ON I_g^δ - CLOSED SETS IN IDEAL TOPOLOGICAL SPACES

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Abstract : In this paper we define and study a new class of generalized closed sets called I_g^δ - closed set in ideal topological space. Also we discuss the characterizations and properties of I_g^δ - closed set.

Keywords: δ - closed set, τ_g^* - closed set, g - local function, I_g^δ - closed set.

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I. INTRODUCTION

Levine introduced the novel idea of generalized closed set in 1970[6] which is a generalization of closed sets. Nowadays ideals are playing very important role in General topology. It was the works of Vaidyanathaswamy [11], Jankovic and Hamlett [4] which initiated the research in applying topological ideals to generalize the most basic properties in General topology. In 2014, g–local functions was introduced by K. Bhavani [1]. Recently using g - local functions I_g^{**} - closed sets was introduced by C. Santhini et al in ideal topological spaces [10]. Following the trend, we have introduced and investigated a new class of generalized closed sets in an ideal topological space called I_g^δ - closed set.

1.1 Preliminaries

A collection $I \subseteq P(X)$ is called an ideal on X if it satisfies the following conditions:

a. $A \in I$ and $B \subseteq A$ implies $B \in I$
b. $A \in I$ and $B \in I$ implies $A \cup B \in I$.

An ideal topological space is a topological space (X, τ) with an ideal I on X and it is denoted by (X, τ, I). For a subset A of X, $A^*(I, \tau) = \{x \in X / A \cap U \notin I$ for every $U \in \tau(x)\}$ is called the local function of A with respect to I and τ [5] and simply we can write A^*. A Kuratowski closure $cl^*(.)$ for a topology $A^*(I, \tau)$, called the ∗- topology and finer than τ, is defined by $cl^*(A) = A \cup A^*$ [11]. A subset A of an ideal topological space (X, τ, I) is ∗- closed [4], if $A^* \subseteq A$. The complement of ∗- closed set is called a ∗- open set. A subset A of a topological space (X, τ) is said to be g - closed [9], if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open. The complement of a g - closed set is called a g - open set. For every subset A of X, $A_g^*(I, \tau) = \{x \in X / U_x \cap A \notin I$, for every g - open set U_x containing x } is called the g - local function of A [2] with respect to I and τ_g and is denoted by A_g^*. Also, $cl_g^*(A) = A \cup A_g^*$ [2] is a Kuratowski closure operator for a topology $\tau_g^* = \{X - A / cl_g^*(A) = A\}$ on X which is finer than τ_g. A subset A of an ideal topological space (X, τ, I) is said to be ∗- perfect [4] if $A^* = A$. A subset A of a topological space (X, τ, I) is said to be τ_g^* - closed [2] if $A_g^* \subseteq A$.

Definition 1.2:

A subset A of a topological space (X, τ) is said to be δ - closed[12] if A = $cl_\delta(A)$, where $cl_\delta(A) = \{x \in X / int(cl(U)) \cap A \neq \phi, U \in \tau$ and $x \in U\}$. The complement of a δ- closed set is called a δ- open set.

Definition1.3:

A subset A of an ideal topological space (X, τ, I) is said to be semi - I - closed (sI- closed)[3] if $int(cl^*(A)) \subseteq A$.

Definition 1.4: A subset A of a topological space (X, τ) is said to be

a. g^* - closed [1] if $cl(A) \subset U$ whenever $A \subset U$ and U is g - open in (X, τ).
b. gδ - closed [9] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is δ - open in (X, τ).
c. rg - closed [9] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in (X, τ).

Definition 1.5: A subset A of an ideal topological space (X, τ, I) is said to be

a. I_g - closed [9] if $A^* \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ, I).
b. I_{gg} - closed [1] if $A_g^* \subset U$ whenever $A \subset U$ and U is g - open in (X, τ, I).
c. ∗- g - closed [8] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is ∗- open in (X, τ, I).
d. I_g - ∗- closed [7] if $A^* \subseteq U$ whenever $A \subseteq U$ and U is ∗- open in (X, τ, I).

e. $I_{g\delta}$ -closed [9] if $A^* \subseteq U$ whenever $A \subseteq U$ and U is δ - open in (X, τ , I)

f. I_{rg} - closed [9] if $A^* \subseteq U$ whenever $A \subseteq U$ and U is regular open in (X, τ, I).

g. I_g^{**} - closed [10] if $A_g^* \subseteq U$ whenever $A \subseteq U$ and U is $*$ - open in (X, τ, I).

Definition 1.6: A subset A of an ideal topological space (X, τ, I) is said to be a χ_I -set [9] if $A = U \cap V$, where U is a δ - open set and V is $*$ - perfect set.

Definition 1.7: A subset A of an ideal topological space (X, τ , I) is said to be a γ_I - set [9] if $A = U \cap V$, where U is a δ - open set and V is a $*$ - perfect set.

2. I_g^{δ} - closed set

In this section a new class of generalized closed sets called I_g^{δ} - closed set is introduced and several characterizations of this notion are discussed.

Definition 2.1: A subset A of an ideal topological space (X, τ, I) is said to be an I_g^{δ} - closed set if $A_g^* \subseteq U$ whenever $A \subseteq U$ and U is δ - open in X.

Example 2.2: Let X = {a, b, c, d}, τ = {ϕ, X, {a}, {b}, {a, b}, {a, b, c}} and I = {ϕ, {a}}. Then {a} is an I_g^{δ} - closed set.

Theorem 2.3: Let (X, τ, I) be an ideal topological space and $A \subseteq X$. Then

a. Every τ_g^* - closed set is I_g^{δ} - closed.

b. Every $*$ - closed set is I_g^{δ} - closed.

c. Every gδ - closed set is I_g^{δ} - closed.

d. Every I_{gg} - closed set is I_g^{δ} - closed.

e. Every I_g - closed set is I_g^{δ} - closed.

f. Every δ- closed set is I_g^{δ} - closed.

g. Every g^* - closed set is I_g^{δ} - closed.

h. Every I_g^{**} - closed set is I_g^{δ} - closed.

Proof:

a. Let A be an τ_g^* - closed set and U be a δ - open set such that $A \subseteq U$. Since A is τ_g^* - closed, $A_g^* \subseteq A$, which implies $A_g^* \subseteq A \subseteq U$ and hence A is an I_g^{δ} - closed set.

b. Let A be a $*$ - closed set and U be a δ - open set such that $A \subseteq U$. Since A is $*$ - closed $A^* \subseteq A$ which implies $A^* \subseteq A \subseteq U$. By theorem 3.7 [2], $A_g^* \subset A^* \subseteq A \subseteq U$. Consequently A is an I_g^{δ} - closed set.

c. Let A be a gδ - closed set and U be a δ- open set such that $A \subseteq U$. Since A is gδ - closed, cl (A) $\subseteq$ U. By theorem 3.7 [2], $A_g^* \subset$ cl (A) $\subseteq$ U and hence A is an I_g^{δ} - closed set.

d. Let A be a I_{gg} - closed set. By theorem 2.14[1], A is τ_g^* - closed set. Then by (a) result follows.

e. Let A be an I_g - closed set and U be a δ- open set such that $A \subseteq U$ and hence $A \subseteq U$ where U is open. Since A is I_g - closed, $A^* \subseteq U$. By theorem 3.7 [2], $A_g^* \subset A^* \subseteq U$. Hence A is an I_g^{δ} - closed set.

f. Let A be a δ - closed set. Since every δ - closed set is $*$ - closed, by (b), result follows.

g. Let A be a g^* - closed set and U be a δ - open set such that $A \subseteq U$. Then $A \subseteq U$ where U is g - open and since A is g^* - closed, cl (A) $\subseteq$ U. By theorem 3.7 [2], $A_g^* \subset$ cl (A) $\subseteq$ U and hence A is an I_g^{δ} - closed set.

h. Let A be an I_g^{**} - closed set and U be a δ - open such that $A \subseteq U$. Since every δ - open set is $*$ - open, $A \subseteq U$ where U is $*$ - open and since A is I_g^{**} - closed, $A_g^* \subseteq U$ and so A is I_g^{δ} - closed.

Remark 2.4: Converses of the above theorem are not true as seen from the following examples.

Example 2.5: Let X = {a, b, c, d}, τ = {ϕ, X, {a}, {d}, {a, d}, {a, c, d}} and I = {ϕ}. Then {c} is I_g^{δ} - closed but not τ_g^* closed.

Example 2.6: Let X = {a, b, c, d}, τ = {ϕ, X, {a}, {d}, {a, d}, {a, c, d}} and I = {ϕ, {a}, {b}, {a, b}}. Then {c} is I_g^{δ} - closed but not $*$ - closed.

Example 2.7: Let X = {a, b, c, d}, τ = {ϕ, X, {a}, {d}, {a, d}, {a, c, d}} and I = {ϕ , {d}}. Then {c} is I_g^δ - closed but not gδ - closed.

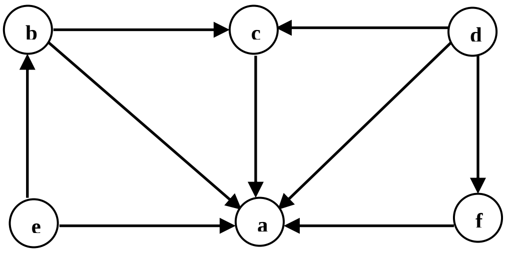
Example 2.8: Let X = {a, b, c, d}, τ = {ϕ, X, {a}, {d}, {a, d}, {a, c, d}} and I = {ϕ}. Then {c} is I_g^δ - closed but not I_{gg} - closed.

Example 2.9: Let X = {a, b, c, d}, τ = {ϕ, X, {a}, {b}, {a, b}, {a, b, c }} and I = {ϕ}. Then {c} is I_g^δ - closed but not I_g - closed.

Example 2.10: Let X = {a, b, c, d}, τ = {ϕ, X, {a}, {c}, {d}, {a, d}, {a, c}, {c, d}, {a, c, d}, {b, c, d}} and I = {ϕ}. Then {b} is I_g^δ - closed but not δ - closed.

Example 2.11: Let X = {a, b, c, d}, τ = {ϕ , X, {a}, {c}, {d}, {a, d}, {a, c}, {c, d}, {a, c, d}, {a, b, c}} and I = {ϕ}. Then {a, c} is I_g^δ - closed but not g^* - closed.

Remark 2.12: From the above results, we get the following diagram where a $\rightarrow$ b represents a implies b but not conversely.



(a) I_g^δ - closed (b) $*$ - closed (c) I_g^{**} - closed (d) g^* - closed (e) δ - closed (f) g- closed.

Remark 2.13: sI - closed sets and I_g^δ - closed sets are independent.

Example 2.14: Let X = {a, b, c, d}, τ = {ϕ, X, {a}, {d}, {a, d}, {a, c, d}} and I = {ϕ}. Then {a} is sI - closed but not I_g^δ - closed. Also {a, b, d} is I_g^δ - closed but not sI -closed.

3. Characterizations of I_g^δ - closed set

Theorem 3.1: Let (X, τ, I) be an ideal topological space and A $\subseteq$ X. If A $\subset A_g^*$, then the following are equivalent.

a. A is I_g^δ - closed.

b. $cl_g^*(A) \cap F = \phi$ whenever $A \cap F = \phi$ and F is δ - closed.

Proof

(a)$\Longrightarrow$ (b) Suppose $A \cap F = \phi$ where F is δ - closed. Then A $\subseteq$ X - F where X - F is δ - open. Since A is I_g^δ - closed, $A_g^* \subseteq X - F$. By theorem 3.10 [2], $cl_g^*(A) = A_g^* \subseteq X - F$. Hence $cl_g^*(A) \cap F = \phi$.

(b)$\Longrightarrow$(a) Let U be a δ -open set such that A $\subseteq$ U. Then $A \cap (X - U) = \phi$, where X - U is δ - closed. By (b) $cl_g^*(A) \cap (X - U) = \phi$ which implies $cl_g^*(A) \subseteq$ U. By theorem 3.10 [2], $A_g^* = cl_g^*(A) \subseteq U$. Consequently A is I_g^δ - closed.

Theorem 3.2: Let (X, τ, I) be an ideal topological space and A $\subseteq$ X. If A $\subset A_g^*$, then the following are equivalent.

a. A is an $I_{g\delta}$ - closed set.

b. A is an I_g^δ - closed set.

Proof

(a) $\Longrightarrow$(b) Let A be an $I_{g\delta}$ - closed set and U be a δ- open set such that A $\subseteq$ U. Since A is $I_{g\delta}$ -closed, $A^* \subseteq U$. By theorem 3.10 [2], $A_g^* = A^* \subseteq U$. Hence A is I_g^δ - closed.

(b)$\Longrightarrow$(a) Suppose A is I_g^δ - closed and U be a δ- open set such that A $\subseteq$ U. Since A is an I_g^δ - closed set, $A_g^* \subseteq U$. By theorem 3.10 [2], $A^* = A_g^* \subseteq U$. Hence A is an $I_{g\delta}$ - closed set.

Theorem 3.3: For a subset A of an ideal topological space (X, τ, I), then the following are equivalent, when A $\subset A_g^*$.

a. A is $*$ - closed.

b. A is a γ_I - set and a I_g^δ - closed set.

Proof:

(a)⟹(b) Let A be ∗- closed and U be a δ - open set such that A ⊆ U. By theorem 3.10 [2], $A_g^* = A^* \subseteq A \subseteq U$ and so A is I_g^δ - closed. Also A = X ∩ A, where X is δ- open and A is ∗- closed implies A is a γ_I - set.

(b)⟹(a) Let A be a γ_I - set and an I_g^δ - closed set. Then A = U ∩ V, where U is δ- open and V is ∗- closed. Since A = U ∩ V, A ⊆ U and A is I_g^δ - closed, implies $A_g^* \subseteq U$. By theorem 3.10 [2], $A^* = A_g^* \subseteq U$. Also since A ⊆ V, by theorem 3.7 [2], $A^* \subseteq V^*$. Since V is ∗- closed, $V^* \subseteq V$. Thus we have $A^* \subseteq V$ which implies $A^* \subseteq U \cap V = A$. Consequently A is ∗- closed.

Theorem 3.4: Let (X, τ, I) be an ideal topological space and A ⊆ X. If $A \subset A_g^*$ and A is I_g^δ -closed, then

a. A is I_{rg} - closed.

b. A is rg - closed.

Proof:

(a) Let A be an I_g^δ - closed set and U be a regular open set such that A ⊆ U. Then A ⊆ U where U is δ- open and since A is I_g^δ - closed, $A_g^* \subseteq U$. By theorem 3.10 [2], $A^* = A_g^* \subseteq U$. Consequently A is an I_{rg} - closed set.

(b) Similar to the proof of (a).

Remark 3.5: The converses of the above theorem are not true as seen from the following examples.

Example 3.6: Let X = {a, b, c, d}, τ = {ϕ, X, {a}, {b}, {a, b}, {a, b, c}} and I = {ϕ}. Then {a, b} is I_{rg} - closed but not I_g^δ - closed.

Example 3.7: Let X= {a, b, c, d}, τ = {ϕ, X, {a}, {d}, {a, d}, {c, d}, {a, c, d}} and I = {ϕ, {b}, {c}, {b, c}}. Then {a, d} is rg - closed but not an I_g^δ - closed set.

Theorem 3.8: Let (X, τ, I) be an ideal topological spaces and let A ⊆ X. If $A \subset A_g^*$, then the following hold.

a. If A is ∗- g - closed, then A is I_g^δ - closed.

b. If A is I_g - ∗- closed, then I_g^δ - closed.

c. If A is ∗- perfect, then A is I_g^δ - closed.

Proof:

(a) Let A be ∗- g - closed and U be a δ - open set such that A ⊆ U. Since every δ- open set is ∗- open, A ⊆ U where U is ∗- open and since A is ∗- g - closed, cl (A) ⊆ U. By theorem 3.10 [2], A_g^* = cl (A) ⊆ U. Hence A is an I_g^δ - closed set.

(b) Similar to the proof of (a).

(c) Let A be ∗- perfect and U be a δ- open set such that A ⊆ U and then $A^* = A \subseteq U$. By the theorem 3.10 [2], $A_g^* = A^* \subseteq U$. Consequently A is an I_g^δ - closed set.

Remark 3.9: The converses of the above theorem are not true as seen from the following examples.

Example 3.10: Let X = {a, b, c, d}, τ = {ϕ, X, {a}, {d}, {a, d}, {a, c, d}} and I = {ϕ}. Then {c} is I_g^δ - closed but not ∗- g - closed.

Example 3.11: Let X = {a, b, c, d}, τ = {ϕ, X, {a}, {b}, {a, b}, {a, b, c}} and I = {ϕ, {a}}. Then {c} is I_g^δ - closed but not I_g - ∗-closed.

Example 3.12: Let X = {a, b, c, d}, τ = {ϕ, X, {a}, {d}, {a, d}, {a, c, d}} and I = {ϕ, {a}, {b}, {a, b}}. Then {a} is I_g^δ - closed but not ∗- perfect.

Theorem 3.13: Let (X, τ, I) be an ideal topological space. If A and B are I_g^δ - closed sets, then A ∪ B is an I_g^δ - closed set.

Proof:

Let U be a δ - open set such that A ∪ B ⊆ U. Then A ⊆ U and B ⊆ U. Since A and B are I_g^δ - closed sets, $A_g^* \subseteq U$ and $B_g^* \subseteq U$ and so $A_g^* \cup B_g^* \subseteq U$. By theorem 3.7 [2], $(A \cup B)_g^* = A_g^* \cup B_g^* \subseteq U$ and so A ∪ B is an I_g^δ - closed set.

Remark 3.14: If A and B are I_g^δ - closed sets then A ∩ B is not I_g^δ - closed.

Example 3.15: Let X = {a, b, c, d}, τ = {ϕ, X, {a}, {d}, {a, d}, {a, c, d}} and I = {ϕ}. Then A = {a, b} and B = {a, c} are I_g^δ - closed whereas A $\cap$ B = {a} is not I_g^δ - closed.

Theorem 3.16: Let (X, τ, I) be an ideal topological space and A $\subseteq$ X. If A is I_g^δ - closed then A - I is I_g^δ - closed, for every I $\in$ I.

Proof:

Let U be a δ- open set such that A - I $\subseteq$ U. Since A is I_g^δ - closed, $A_g^* \subseteq$ U. By theorem 3.7 [2], $(A - I)_g^* = A_g^* \subseteq$ U. Consequently A - I is an I_g^δ - closed set.

Theorem 3.17: Let (X, τ, I) be an ideal topological space and I $\in$ I, then I is an I_g^δ - closed set.

Proof:

By theorem 3.12 [2], I is τ_g^* - closed and by theorem 2.3, the results follows.

Theorem 3.18: Let (X, τ, I) be an ideal topological space. If A and B are subsets of X such that A $\subseteq$ B $\subseteq$ A_g^* and A is an I_g^δ - closed set, then B is an I_g^δ - closed set.

Proof:

Let U be a δ - open set such that B $\subseteq$ U. Since A is I_g^δ - closed, $A_g^* \subseteq$ U. Also by theorem 3.7 [2] $B_g^* \subseteq (A_g^*)_g^* \subseteq A_g^* \subseteq$ U and hence B is an I_g^δ - closed set.

Theorem 3.19: If a subset A of (X, τ, I) is I_g^δ - closed, then A_g^* - A contains no nonempty δ- closed set.

Proof:

Let F be a δ - closed subset of A_g^* - A. Then A $\subseteq$ X - F, where X - F is δ - open. Since A is I_g^δ - closed, $A_g^* \subseteq$ X - F. Then F $\subseteq$ X - A_g^* which implies F $\subseteq A_g^* \cap (X - A_g^*) = \phi$.

Theorem 3.20: If a subset A of an ideal topological space (X, τ, I) is a χ_I -set and an I_g^δ - closed set, and also if A $\subset A_g^*$, then A is $*$- closed.

Proof:

Since A is a χ_I - set, A = U $\cap$ V, where U is δ- open and V is $*$- perfect. Now A $\subseteq$ U and A is I_g^δ - closed implies $A_g^* \subseteq$ U. By theorem 3.10 [2], $A^* = A_g^* \subseteq$ U. Also A $\subseteq$ V. By theorem 3.7[2], $A^* \subseteq V^*$. Since V is $*$- perfect, V^* = V. Then we have $A^* \subseteq$ V which implies $A^* \subseteq$ U $\cap$ V = A and so A is $*$- closed.

Theorem 3.21: Let (X, τ, I) be an ideal topological space and A $\subseteq$ X. If A is I_g^δ - closed then A $\cup$ (X - A_g^*) is I_g^δ - closed.

Proof:

Let U be a δ - open set such that A $\cup$ (X - A_g^*) $\subseteq$ U. Then X - U $\subseteq$ X - (A $\cup$ (X - A_g^*)) = A_g^* - A. Since A is I_g^δ - closed, by theorem 3.19, A_g^* - A = ϕ. Then X - U = ϕ and so X = U which implies A $\cup$ (X - A_g^*) $\subseteq$ X and so $(A \cup (X - A_g^*))_g^* \subseteq X_g^* \subseteq$ X = U. Consequently A $\cup$ (X - A_g^*) is an I_g^δ - closed set.

Theorem 3.22: Let (X, τ, I) be an ideal topological space. Then either {x} is δ- closed (or) $\{x\}^c$ is I_g^δ - closed.

Proof:

Suppose {x} is not δ- closed. Then $\{x\}^c$ is not δ- open. Then the only δ- open set containing $\{x\}^c$ is X and hence $(\{x\}^c)_g^* \subseteq$ X which implies $\{x\}^c$ is I_g^δ - closed.

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