplace transform

1. INTRODUCTION

The application of magneto hydrodynamic principles in medicine and biology is of growing interest in recent times sud *et al* [1]. Since by Lenz's law, the Lorentz force opposes the motion of conducting fluid and blood is electrically conducting [2], the magnetic field deaccelerates the blood flow in arterial system. Thus it is useful for the treatment of certain cardiovascular disorders and in the diseases with accelerated blood circulation like hemorrhages and hypertension etc. [3]. Kolin [4] first introduced the idea of electromagnetic field in medical research which may be used in regulating blood movement in human system by the application of magnetic field. Mathematical simulation of blood flow through a bifurcation in the presence of uniform transverse magnetic field has been presented by Suri and Suri [5] while Sanyal and Maji [6] generalised the problem taking the blood to be micropolar conducting fluid. It was observed by the authors that the constant magnetic field slows down the velocity of blood.

Now foreign particles appear in blood flow under allergic conditions. Blood corpuscles can also be treated as small particles suspended in the fluid. Keeping this in view, Naidu and Devsen [7] obtained solutions for the pulsatile motion of a suspension of particles in rigid and complaint tubes while numerical treatment of the problem has been studied by Sankara subramanian and Naidu [8].

The present paper deals with the unsteady motion of blood with a suspension of small uniformly sized spherical particles through a bifurcation in the presence of a uniform transverse magnetic field. The bifurcation model has been studied to investigate the effect of magnetic field on the nature of local disturbances and high shear forces which cause certain cardiovascular lesions like atherosclerotic plaques, aneurysms and intimal cushions etc. near bifurcation. The results have also been discussed graphically.

2. MATHEMATICAL MODEL AND BASIC EQUATIONS

The blood circulation system is one of three-dimensional elastic tubes having varying cross-sections and angle of bifurcation. However, for convenience, we discuss the effects of constant magnetic field on unsteady blood flow with suspended particles through a two-dimensional non-conducting parallel plates with equally branched channel. The thickness of the bifurcating wall is assumed to be of negligibly small magnitude [4]. We also suppose that the blood is a Newtonian, incompressible, homogeneous and viscous fluid and its viscosity is constant. A uniform magnetic field B_0 is applied in a direction perpendicular to the direction of the flow. The geometry of the bifurcation is shown in fig. 1 (vide Zamir and Roach [9], Suri and Suri [5]).

The unsteady flow of blood in the presence of constant magnetic field with a suspension of small particles, uniformly distributed in the fluid (taking the number density N to be constant) is governed by the two-dimensional boundary layer equations

$$\frac{\partial u^*}{\partial t^*} + \frac{1}{\rho}\frac{\partial p^*}{\partial x^*} = \frac{\mu}{\rho}\frac{\partial^2 u^*}{\partial y^{*2}} - \frac{\sigma {B_0}^2 u^*}{\rho^*} + \frac{KN}{\rho}(U^* - u^*) \quad (1)$$

$$\left(\frac{m}{K}\frac{\partial}{\partial t^*} + 1\right) U^* = u^* \quad (2)$$

where we have assumed that the magnetic Reynolds number is so small that the induced electric and magnetic fields can be neglected. In the above, u^* and U^* are the velocities along the $x^* -$ direction of the fluid and the suspension respectively; p^*, the pressure; t^*, the time; (x^*, y^*) are axes coordinates; ρ, the density; σ , the electrical conductivity; m, the mass of each particle; B_0, the constant magnetic field applied along u^*- direction; K, the Stokes drag coefficient and μ is the viscosity of blood.

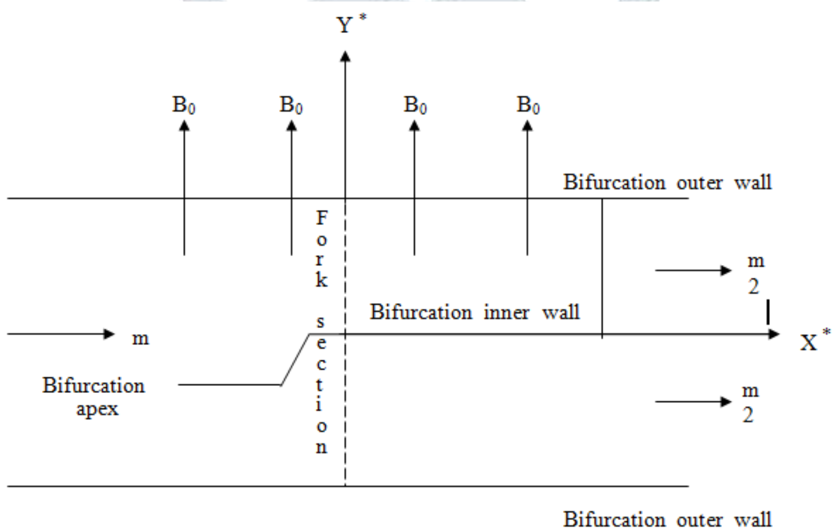

Figure 1 : Geometry of the Blood Flow in presence of External Magnetic Field

Introducing the following non-dimensional quantities,

$$x = \frac{x^*}{b}, \quad y = \frac{y^*}{b}, \quad t = \frac{t^*}{b^2\rho}, \quad U = \frac{\rho b U^*}{\mu}$$

$$u = \frac{\rho b u^*}{\mu}, \quad \beta = \frac{NKb^2}{\mu}, \quad \gamma = \frac{m\mu}{Kb^2\rho} \tag{3}$$

$$h(x,t) = \frac{b^3\rho\left(\frac{dp^*}{dx^*}\right)}{\mu^2}, \quad M = bB_0\sqrt{\frac{\sigma}{\mu}} = \text{Hartmann number.}$$

where *b* is the branch diameter, the equations (1) and (2) reduce respectively to

$$\frac{\partial u}{\partial t} + h = \frac{\partial^2 u}{\partial y^2} - M^2 u + \beta(U - u), \tag{4}$$

$$\left(\gamma\frac{\partial}{\partial t} + 1\right) U = u. \tag{5}$$

The initial conditions for the problem are assumed to be

$$u = u_0 \text{ and } U = 0 \text{ at } t = 0 \tag{6}$$

and the boundary conditions are

$$x < 0\ ;\ u = U = 0 \text{ for } y = \pm 1 \tag{7}$$

and

$$x \geq 0\ ;\ u = U = 0 \text{ for } y = 0\ ,\ \pm 1 \tag{8}$$

The law of conservation of mass (Sud *et al* [3]) in dimensionless form is

$$\int_0^1 u\, dy = 1 \tag{9}$$

3. MOMENTUM AND MOMENTUM INTEGRAL EQUATIONS

Since the pressure gradient parameter (h) is unknown and the boundary layer solutions of the equations (4) and (5) do not satisfy the boundary conditions on y = + 1 in the branches, we can express the problem based on integral form of boundary layer equations. Also the flow configuration is symmetric about y = 0, so we restrict our analysis to one of the branches i.e., the region bounded by

$$0 \leq x \leq +\infty, 0 \leq y \leq +1,$$

Taking Laplace transform on both sides of the momentum equation and continuity equation (4) and (5), we get

$$s\,\bar{u} - u_0 + h = \frac{d^2\bar{u}}{dy^2} - M^2\bar{u} + \beta\,(\bar{U} - \bar{u}) \tag{10}$$

and

$$(\gamma s + 1)\bar{U} = \bar{u} \tag{11}$$

where s is the Laplace transform parameter.

Eliminating $\bar{U}$ between (11) and (12), we get

$$(s + D^2)\bar{u} - u_0 + \bar{h} = \frac{d^2\bar{u}}{dy^2} \tag{12}$$

where $D^2 = M^2 + \frac{\beta\,\gamma\,s}{\gamma\,s+1}$ (13)

The equation of the moment of momentum in the direction of the flow is given by

$$y\,(s + D^2)\bar{u} - y\,u_0 + y\,\bar{h} = y\,\frac{d^2\bar{u}}{dy^2} \tag{14}$$

Integrating each term of the equations (13) and (14) from y = 0 to y = 1 to get integral form of these equations and noting that

$$\int_0^1 \frac{d^2\bar{u}}{dy^2}\,dy = \left(\frac{d\bar{u}}{dy}\right)_{y=1} - \left(\frac{d\bar{u}}{dy}\right)_{y=0} \tag{15}$$

$$\int_0^1 y\,\frac{d^2\bar{u}}{dy^2}\,dy = \left(\frac{d\bar{u}}{dy}\right)_{y=1} \tag{16}$$

we get respectively

$$(s + D^2)\int_0^1 \bar{u}\,dy - \int_0^1 u_0\,dy + \bar{h} = \left(\frac{d\bar{u}}{dy}\right)_{y=1} - \left(\frac{d\bar{u}}{dy}\right)_{y=0} \tag{17}$$

$$(s + D^2)\int_0^1 y\,\bar{u}\,dy - \int_0^1 y\,u_0\,dy + \frac{\bar{h}}{2} = \left(\frac{d\bar{u}}{dy}\right)_{y=1} \tag{18}$$

Equations (17) and (18) can be solved by separating independent variables in u by assuming u as a polynomial in y w coefficients which are functions of x and t only as follows (Suri *et al* [5]).

$$u(y, t) = f_0(t) + f_1(t)y + f_2(t)\,y^2 + g(t)\,y^3 \tag{19}$$

where, f_0, f_1, f_2 and g are arbitrary functions of x and t.

Using Laplace-transform on both sides of the equation (19) we get

$$\bar{u}\,(y, t) = \bar{f}_0(t) + \bar{f}_1(t)y + \bar{f}_2(t)\,y^2 + \bar{g}(t)\,y^3 \tag{20}$$

Again using Laplace-transform on boundary conditions (8) and (9) and applying these results into (20), we find the coefficients $\bar{f}_0$, $\bar{f}_1$ $\bar{f}_2$ in terms of $\bar{g}$ as

$$\bar{f}_0 = 0\,, \quad \bar{f}_1 = \left(\frac{\bar{g}}{2} + \frac{6}{s}\right), \quad \bar{f}_2 = -\left(\frac{3}{2}\,\bar{g} + \frac{6}{s}\right) \tag{21}$$

Therefore the equation (20) gives

$$\bar{u} = \frac{1}{s}\,(6y - 6y^2) + \bar{g}\left(\frac{y}{2} - \frac{3}{2}\,y^2 + y^3\right) \tag{22}$$

Substituting (22) in equations (17) and (18) we get

$$\bar{h} = -\left(\frac{D^2 + 12}{s}\right) \text{ and } \bar{g} = \frac{g_0}{s + (D^2 + 60)} \tag{23}$$

where g_0 is the value of g at t = 0. Laplace inversion of these lead to

$$h = -\left\{\beta e^{-\frac{t}{\gamma}} + (M^2 + 12)\right\} \tag{24}$$

and

$$g = g_0\,e^{-\frac{1}{2}\left(\alpha + \beta + \frac{1}{\gamma}\right)t}\left\{\cos\lambda t - \frac{1}{2\lambda}\left(\alpha + \beta + \frac{1}{\gamma}\right)\sin\lambda t\right\}, \tag{25}$$

where $\alpha = M^2 + 60,\ \lambda^2 = \frac{\alpha}{\gamma} - \frac{1}{4}\left(\alpha + \beta + \frac{1}{\gamma}\right)^2$ (26)

Hence the equation (22) is given by

$$u = (6y - 6y^2) + \left(\frac{y}{2} - \frac{3}{2}\,y^2 + y^3\right) g_0 exp\left\{-\frac{1}{2}\left(\alpha + \beta + \frac{1}{\gamma}\right)t\right\}$$
$$\times\left\{\cos\lambda t - \frac{1}{2\lambda}\left(\alpha + \beta + \frac{1}{\gamma}\right)\sin\lambda t\right\} \tag{28}$$

and

$$h = -\left\{\beta e^{-\frac{t}{\gamma}} + (M^2 + 12)\right\} \tag{29}$$

Using (22) we have from the equation (12)

$$\bar{U} = (6y - 6y^2)\left\{\frac{1}{s} - \frac{1}{s + \frac{1}{\gamma}}\right\} + \frac{g_0}{\lambda\gamma}\left(\frac{y}{2} - \frac{3}{2}\,y^2 + y^3\right)$$
$$\times \frac{1}{\left\{s + \frac{1}{2}\left(\alpha + \beta + \frac{\gamma}{2}\right)\right\}^2 + \lambda^2} \tag{30}$$

Laplace inversion of which leads to the velocity of suspension as

$$U = (6y - 6y^2)\left(1 - e^{-\frac{t}{\gamma}}\right) + \frac{g_0}{\lambda\gamma}\left(\frac{y}{2} - \frac{3}{2}\,y^2 + y^3\right)$$
$$\times e^{-\frac{1}{2}\left(\alpha + \beta + \frac{1}{\gamma}\right)t}.\ \sin\lambda t \tag{31}$$

In addition, the shear-stress parameters τ_0 and τ_1 at the inner and outer walls of bifurcation can be obtained from (24) in non dimensional form respectively as

$$\tau_0 = \left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{1}{2}\,g_0\ exp\left\{-\frac{1}{2}\left(\alpha + \beta + \frac{1}{\gamma}\right)t\right\}$$
$$\times\left\{\cos\lambda t - \frac{1}{2\lambda}\left(\alpha + \beta + \frac{1}{\gamma}\right)\sin\lambda t\right\} + 6 \tag{35}$$

$$\tau_1 = \left(\frac{\partial u}{\partial y}\right)_{y=1} = \frac{1}{2}\,g_0\ exp\left\{-\frac{1}{2}\left(\alpha + \beta + \frac{1}{\gamma}\right)t\right\}$$
$$\times\left\{\cos\lambda t - \frac{1}{2\lambda}\left(\alpha + \beta + \frac{1}{\gamma}\right)\sin\lambda t\right\} - 6 \tag{36}$$

4. PARTICULAR CASE

If we assume $\beta \to 0, \gamma \to \infty$, then the solutions for the non-dusty case are given by

$$u = (6y - 6y^2) + g_0\left(\frac{y}{2} - \frac{3}{2}\,y^2 + y^3\right).\exp\{-(M^2 + 60)\} \tag{37}$$

$$U = 0. \tag{38}$$

and the shear-stress parameters τ_0 are τ_1 are

$$\tau_0 = \frac{1}{2}\,g_0\ exp\{-(M^2 + 60)t\} + 6 \tag{39}$$

$$\tau_1 = \frac{1}{2}\,g_0\ exp\{-(M^2 + 60)t\} - 6 \tag{40}$$

These results are in excellent agreement with those obtained by Suri and Suri [5].

5. NUMERICAL RESULTS AND DISCUSSIONS

To get a physical insight into the problem, we discuss the behaviour of the velocity of blood and of suspension. In figure - 2, the effect of the magnetic field on the blood particle has been shown. It is noted that the magnetic field deaccelerates the flow from the center of the channel while it is accelerated near the wall. The figure - 3 shows that the presence of dust particles decreases the blood velocity near the centre of the channel, but is increased as we pass on to the wall.

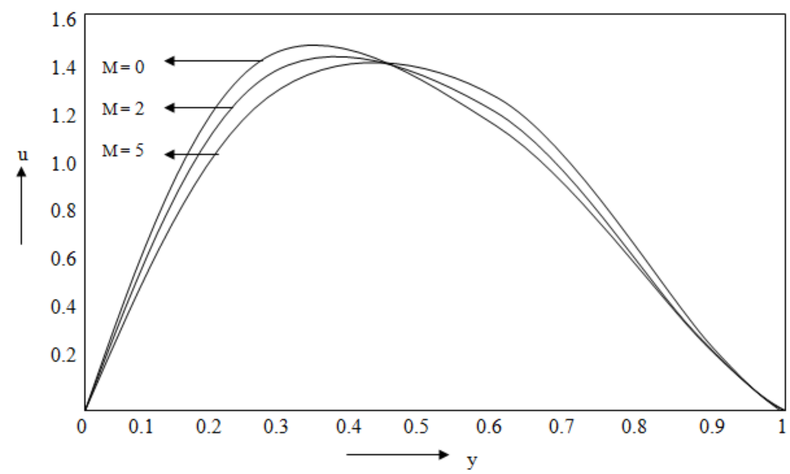

Figure 2 : Effect of Dust Particles on Blood Flow

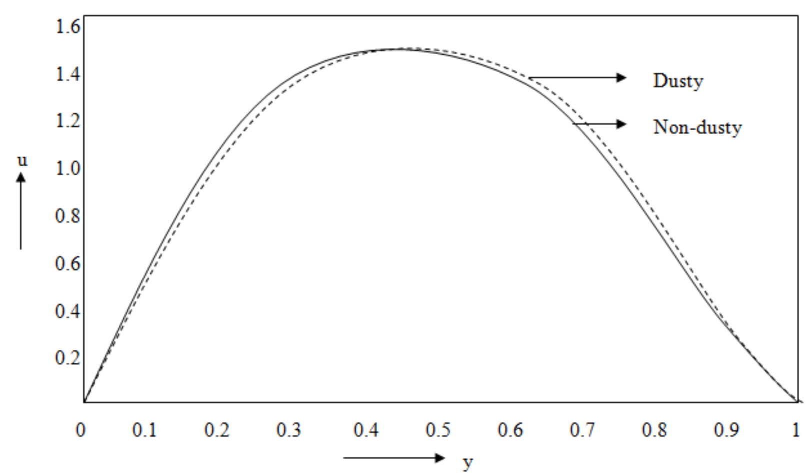

Figure 3 : Effect of Dust Particles on Blood Flow

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