ON LOCALLY Q-SETS IN TOPOLOGY

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Abstract : Bourbaki studied the concept of locally closed sets in topological spaces. A subset B of a topological space is locally closed if it is the intersection of a closed set with an open set. Following this topologists introduced the new notions of locally closed sets by replacing open sets by nearly open sets and generalized open sets and/or by replacing the closed sets by nearly closed sets and generalized closed sets. Levine studied the concept of Q-sets in 1961 and Thangavelu and Rao introduced the notion of q-sets in 2002. Recently, the author further investigated the properties of q-sets and Q-sets in topology. The purpose of this paper is to introduce a new class of sets called locally Q-sets and studied their properties.

Keywords: α-open, regular open, q-set, Q-set, continuous
MSC 2010: 54A05

I. INTRODUCTION

Levine[3] studied the concept of Q-sets in 1961. The concept of a q-set [6] was introduced in 2002. Following this several mathematicians extended these notions to bitopology, fuzzy topology and ideal topology. The author [2] investigated the further properties of q-sets and Q-sets. In this paper the concepts of a locally q-set and a locally Q-set are introduced and their basic properties have been studied.

II. PRELIMINARIES

Throughout this paper (X, τ) and (Y,σ) denote topological spaces. X\A is the complement of a subset A of X. The interior and closure operators are respectively denoted by *int* A and *cl* A. The following definitions and lemmas will be useful in sequel. The phrase "iff" is used for the phrase "if and only if". The following definitions and lemmas will be useful in the paper.

Definition 2.1:A subset A of a topological space X is called

(i) α-open [4] if A$\subseteq$ *int cl int* A and α-closed if *cl int cl* A$\subseteq$ A.
(ii) regular open [5] if A = *int cl* A and regular closed if A = *cl int* A.
(iii) a q-set [6] if *int cl* A $\subseteq$ *cl int* A.
(iv) a Q-set [3] if *int cl* A = *cl int* A.

The following lemma that is established in [1] will be useful**.**
Lemma 2.2: If A is a subset of X then α*int* α*cl* A = *int cl* A and α*cl* α*int* A = *cl int* A.
Diagram 2.3: Regular clopen $\Leftrightarrow$ clopen $\Leftrightarrow$ α-clopen $\Rightarrow$Q-set $\Rightarrow$q-set.
Lemma 2.4: A is a Q-set if and only if X \ A is a Q-set.
Lemma 2.5: If B is open and A is a q-set then A $\cap$ B and B\ A are q-sets.

III. LOCALLY Q-SETS

The intersection of a Q-set with an open set need not be an open set and need not be a Q-set as shown below.

Let X={a, b, c}and τ ={ϕ, {b}, {c}, {b, c}, X}. Then Q(τ) = {ϕ, {a}, {b, c}, X} = the collection of Q-sets in X. Now X is open and {a} is a Q-set. X $\cap$ {a}={a} is a Q-set but not open. Also X is a Q-set and {b} is open. X $\cap$ {b}={b} is open but not a Q-set.

This discussion motivates us to have the following definition.

Definition 3.1: Let (X, τ) be a topological space. A subset of X is called a locally Q-set if it is the intersection an open set with a Q-set. Also a subset of X is called a locally q-set if it is the intersection an open set with a q-set.

The discussion prior to Definition 3.1 shows that a locally Q-set need not be an open set and need not be a Q-set . However from Lemma 2.5. we see that every locally q-set is a q-set. Since every q-set is a locally q-set it follows that a subset A of X is a locally q-set if and only if it is a q-set. In fact, the class of all locally q-sets coincides with the class of all q-sets in any topological space.

Therefore we are interested to study the locally Q-sets rather than locally q-sets. The proof for the next proposition is straight forward.

Proposition 3.2: Every Q-set is a locally Q-set and every open set is a locally Q-set.

Proposition 3.3 : A subset B of a space X is a locally Q-set if and only if X\B is a union of a closed set with a Q-set.

Proof: A subset B of a space X is a locally Q-set if and only if B = G $\cap$ A where G is open in X and A is a Q-set in X. Since X\B = (X\G) $\cup$ (X\A) and since X\A is a Q-set whenever A is a Q-set, it follows that B is a locally Q-set if and only if X\B is a union of a closed set with a Q-set.

Proposition 3.4: Suppose G is clopen or regular clopen or α-clopen and A is a Q-set. Then G $\cap$ A is both a Q-set and a locally Q-set.

Proof: Follows from the fact that *int cl* A = *cl int* A = A for every clopen set A.

Proposition 3.5: In an extremally disconnected space, every q-set is a Q-set and hence every q-set is a locally Q-set.

Proof: Let X be an extremally disconnected space. The for any subset A of X,
int cl int A = *cl int* A and *int cl* A = *cl int cl* A that implies *cl int* A $\subseteq$ *int cl* A. If A is a q-set then *int cl*A $\subseteq$ *cl int* A that together previous inclusion implies that *int cl* A = *cl int* A. This shows that every q-set in an extremally disconnected space is a Q-set and hence a locally Q-set.

Proposition 3.6: In a locally indiscrete space, every locally Q-set is a Q-set.

Proof: In an locally indiscrete space every open set is closed and every open set is clopen. Suppose G $\cap$ A is a locally Q-set in X where G is open in X and A is a Q-set in X. Since A is a Q-set, *cl* A = *int cl* A = *cl int* A = *int* A that implies *cl* A = *int* A = A . Therefore in a locally indiscrete space every Q-set is clopen. Now since G is clopen and since A is clopen, G $\cap$ A is also clopen that implies G $\cap$ A is a Q-set.

Proposition 3.7: Let X and Y be any two topological spaces . Let X $\times$Y be the product space. Let U and V be the subsets of X and Y respectively. Then U$\times$V is a locally Q-set in X$\times$Y if and only if U is a locally Q-set in X and V is a locally Q-set in Y.

Proof: Suppose U$\times$V is a locally Q-set in X$\times$Y. Then U$\times$V = (G$\times$H) $\cap$ (A$\times$B) where G$\times$H is open in X$\times$Y and A$\times$B is a Q-set in X$\times$Y. Now U$\times$V = (G $\cap$ A)$\times$(H $\cap$ B). Since G is open in X and A is a Q-set in X and since H is open in Y and B is a Q-set in Y, G $\cap$ A is a locally Q-set in X and (H $\cap$ B) is a locally Q-set in Y that implies U is a locally Q-set in X and V is a locally Q-set in Y. Conversely let U be a locally Q-set in X and V be a locally Q-set in Y. Then U = G $\cap$ A and V = H $\cap$ B where G is open in X and A is a Q-set in X; H is open in Y and B is a Q-set in Y.

Since U$\times$V = (G $\cap$ A)$\times$(H $\cap$ B) = (G$\times$H) $\cap$ (A$\times$B) and since G$\times$H is open in X$\times$Y and A$\times$B is a Q-set in X$\times$Y it follows that U$\times$V is a locally Q-set in X $\times$Y.

Proposition 3.8: Every locally Q-set in (X, τ) is a locally Q-set in (X, τ^{α}).

Proof: Follows from the fact that the class of all Q-sets in (X, τ) is same as the class of all Q-sets in (X, τ^{α}) and from the fact that every open set is α-open.

Definition 3.9: Let f:(X, τ)$\rightarrow$(Y, σ) be a function. f is locally Q-continuous from (X, τ) to (Y, σ) if the inverse image of every open set in (Y, σ) is a locally Q-set in (X, τ).

Proposition 3.10: Suppose f:(X, τ)$\rightarrow$(Y, σ) is a bijection. Then f:(X, τ)$\rightarrow$(Y, σ) is locally Q-continuous if and only if f^{-1}(H) is a union of a closed set in X with a Q set in X for every closed set H in Y.

Proof: $f:(X, \tau)\rightarrow(Y, \sigma)$ is locally Q-continuous if and only if $f^{-1}(Y\backslash H)$ is a locally Q-set in X for every closed set H in Y if and only if $X\backslash f^{-1}(H)$ is a locally Q-set in X for every closed set H in Y that is if and only if $X\backslash f^{-1}(H) = G \cap A$ for every closed set H in Y where G is open in X and A is a locall Q-set in X. Now $X\backslash f^{-1}(H) = G \cap A$ iff $f^{-1}(H) = X\backslash(G \cap A) = (X\backslash G) \cup (X\backslash A)$. Therefore it follows that $f:(X, \tau)\rightarrow(Y, \sigma)$ is locally Q-continuous iff $f^{-1}(H)$ is a union of a closed set in X with Q set in X for every closed H in Y.

Proposition 3.11: If $f:(X, \tau)\rightarrow(Y, \sigma)$ is locally Q-continuous then for every $x\in X$ and for every open set V in Y with $f(x)\in V$ there is a locally Q-set $G \cap A$ in X such that $x \in G \cap A$ and $f(G \cap A) \subseteq V$.

Proof: Suppose $f:(X, \tau)\rightarrow(Y, \sigma)$ is locally Q-continuous. Let $x \in X$ and V be open in Y with $f(x) \in V$. Then $f^{-1}(V)$ is a locally Q-set in X. Let $f^{-1}(V) = G \cap A$ where G is open in X and A is a Q-set in X. Clearly $x \in G \cap A = f^{-1}(V)$ that implies $f(G \cap A) \subseteq V$.

Remark: The converse of the above proposition is not true. However the converse is true when the class of locally Q-sets in X is closed under arbitrary union as seen in the next proposition.

Proposition 3.12: Let (X, τ) be a space such that the class of locally Q-sets is closed under arbitrary union. Then for any function $f:(X, \tau)\rightarrow(Y, \sigma)$, the following are equivalent.

(i) $f:(X, \tau)\rightarrow(Y, \sigma)$ is locally Q-continuous.

(ii) $f^{-1}(H)$ is a union of a closed set in X with a Q-set in X for every closed set H in Y.

(iii) for every $x \in X$ and for every open set V in Y with $f(x) \in V$ there is a locally Q-set $G \cap A$ in X such that $x \in G \cap A$ and $f(G \cap A) \subseteq V$.

Proof: (i)$\Leftrightarrow$ (ii) follows from Proposition 3.10 and (i)$\Rightarrow$ (iii) follows from Proposition 3.11. (iii)$\Rightarrow$ (i) follows from the given condition that the class of locally Q-sets in X is closed under arbitrary union.

Conclusion: In this paper, Q-sets and locally Q-sets were discussed in α-space and extremely disconnected spaces. In fact the class of all Q-sets and the class of all locally Q-sets are the same in locally indiscrete spaces. Further locally Q-sets were studied in product space of two spaces. The weak form of Q-continuity namely locally Q-continuity was characterized .

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