Augmentation Of Fixed Point Theorem in Complete Fuzzy Metric Space With Altering Distance Function

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ABSTRACT

In present manuscript, we have built up a fixed point theorem in complete fuzzy metric space with altering distance function and contractive type condition for carrying off our motive . The authors tried to explain that changing space , conditions and inequality , the main purpose can be achieved

KEYWORDS AND PHRASES

Altering Distance Function ,Control Function, Contractive Condition,Complete Fuzzy Metric Space , Fixed Point Theorem.

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1. INTRODUCTION AND PRELIMINARIES

The concept of fuzzy set theory was first initiated by Zadeh [21] in the year 1965, after this Zadeh [22] in year 1968 gave some more significant definitions, lemmas which was used in present scenario. Proceeding in the same manner and following the renowned researchers, scientists and authors we have extended and generalized the result of Naschie [3],Gupta et.al [10], Grabeic [5] and also some other result of literature such as Vasuki [19] , Gregori and Sapena [6], Gupta and Mani [8] and Sayyed et.al. [14],Gupta,et.al.[9] , Soni and Shukla [17],Beg et.al.[2] and Sgrivastava et.al. [16].

First we give some basic definitions which were described and noted by many mathematicians .

Definition 1.1 (Schweizer [15]) . A binary operation $*$: [0,1] x [0,1] → [0,1] is a continuous triangular norm in other words called t-norm if for all a, b,c, e ∈ [0,1] then following conditions are satisfied

(i) * is commutative and associate,

(ii) a * 1 =a,

(iii) * is continuous, and

(iv) a * b ≤ c * e whenever a ≤ c and b ≤ e.

Kramosil and Michalek [11] is defined fuzzy metric space in the sense of -

Definition 1.2 (Kramosil & Michalek[11]) . The triplet (X, M, *) is said to be fuzzy metric space if X is an arbitrary set , * is continuous t-norm , and M is fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions:

(i) M(x,y,0) = 0,

(ii) M(x,y,t) = 1, ∀ t > 0 iff x = y,

(iii) M(x, y, t) = M(y, x, t),

(iv) M(x, y, t) * M(y, z, s) ≤ M(x, z, t+s) ∀ x, y, z ∈ X and t, s > 0,

(v) M(x, y, .) : [0, ∞) → [0, 1] is left continuous, and

(vi) $\lim_{t\to\infty} M\,(x, y, t) = 1$ ∀ x, y ∈ X .

The triplet M(x,y,t) can be taken as the degree of nearness between x and y with respect to t ≥ 0.

LEMMA 1.1 For every x, y ∈ X, the mapping M(x, y, .) is non-decreasing on (0, ∞).

Furher more Grabiec [5] extended the fixed point theorem of Banach [1] to fuzzy metric space in sense of Kramosil and Michalek [11].

THEOREM 1.1 (Grabiec [5] . *Let (X, M, *) be a complete fuzzy metric space satisfying*

(i) $\lim_{t\to\infty} M\,(x, y, t) = 1$, and

(ii) M(Fx, Fy, kt) ≥ M(x, y, t), ∀ x, y ∈ X, *where $0 < k < 1$. Then F has a unique fixed point.*

Then Vasuki [19] generalized Grabiecs result for common fixed point theorem for n sequence of mapping in a fuzzy metric space. Gregori and Sapena [7] gave fixed point theorems for complete fuzzy metric space in the sense of George and Veeraman [4] and also for Kramosil and Michaleks [11] fuzzy metric space which are complete in Grabeics sense.

George and Veeramani [4] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [11] with the help of t-norm and gave the following definition.

Definition 1. 3 (George & Veeramani [4]) The triplet (X, M, *) is said to be fuzzy metric space if X is an arbitrary set, * is continuous t-norm, and M is fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions:

(i) $M(x,y,0) > 0$,

(ii) $M(x,y,t) = 1$, $\forall\, t > 0$ iff $x = y$,

(iii) $M(x, y, t) = M(y, x, t)$,

(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t+s)$ $\forall\, x, y, z \in X$ and $t, s > 0$, and

(v) $M(x, y, .) : [0, \infty) \to [0, 1]$ is continuous.

By introducing this definition, they also succeeded in introducing a Hausdorff topology on such fuzzy metric spaces which is widely used these days by researchers in their respective field of research. George and Veeramani [4] have pointed out that the definition of Cauchy sequence given by Grabeic is weaker and hence it is essential to modify that definition to get better results in fuzzy metric space.

Consequently, some more metric fixed point results were generalized to fuzzy metric spaces by various authors such as Subrahmanyam [18], Vasuki [19], Saini, Gupta, and Singh [12,13], Vijayaraju [20], and Gupta and Mani [8,9].

Now we give some important definitions given by Grabiec [5] and lemmas that are used in sequel.

Definition 1.4 A sequence $\{X_n\}$ in a fuzzy metric space (X, M, *) is said to be convergent to $x \in X$ if $\lim_{t\to\infty} M(X_n, x, t) = 1$ $\forall\, t > 0$.

Definition 1. 5 . A sequence $\{X_n\}$ in a fuzzy metric space (X, M, *) is called Cauchy Sequence if $\lim_{n\to\infty} (X_{n+p}, X_n, t) = 1$ $\forall\, t > 0$ and each $p > 0$.

Definition 1.6 . A fuzzy metric space (X, M, *) is said to be complete if every Cauchy sequence in X converges in X.

EXAMPLE 1.1 (Gregori et al.[7]) Let (X, d) be a bounded metric space with d (x, y) < k for all x, y ∈ X. Let g : $R^+ \to (k, \infty)$ be an increasing continuous function. Define a function M as

M (x, y, t) = $1 - \frac{d(x,y)}{g(t)}$. Then (X, M, *) is a fuzzy metric space on X where * is a Lukasievicz t-norm, i.e, * (a, b) = max {a+b-1, 0}.

LEMMA 1.2 If there exists k ∈ (0 ,1) such that M (x, y, kt) ≥ M (x, y, t) for all x, y ∈ X and t ∈ (0, ∞), then x=y.

In our result, we define a class Φ of all mappings ξ : [0 ,1] → [0 ,1] satisfying the following conditions:

(i) ξ is increasing on [0,1], and

(ii) ξ(t) > t, ∀ t ∈ (0, 1] and ξ(t) = t if and only if t = 1.

***Definition 2.7.* [Zadeh [21]]** : A function ∅ : [0,1] → [0,1] is called an altering distance function if it satisfies the following properties:

(2.7.1) ∅ is strictly decreasing and continuous;

(2.7.2) ∅ (λ) = 0 if and only if λ = 1.

It is obvious that $\lim_{\lambda\to 1} \emptyset(\lambda) = \emptyset(1) = 0$.

2. MAIN RESULTS

THEOREM 2.1. Let E: X → X be a mapping in a complete fuzzy metric space $(X, M, *)$ with satisfying the condition :

$$M(Ex, Ey, kt) \geq \xi\{\lambda(x, y, t)\} \quad \text{--- (2.1.1)}$$

Where,

$$\lambda(x, y, t) = a\,\frac{M(x,Ex,t)*M(y,Ey,t)}{M(x,y,t)} + a^*\,\frac{M(x,Ey,t)*M(y,Ex,t)}{M(x,y,t)} + a^{**}\,M(x, Ex, t)$$
$$+\, b\,M(y, Ey, t) + b^*\,M(x, y, t)$$

…(2.1.2)

For all x, y ∈ X, ξ ∈ ∅ is an altering distance function and a,a*,a**,b and b* ∈ (0, 1) with

$0 \le a + a^{**} + b + b^{*} \le 1$. Then E has a unique fixed point.

PROOF: Let x ∈ X be any arbitrary point in X. Construct a sequence $\{x_n\} \in X$ such that $E\, x_n = x_{n+1}$ for all n ∈ N.

First we claim that the sequence $\{x_n\}$ is a Cauchy Sequence. For this taking $x = x_{n-1}$ and $y = x_n$ in equation 2.1.1, we get

$$M(Ex_{n-1}, Ex_n, kt) \ge \xi\,\{\lambda(x_{n-1}, x_n, t)\} \qquad \text{---} \quad (2.1.3)$$

By using equation 2.1.2, we claim that

$$\lambda\,(x_{n-1}, x_n,\ t) = a\,\frac{M(x_{n-1},Ex_{n-1},t)*M(x_n,Ex_n,t)}{M(x_{n-1},x_n,t)} + a^{*}\,\frac{M(x_{n-1},Ex_n,t)*M(x_n,Ex_{n-1},t)}{M(x_{n-1},x_n,t)}$$

$$+\, a^{**}\, M(x_{n-1}, Ex_{n-1}, t) + b\; M(x_n, Ex_n, t) + b^{*}\; M(x_{n-1}, x_n, t)$$

$$= a\,\frac{M(x_{n-1},x_n,t)*M(x_n,x_{n+1},t)}{M(x_{n-1},x_n,t)} + a^{*}\,\frac{M(x_{n-1},x_{n+1},t)*M(x_n,x_n,t)}{M(x_{n-1},x_n,t)}$$

$$+\, a^{**}\, M(x_{n-1}, x_n, t) + b\, M(x_n, x_{n+1}, t) + b^{*}\; M(x_{n-1}, x_n, t)$$

By using definition of altering distance function , we can eaisly write

$$= a\, M(x_n, x_{n+1}, t) + a^{**}\, M(x_{n-1}, x_n, t) + b\, M(x_n, x_{n+1}, t) + b^{*}\; M(x_{n-1}, x_n, t)$$

Or

$$\lambda\,(x_{n-1}, x_n,\ t) \le \frac{a+b}{1-a^{**}-b^{*}} \quad M(x_n, x_{n+1}, t)$$

Hence, our claim follows immediately from Lemma 1.2. Now suppose,

$M(x_n, x_{n+1}, t) \ge M(x_{n-1}, x_n, t)$ Then again from equation 2.1.3, we write

$M(x_n, x_{n+1}, kt) \ge \xi\,\{M(x_{n-1}, x_n, t)\} > M(x_{n-1}, x_n, t)$

Now by simple induction, for all n and t > 0, we get

$M(x_n, x_{n+1}, kt) \ge M\left(x, x,\ \frac{t}{k^{n-1}}\right)$

Now for any positive integer r, we have

$$\mathrm{M}(x_n, x_{n+r}, t) \geq \mathrm{M}\left(x_n, x_{n+1}, \frac{t}{r}\right) * \ \ldots \ * \mathrm{M}\left(x_{n+p-1}, x_{n+p}, \frac{t}{r}\right)$$

Using equation , we get

$$\mathrm{M}(x_n, x_{n+r}, t) \geq \mathrm{M}\left(x_n, x_1, \frac{t}{rk^n}\right) * \ \ldots \ * \mathrm{M}\left(x, x_1, \frac{t}{rk^n}\right)$$

Taking $\lim_{n\to\infty}$, we get

$$\lim_{n\to\infty} M(x_n, x_{n+r}, t) = 1$$

This implies $\{x_n\}$ is a Cauchy sequence, therefore there exists a point u ∈ X such that

$$\lim_{n\to\infty} x_n = \mathrm{u} .$$

Now we shall claim that u is a fixed point of E. For proving this we consider

$$\mathrm{M}(u, Eu, t) \geq \mathrm{M}(Ex_n, Eu, t) * \mathrm{M}(u, x_{n+1}, t) \geq \left\{\lambda\left(\mathrm{x_n}, u, \frac{t}{2k}\right)\right\} * \mathrm{M}(u, x_{n+1}, t) \qquad \ldots 2.1.4$$

Again from equation 2.1.2 , we can describe

$$\lambda\left(\mathrm{x_n}, u, \frac{t}{2k}\right) = \mathrm{a} \ \frac{\mathrm{M}\left(x_n, Ex_n, \frac{t}{2k}\right) * \mathrm{M}\left(u, Eu, \frac{t}{2k}\right)}{\mathrm{M}\left(x_n, u, \frac{t}{2k}\right)} + \mathrm{a}^* \ \frac{\mathrm{M}\left(x_n, Eu, \frac{t}{2k}\right) * \mathrm{M}\left(u, Ex_n, \frac{t}{2k}\right)}{\mathrm{M}\left(x_n, u, \frac{t}{2k}\right)}$$

$$+ \mathrm{a}^{**} \ \mathrm{M}\left(x_n, Ex_n, \frac{t}{2k}\right) + \mathrm{b} \ \mathrm{M}\left(u, Eu, \frac{t}{2k}\right) + \mathrm{b}^* \ \mathrm{M}\left(x_n, u, \frac{t}{2k}\right)$$

Taking $\lim_{n\to\infty}$ in above, we get

$$\lambda\left(\mathrm{u}, u, \frac{t}{2k}\right) = \mathrm{a} \ \frac{\mathrm{M}\left(u, Eu, \frac{t}{2k}\right) * \mathrm{M}\left(u, Eu, \frac{t}{2k}\right)}{\mathrm{M}\left(u, u, \frac{t}{2k}\right)} + \mathrm{a}^* \ \frac{\mathrm{M}\left(u, Eu, \frac{t}{2k}\right) * \mathrm{M}\left(u, Eu, \frac{t}{2k}\right)}{\mathrm{M}\left(u, u, \frac{t}{2k}\right)}$$

$$+ \mathrm{a}^{**} \ \mathrm{M}\left(u, Eu, \frac{t}{2k}\right) + \mathrm{b} \ \mathrm{M}\left(u, Eu, \frac{t}{2k}\right) + \mathrm{b}^* \ \mathrm{M}\left(u, u, \frac{t}{2k}\right)$$

Or

$$\lambda\left(\mathrm{u}, u, \frac{t}{2k}\right) \leq \frac{\mathrm{a+b}}{1-\mathrm{a}^{**}-\mathrm{b}^*} \ \mathrm{M}\left(u, Eu, \frac{t}{2k}\right)$$

Hence from equation 2.1.4, we get

$$\mathrm{M}(u, Eu, t) \geq \xi\left\{\mathrm{M}\left(u, Eu, \frac{t}{2k}\right)\right\} * \mathrm{M}(x_{n+1}, u, t) > \mathrm{M}\left(u, Eu, \frac{t}{2k}\right) * \mathrm{M}(x_{n+1}, u, t) \qquad \ldots 2.1.5$$

Taking $\lim\limits_{n\to\infty}$ in equation 2.1.5 and using Lemma 1.2, we get Eu = u .

FOR UNIQUENESS: Now we shall show that u is a unique fixed point of E. Suppose not, there exists a point z ∈ X such that Ez = z consider

$$1 \geq \mathrm{M}(z,u,t) = \mathrm{M}(Ez,Eu,t) \geq \xi\left\{\lambda\left(z,u,\frac{t}{k}\right)\right\} \qquad \ldots 2.1.6$$

Where,

$$\lambda\left(z,u,\frac{t}{k}\right) = \mathrm{a}\,\frac{\mathrm{M}\left(z,Ez,\frac{t}{k}\right)*\mathrm{M}\left(u,Eu,\frac{t}{k}\right)}{\mathrm{M}\left(z,u,\frac{t}{k}\right)} + \mathrm{a}^*\,\frac{\mathrm{M}\left(z,Eu,\frac{t}{k}\right)*\mathrm{M}\left(u,Eu,\frac{t}{k}\right)}{\mathrm{M}\left(z,u,\frac{t}{k}\right)}$$

$$+\ \mathrm{a}^{**}\,\mathrm{M}\left(z,Ez,\frac{t}{k}\right) + \mathrm{b}\ \mathrm{M}\left(u,Eu,\frac{t}{k}\right) + \mathrm{b}^*\,\mathrm{M}\left(z,u,\frac{t}{k}\right)$$

$$= \mathrm{a}\,\frac{\mathrm{M}\left(z,z,\frac{t}{k}\right)*\mathrm{M}\left(u,u,\frac{t}{k}\right)}{\mathrm{M}\left(z,u,\frac{t}{k}\right)} + \mathrm{a}^*\,\frac{\mathrm{M}\left(z,u,\frac{t}{k}\right)*\mathrm{M}\left(z,z,\frac{t}{k}\right)}{\mathrm{M}\left(z,u,\frac{t}{k}\right)} + \mathrm{a}^{**}\,\mathrm{M}\left(z,z,\frac{t}{k}\right)$$

$$+\ \mathrm{b}\ \mathrm{M}\left(u,u,\frac{t}{k}\right) + \mathrm{b}^*\ \mathrm{M}\left(z,u,\frac{t}{k}\right)$$

This implies that either $\lambda\left(z,u,\frac{t}{k}\right) = 1$ or $\lambda\left(z,u,\frac{t}{k}\right) = \mathrm{M}\left(z,u,\frac{t}{k}\right)$

Using it in equation 2.1.6, we get z = u.

Thus u is a unique fixed point of E. This complete the proof of Theorem 2.1.

COROLLARY 2.1: Let $(X,M,*)$ be a complete fuzzy metric space and E:X → X be a mapping satisfying

$$\mathrm{M}\ (Ex,Ey,kt) \geq \lambda\ (x,y,t)$$

where,

$$\lambda\ (x,y,t) = \mathrm{a}\ \frac{\mathrm{M}(x,Ex,t)*\mathrm{M}(\mathrm{y},Ey,t)}{M(x,y,t)} + \mathrm{a}^*\ \frac{\mathrm{M}(x,Ey,t)*\mathrm{M}(\mathrm{y},Ex,t)}{M(x,y,t)} + \mathrm{a}^{**}\ M\ (x,Ex,t)$$

$$+\ \mathrm{b}\ M\ (y,Ey,t) + \mathrm{b}^*\ M(x,y,t)$$

For all x, y ∈ X, ξ ∈ ∅ is an altering distance function and a,a*,a**,b and b* ∈ (0, 1) with

0 ≤ a + a** + b + b* ≤ 1 . Then E has a unique fixed point.

The proof of the result follows immediately from Theorem 2.1 by taking ξ(t) = t.

3. APPLICATION:

In this section, we give an application related to our result.

Let us define $\psi : [0, \infty] \rightarrow [0, \infty]$, as

$\Psi(t) = \int_0^t \Psi(t)dt\ \ \forall\, t > 0$, be a non-decreasing and continuous function. Moreover, for each

$\epsilon > 0$, $\Psi(\varepsilon) > 0$ It also implies that $\Psi(t) = 0$ if and only if $t = 0$.

THEOREM 3.1. Let (X, M, *) be a complete fuzzy metric space and $E : X \rightarrow X$ be a mapping satisfying

$$\int_0^{M(Ex,Ey,kt)} \Psi(t)\, dt > \xi\left\{\int_0^{\lambda\,(x,y,t)} \Psi(t)\ dt\right\}$$

Where,

$$\lambda\,(x,y,t) = a\ \frac{M(x,Ex,t)*M(y,Ey,t)}{M(x,y,t)} + a^*\ \frac{M(x,Ey,t)*M(y,Ex,t)}{M(x,y,t)} + a^{**}\ M\,(x,Ex,t)$$

$$+\, b\ \ M\,(y,Ey,t) + b^*\ M(x,y,t)$$

For all $x, y \in X$, $\xi \in \emptyset$ is an altering distance function and a,a^*,a^{**},b and $b^* \in (0, 1)$ with

$0 \leq a + a^{**} + b + b^* \leq 1$. Then E has a unique fixed point.

Proof: By taking $\Psi(t) = 1$ and applying Theorem 2.1, we obtain the result.

REFERENCES

[1] Banach,S., sur les oprations dans les ensembles abstraits et leur application aux quations intgrales [On operations in the abstract sets and their applications to the integral equations. Fundamenta mathematicae, 3, (1922),133-181.

[2] Beg,I.,Sedghi,S. and Shobe,N., Fixed point theorems in fuzzy metric spaces, International journal of Analysis,,Vol.2013, article ID 934145.

[3] EI Naschie,M.S., A review of E-infinity theory and the mass spectrum of high energy particle physics. Chaos Salitons Fractals, 19, (2004) ,209-236.

[4] George,A.,& Veeramani,P. On some results in fuzzy metric spaces, Fuzzy Sets and Systems , 64 (1994) ,395-399.

[5] Grabiec,M. Fixed point in fuzzy metric spaces, Fuzzy Sets and Systems ,27(1988), 385-389.

[6] Gregori,V., Morillas,S., & Sapena, A., Examples of fuzzy metrics and applications , Fuzzy Sets and Systems , 170(2011), 95-111.

[7] Gregori,V. & Sapena ,A., On fixed point theorems in fuzzy metric spaces, Fuzzy Sets and Systems ,125 (2002) , 245-252.

[8] Gupta,V.,& Mani,N., ,Existence and uniqueness of fixed point in fuzzy metric spaces and its applications ,Proceeding of the second international conference on soft computing for problem solving; Advances in intelligent systems & computing , 236 (2014 a) , 217-224.

[9] Gupta,V.& Mani,N. Common fixed points by using E.A. property in fuzzy metric spaces. Proceeding s of the Third International Conference on Soft Computing for Problem Solving: Advances in Intelligent Systems and Computing, 259 (2014b), 45-54.

[10] Gupta, V., Saini, R.K., Mani, N., & Tripathi, A.KFixed point theorems using control function in fuzzy metric spaces. Cogent Mathematics, .(2015).

[11] Karmosill,I., & Michalek,J, Fuzzy metric and statistical metric spaces. Ky-bernetica,11. (1975), 326-334.

[12] Saini, R.K., Gupta, V., & Singh, S.B., Fuzzy version of some fixed points theorems on expansion type maps in fuzzy metric spaces. Thai Journal of Mathematics, 5, (2007), 245-252.

[13] Saini, R.K., Kumar, M., Gupta, V., & Singh, S.B., Common coincidence points of R-weakly commuting fuzzy maps. Thai Journal of Mathematics,6, (2008), 109-115.

[14] Sayyed,S.A., Vyas,L., Sayyed,F. And Badshah,V.H., “Some Results On Fixed Point Theorems For Selfmaps” Ultra Engineer , Vol.1 (2), (2012).143-145.

[15] Schweizer, B., & Sklar, A., Statistical metric spaces. Pacific Journal of Mathematics,10, (1960), 313-334.

[16] Shrivastava,M., Qureshi,K. and Singh, A.D., Some fixed point theorems in fuzzy metric spaces, American journal of Engineering Research, vol.5 no.7, (2016), 137-141.

[17] Soni, S. and Shukla, M.K., Some fixed point theorems in fuzzy metric space for expansion mappings, International Journal of Advanced Research in Computer Science, Vol.9,No. (2018), 280-283.

[18] Subrahmanyam, P. V., A common fixed point theorem in fuzzy metric space. Information Sciences,83, .(1995), 109-112.

[19] Vasuki,R. A common fixed point theorem in fuzzy metric space. Fuzzy Sets and Systems , 97, (1998), 395-397.

[20] Vijayaraju, P.,& Sajath, Z. M. I., Some common fixed point theorems in fuzzy metric spaces. International Journal of Mathematical Analysis, 3, (2009), 701-710.

[21] Zadeh, L. A., Fuzzy sets. Information and Control, 8, (1965), 338-353.

[22] Zadeh, L. A. Probability measures of fuzzy events, J.Math.Anal.Appl.,23, (1968), 421-47