Numerical solution of a fingering phenomenon in double phase flow in inclined porous medium

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Abstract: During secondary oil recovery process, with the injection of water into the petroleum reservoir, an important phenomenon called 'fingering' arises. This phenomenon is of a great practical importance and has been extensively studied by the researchers and scientists working in the petroleum production industry. The homogeneous porous medium is considered with a small inclination with the horizontal. The governing non-linear equation is based on the mass conservation principle, important Darcy's law, some specific standard relationships and considered basic assumptions. The numerical solution is obtained by using Crank-Nicolson finite difference scheme for the specified initial and boundary conditions. The obtained numerical results behave well with the physical phenomena and the basic properties of the parameters are preserved.

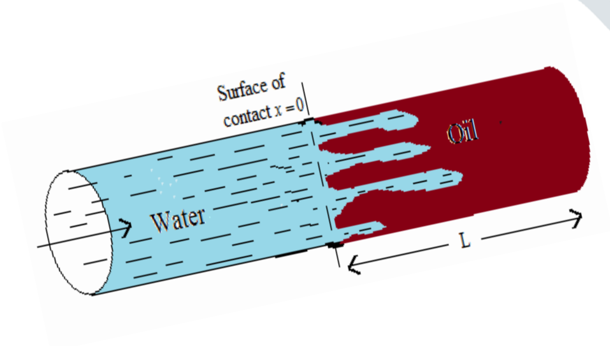
Index Terms – **Darcy's law, capillary pressure, homogeneous porous medium, Crank-Nicolson scheme.**

I. INTRODUCTION

The present study describes fingering phenomenon in double phase flow of two immiscible fluids (water and oil) through homogeneous inclined porous medium. In a typical reservoir history, oil is first produced with reservoir pressures acting as the direct driving agent (primary production). Later, as decreased pressure gradients become unable to displace remaining oil, a recovery agent, commonly water, is injected into some of the wells to displace the oil (secondary production) [3]. When water is injected in oil formatted homogeneous porous medium to recover the part of remaining oil, the important phenomenon viz. fingering occurs due to the force of injecting water, difference of viscosity and wettability of the injected water and native oil. When water is injected into the oil-filled region instead of regular displacement of common interface, the water will shoot through the porous medium at relatively high speed through inter connected capillaries [3]. Consequently, it appears in the irregular shape of trembling fingers in the oil field, so it is known as fingering phenomenon (Figure 4.1(a)). In the statistical treatment of fingers, only the average behavior of the two fluids involved is taken into consideration. The saturation of the displacing fluid in porous medium represents the average cross sectional area occupied by fingers. The displacement of native oil by another fluid of lesser viscosity-injected water provides well-developed fingers [3, 9]. An extensive study on this phenomenon with different view point is available in the standard literature.

II. MATHEMATICAL FORMULATION

The fingering phenomenon in homogeneous porous medium has been studied under the essential basic assumptions: Immiscible flow of two incompressible fluids, injected water and native oil, porous medium is isotropic, incompressible; finite in extent to a small inclination with the horizontal, bottom boundary is horizontal and impermeable, mass is conserved and gravity force is neglected. For mathematical study, we take vertical cross sectional area of the cylindrical piece of porous matrix, which is rectang le, and open end will be common interface $x=0$ from where water is injected. The length x of the fingers is to be measured in the direction of displacement.



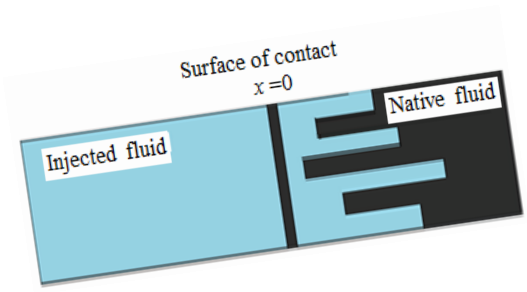


Figure: 4.1(a) Fingers in the cylindrical porous medium 4.1(b) Schematic representation of fingers

In order to observe the saturation of the water, Scheidegger and Johnson suggested replacing irregular fingers by the rectangular schematic fingers [7, 9] (Figure 4.1(b)). The seepage velocities of injected water V_w and native oil V_o are expressed by Darcy's law as

$$V_w = -\frac{k_w}{\delta_w} K\left(\frac{\partial P_w}{\partial x} + \rho_w g \sin\theta\right) \tag{4.1}$$

$$V_o = -\frac{k_o}{\delta_o} K\left(\frac{\partial P_o}{\partial x} + \rho_o g \sin\theta\right) \tag{4.2}$$

where, K is the permeability of the homogeneous porous medium. k_w and k_o are the relative permeabilities of injected water and native oil, which are the functions of the phase saturations of water and oil, S_w and S_o respectively. P_w and P_o are the pressures of water and oil respectively. δ_w and δ_o are the constant kinematic viscosities of injecting water and displacing oil respectively. ρ_w and ρ_o are the densities of the water and oil respectively. θ is the angle of inclination of the considered homogeneous porous medium [1, 2, 3]. The injected water and displaced oil also satisfy the equation of continuity for their constant phase densities as

$$\phi\frac{\partial S_w}{\partial t} + \frac{\partial V_w}{\partial x} = 0 \tag{4.3}$$

$$\phi\frac{\partial S_o}{\partial t} + \frac{\partial V_o}{\partial x} = 0 \tag{4.4}$$

where, ϕ is the porosity of the homogeneous porous medium [2, 6].
Substituting the value of the seepage velocities V_w and V_o from the equations (4.1) and (4.2) into the equations (4.3) and (4.4) respectively,

$$\phi\frac{\partial S_w}{\partial t} = \frac{\partial}{\partial x}\left(\frac{k_w}{\delta_w} K \frac{\partial P_w}{\partial x}\right) + \rho_w g \sin\theta \frac{\partial}{\partial x}\left(\frac{k_w}{\delta_w} K\right) \tag{4.5}$$

$$\phi\frac{\partial S_o}{\partial t} = \frac{\partial}{\partial x}\left(\frac{k_o}{\delta_o} K \frac{\partial P_o}{\partial x}\right) + \rho_o g \sin\theta \frac{\partial}{\partial x}\left(\frac{k_o}{\delta_o} K\right) \tag{4.6}$$

As an empirical fact; we accept that capillary pressure P_c is a unique function of wetting phase saturation [3,7].

$$P_c\left(S_w\right) = P_o - P_w \tag{4.7}$$

Eliminating $\partial P_w / \partial x$ from equations (4.5) and (4.7), we obtain

$$\phi\frac{\partial S_w}{\partial t} = \frac{\partial}{\partial x}\left(K\frac{k_w}{\delta_w}\left(\frac{\partial P_o}{\partial x} - \frac{\partial P_c}{\partial x}\right)\right) + \rho_w g \sin\theta \frac{\partial}{\partial x}\left(\frac{k_w}{\delta_w} K\right) \tag{4.8}$$

For more simplification adding (4.6) and (4.8) and using the relation $S_w + S_o = 1$, we have

$$\frac{\partial}{\partial x}\left(K\left(\frac{k_w}{\delta_w} + \frac{k_o}{\delta_o}\right)\frac{\partial P_o}{\partial x} - K\frac{k_w}{\delta_w}\left(\frac{\partial P_c}{\partial x}\right)\right) + \rho_w g \sin\theta \frac{\partial}{\partial x}\left(\frac{k_w}{\delta_w} K\right) + \rho_o g \sin\theta \frac{\partial}{\partial x}\left(\frac{k_o}{\delta_o} K\right) = 0 \tag{4.9}$$

Now, integrating the equation (4.9) with respect to x, we get

$$K\left(\frac{k_w}{\delta_w} + \frac{k_o}{\delta_o}\right)\frac{\partial P_o}{\partial x} - K\frac{k_w}{\delta_w}\frac{\partial P_c}{\partial x} + \rho_w g \sin\theta\left(\frac{k_w}{\delta_w} K\right) + \rho_o g \sin\theta\left(\frac{k_o}{\delta_o} K\right) = -A(t) \tag{4.10}$$

where, $A(t)$ is a constant of integration. Further simplification of the equation (4.10) yields

$$\frac{\partial P_o}{\partial x} = \frac{-A(t)}{K\left(\frac{k_w}{\delta_w} + \frac{k_o}{\delta_o}\right)} + \frac{\frac{\partial P_c}{\partial x}}{1 + \frac{k_o}{k_w}\frac{\delta_w}{\delta_o}} - g \sin\theta \frac{\left(\rho_w \frac{k_w}{\delta_w} + \rho_o \frac{k_o}{\delta_o}\right)}{\left(\frac{k_w}{\delta_w} + \frac{k_o}{\delta_o}\right)} \tag{4.11}$$

Using (4.11) in (4.8) and simplifying, we get

$$\phi\frac{\partial S_w}{\partial t}+\frac{\partial}{\partial x}\left[\frac{K\dfrac{k_o}{\delta_o}\dfrac{\partial P_c}{\partial x}}{1+\dfrac{k_o}{k_w}\dfrac{\delta_w}{\delta_o}}+\frac{A(t)}{1+\dfrac{k_o}{k_w}\dfrac{\delta_w}{\delta_o}}-g\sin\theta\cdot K\cdot\frac{k_w}{\delta_w}\left(\frac{k_w\rho_w}{\delta_w}+\frac{k_o\rho_o}{\delta_o}\right)\right] \tag{4.12}$$

$$=\rho_w g\sin\theta\frac{\partial}{\partial x}\left(K\frac{k_w}{\delta_w}\right)$$

For more simplification using an average pressure $\bar{P}$, which may be defined by, the individual phase pressures can then be expressed in terms of the mean pressure and capillary pressure by (using the relation (4.7))

$$\bar{P}=P_{avg}=\frac{P_o+P_w}{2}\ \text{Now, we have,}\ P_o=\frac{P_o+P_w}{2}+\frac{P_o-P_w}{2}=\bar{P}+\frac{1}{2}P_c\ ;\ \ P_w=\bar{P}-\frac{1}{2}P_c \tag{4.13}$$

$$\frac{\partial P_o}{\partial x}=\frac{1}{2}\frac{\partial P_c}{\partial x} \tag{4.14}$$

Following the Scheidegger and Johnson approximation $\frac{k_o\delta_w+k_w\delta_o}{\delta_o\delta_w}\approx\frac{k_w}{\delta_w}$, [9] and using the equations (4.10) and (4.14) in (4.12), we get,

$$\phi\frac{\partial S_w}{\partial t}+\frac{1}{2}\frac{\partial}{\partial x}\left(K\frac{k_w}{\delta_w}\frac{\partial P_c}{\partial S_w}\frac{\partial S_w}{\partial x}\right)=\rho_w g\sin\theta\frac{\partial}{\partial x}\left(K\cdot\frac{k_w}{\delta_w}\right) \tag{4.15}$$

Using the standard relationships $k_w=S_w$ and $P_c=-\beta S_w$ in equation (4.15), the governing equation for the fingering phenomenon in inclined homogeneous porous medium is given by

$$\phi\frac{\partial S_w}{\partial t}-\frac{\beta}{2}\frac{K}{\delta_w}\frac{\partial}{\partial x}\left(S_w\frac{\partial S_w}{\partial x}\right)=\rho_w g\sin\theta\frac{K}{\delta_w}\frac{\partial S_w}{\partial x} \tag{4.16}$$

Setting the dimensionless variables $X=\frac{x}{L}$ and $T=\frac{K\beta}{2\delta_w L^2\phi}t$ the equation (4.16) reduces to

$$\frac{\partial S_w}{\partial T}=\frac{\partial}{\partial X}\left(S_w\frac{\partial S_w}{\partial X}\right)+C\frac{\partial S_w}{\partial X} \tag{4.17}$$

where, $C=\frac{2L\rho_w g\sin\theta}{\beta}$

III. INITIAL AND BOUNDARY CONDITIONS

$$S_w(X,0)=0\ \ ;\ \ 0\le X\le 1 \tag{4.18}$$

$$S_w(0,T)=1=S_{w0}\ \ ;\ \ T>0 \tag{4.19}$$

$$S_w(1,T)=0=S_{w1}\ \ ;\ \ T>0 \tag{4.20}$$

It is assumed that initially the porous medium is fully saturated with oil. Since, $S_w+S_o=1$, we have the initial condition (4.18). It is assumed that the entire oil from the initial boundary of the formation, $x=0$, has been displaced for all time by the impact of the injected water.

IV. CRANK-NICOLSON SCHEMES

Equation (4.17) is numerically solved by using Crank-Nicolson finite difference technique for the initial condition (4.18) and boundary conditions (4.19) and (4.20). Applying forward projection for $S_{w_i,\,n+1/2}$ and introducing, the ratio $r=\Delta T/(\Delta X)^2$, the resulting finite difference scheme for the governing equation (4.17) is given [8, 6, 5]: For $2\le i\le(R-1)$,

$$\begin{aligned}&\left[S_{w_{i,n+1/2}}-\frac{1}{4}\left(S_{w_{i+1,n+1/2}}-S_{w_{i-1,n+1/2}}\right)-\frac{C}{4\Delta X}\right]S_{w_{i-1,n+1}}+\left[-2S_{w_{i,n+1/2}}-\frac{2}{r}\right]S_{w_{i,n+1}}\\&\qquad+\left[S_{w_{i,n+1/2}}+\frac{1}{4}\left(S_{w_{i+1,n+1/2}}-S_{w_{i-1,n+1/2}}\right)+\frac{C}{4\Delta X}\right]S_{w_{i+1,n+1}}\\&=-\left[S_{w_{i,n+1/2}}-\frac{1}{4}\left(S_{w_{i+1,n+1/2}}-S_{w_{i-1,n+1/2}}\right)-\frac{C}{4\Delta X}\right]S_{w_{i-1,n}}+\left[2S_{w_{i,n+1/2}}-\frac{2}{r}\right]S_{w_{i,n}}\\&\qquad-\left[S_{w_{i,n+1/2}}+\frac{1}{4}\left(S_{w_{i+1,n+1/2}}-S_{w_{i-1,n+1/2}}\right)+\frac{C}{4\Delta X}\right]S_{w_{i+1,n}}\end{aligned}\tag{4.21}$$

With

$$S_{w_{i,n+1/2}}=S_{w_{i,n}}+\frac{r}{2}\left[\begin{aligned}&S_{w_{i,n}}\left(S_{w_{i+1,n}}-2S_{w_{i,n}}+S_{w_{i-1,n}}\right)+\frac{1}{4}\left(S_{w_{i+1,n}}-S_{w_{i-1,n}}\right)^2\\&\qquad+\frac{C\Delta X}{2}\left(S_{w_{i+1,n}}-S_{w_{i-1,n}}\right)\end{aligned}\right]\tag{4.21.1}$$

For $i=1$:

$$\begin{aligned}&-\left[\frac{11}{4}S_{w_{1,n+1/2}}-\frac{1}{4}S_{w_{2,n+1/2}}+\frac{S_{w0}}{2}+\frac{2}{r}-\frac{C}{4\Delta X}\right]S_{w_{1,n+1}}\\&\qquad+\left[\frac{5}{4}S_{w_{1,n+1/2}}+\frac{1}{4}S_{w_{2,n+1/2}}-\frac{S_{w0}}{2}+\frac{C}{4\Delta X}\right]S_{w_{2,n+1}}\\&=\left[\frac{11}{4}S_{w_{1,n+1/2}}-\frac{1}{4}S_{w_{2,n+1/2}}+\frac{S_{w0}}{2}-\frac{2}{r}-\frac{C}{4\Delta X}\right]S_{w_{1,n}}\\&\qquad-\left[\frac{5}{4}S_{w_{1,n+1/2}}+\frac{1}{4}S_{w_{2,n+1/2}}-\frac{S_{w0}}{2}+\frac{C}{4\Delta X}\right]S_{w_{1,n}}\\&\qquad-4S_{w0}\left[\frac{3}{4}S_{w_{1,n+1/2}}-\frac{1}{4}S_{w_{2,n+1/2}}+\frac{S_{w0}}{2}-\frac{C}{4\Delta X}\right]\end{aligned}\tag{4.22}$$

With

$$S_{w_{1,n+1/2}}=S_{w_{1,n}}+\frac{r}{2}\left[\begin{aligned}&S_{w_{1,n}}\cdot\left(S_{w_{2,n}}-3S_{w_{1,n}}+2S_{w0}\right)+\frac{1}{4}\left(S_{w_{2,n}}+S_{w_{1,n}}-2S_{w0}\right)^2\\&\qquad+\frac{C\Delta X}{2}\left(S_{w_{2,n}}+S_{w_{1,n}}-2S_{w0}\right)\end{aligned}\right]\tag{4.22.1}$$

For $i=R$:

$$S_{w_{R,n+1/2}}=S_{w_{R,n}}+\frac{r}{2}\left[\begin{aligned}&S_{w_{R,n}}\cdot\left(S_{w_{R-1,n}}-3S_{w_{R,n}}+2S_{w1}\right)+\frac{1}{4}\left(2S_{w1}-S_{w_{R-1,n}}-S_{w_{R,n}}\right)^2\\&\qquad+\frac{C\Delta X}{2}\left(2S_{w1}-S_{w_{R-1,n}}-S_{w_{R,n}}\right)\end{aligned}\right]\tag{4.23.1}$$

Used in the following equation

$$\begin{aligned}&\left[\frac{5}{4}S_{w_{R,n+\frac{1}{2}}}+\frac{1}{4}S_{w_{R-1,n+\frac{1}{2}}}-\frac{S_{w1}}{2}-\frac{C}{4\Delta X}\right]S_{w_{R-1,n+1}}\\&\qquad-\left[\frac{11}{4}S_{w_{R,n+\frac{1}{2}}}-\frac{1}{4}S_{w_{R-1,n+\frac{1}{2}}}+\frac{2}{r}+\frac{S_{w1}}{2}+\frac{C}{4\Delta X}\right]S_{w_{R,n+1}}\\&=-\left[\frac{5}{4}S_{w_{R,n+\frac{1}{2}}}+\frac{1}{4}S_{w_{R-1,n+\frac{1}{2}}}-\frac{S_{w1}}{2}-\frac{C}{4\Delta X}\right]S_{w_{R-1,n}}\\&\qquad+\left[\frac{11}{4}S_{w_{R,n+\frac{1}{2}}}-\frac{1}{4}S_{w_{R-1,n+\frac{1}{2}}}-\frac{2}{r}+\frac{S_{w1}}{2}+\frac{C}{4\Delta X}\right]S_{w_{R,n}}\\&\qquad-4S_{w1}\left[\frac{3}{4}S_{w_{R,n+\frac{1}{2}}}-\frac{1}{4}S_{w_{R-1,n+\frac{1}{2}}}+\frac{S_{w1}}{2}+\frac{C}{4\Delta X}\right]\end{aligned} \tag{4.23}$$

V. NUMERICAL RESULTS AND DISCUSSION

Table 4.1 Saturation of water $S_w(X,T)$ for fingering phenomenon in inclined porous medium for $C=0.0001$

X	$T=0.001$	$T=0.01$	$T=0.05$	$T=0.10$	$T=0.20$	$T=0.25$	$T=0.30$	$T=0.3125$
0	1	1	1	1	1	1	1	1
0.1	0.002245	0.57684	0.82132	0.87514	0.91201	0.92125	0.92803	0.92946
0.2	0.000000	0.06078	0.57774	0.70917	0.79779	0.81970	0.83569	0.83905
0.3	0.000000	0.00000	0.31140	0.52748	0.67495	0.71116	0.73747	0.74300
0.4	0.000000	0.00000	0.06064	0.33359	0.54391	0.59583	0.63349	0.64138
0.5	0.000000	0.00000	0.00000	0.13856	0.40563	0.47418	0.52400	0.53444
0.6	0.000000	0.00000	0.00000	0.00292	0.26234	0.34721	0.40954	0.42264
0.7	0.000000	0.00000	0.00000	0.00000	0.12024	0.21717	0.29119	0.30691
0.8	0.000000	0.00000	0.00000	0.00000	0.00876	0.09063	0.17150	0.18933
0.9	0.000000	0.00000	0.00000	0.00000	0.00000	0.00244	0.05858	0.07591
1	0	0	0	0	0	0	0	0

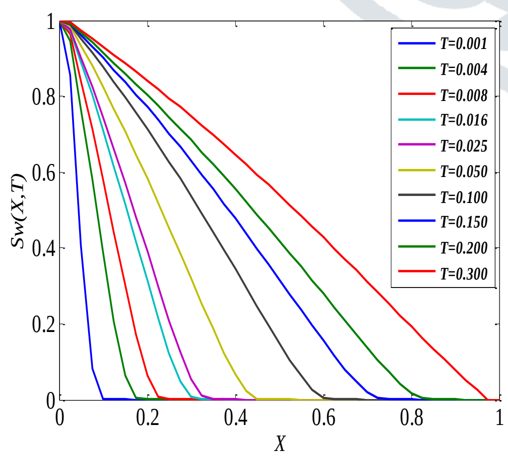


Figure 4.2 Water saturation profile for different times for $C=0.0001$

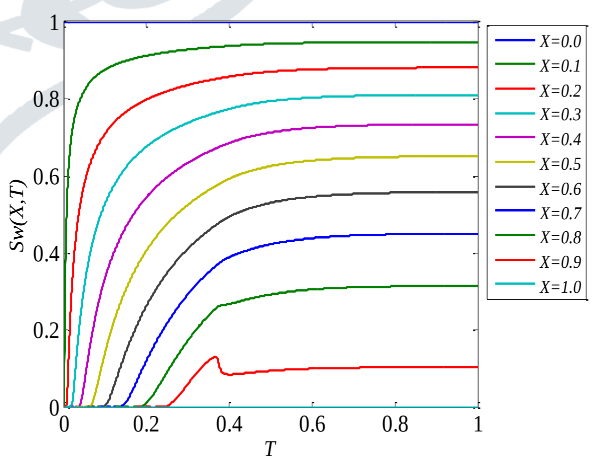


Figure 4.3 Water saturation profile for different distances for $C=0.0001$

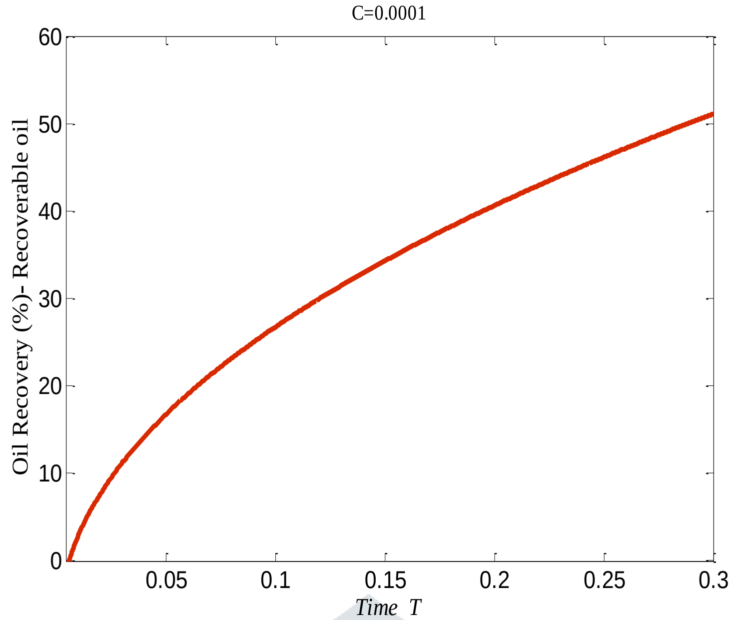


Figure 4.4 Oil recovery factor (%) in fingering phenomenon for C = 0.0001

The numerical solution of the governing nonlinear equation (4.17) representing fingering phenomenon in an inclined porous medium is obtained for the suitable initial and boundary conditions by developing Crank-Nicolson finite difference schemes. For the dimensionless number C = 0.0001(depends on the measure of angle of inclination) the obtained numerical results behave physically well for the various values of time and space. It is observed that the saturation of water decreases with the space variable and increases with time. Its graphical representation given in Figure 4.1, shows that the water saturation profiles are increasing with the time. For the saturation of water $S_w(X,T)$ the oil recovery factor (Figure 4.4) has been calculated which is close to 52 %, it evident that substantial amount of oil is displaced towards the production set up. It is found that for $T > 0.3125$ the numerical values of saturation of water are not physically consistent.

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