SECOND ORDER APPROXIMATION OF PERIODIC SOLUTIONS OF MOTION AROUND THE EQUILIBRIUM POINTS

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ABSTRACT

Studies on periodic orbits around these equilibrium points have aided meaningful developments in the fields of celestial mechanics and space explorations. In this paper, we analysed a second order approximation of periodic solutions of motion around the equilibrium points.

Keywords : restricted three-body problem, collinear equilibrium points, periodic orbits.

1. INTRODUCTION

Studies on periodic orbits around these equilibrium points have aided meaningful developments in the fields of celestial mechanics and space explorations. Radzievskii (1950) [1] investigated the restricted problem of three bodies by taking account of the light pressure. Simmon et.al. (1985) [2] formulated the photogravitational restricted three body problem by considering both primaries as sources of radiation. They studied the existence and linear stability of the equilibrium points. Jain et.al. (2006) [3] has been investigated the motion of the particle only in the case of the periodic solutions 'around' the out-of-plane equilibrium points. Although the interest was justified because of the unusual character of this case, it would be no less significant to examine how the categories of motion which are already known from the classical problem are modified by the existence of the radiation force. Recently, Singh et.al. (2016) [4] examined periodic motions around the collinear points of the restricted three-body problem. In this paper, we analysed a second order approximation of periodic solutions of motion around the equilibrium points.

In the usual barycentric, rotating and dimensionless coordinate system *OXYZ,* with the two main bodies having masses m_1 and m_2 and radiation pressure parameters q_1 and q_2 ($q_i \le 1$, $i = 1, 2$) respectively, the equations of motion of the third body m are

$$\begin{aligned} \ddot{X} - 2\dot{Y} &= X - \frac{Q_1}{Q_1^3}(X + m_2) - \frac{Q_2}{Q_2^3}(X - m_1) = f_1, \\ \ddot{Y} + 2\dot{X} &= Y\left[1 - \frac{Q_1}{Q_1^3} - \frac{Q_2}{Q_2^3}\right] = f_2, \\ \ddot{Z} &= Z\left[-\frac{Q_1}{Q_1^3} - \frac{Q_2}{Q_2^3}\right] = f_3, \end{aligned} \tag{1}$$

where $Q_1 = q_1 m_1$, $Q_2 = q_2 m_2$, $r_1 = \sqrt{\{X + m_2)^2 + Y^2 + Z^2}$, $r_2 = \sqrt{\{X - m_1)^2 + Y^2 + Z^2}$.

The Jacobian integral of motion also exists $-\frac{1}{2}(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) + \frac{1}{2}(X^2 + Y^2) + \frac{Q_1}{r_1} + \frac{Q_2}{r_2} = C.$

2. MOTION AROUND THE COLLINEAR EQUILIBRIUM POINTS

The positions of the points L_j, $j = 1, 2, 3$, are given by the solutions of the algebraic system

$f_1 = 0$, $Y = Z = 0$.

Equivalently, they are the roots of the equation

$$X - \frac{Q_1}{|X + m_2|^3}(X + \mu) - \frac{Q_2}{|X + m_1|^3}(X - m_1) = 0. \tag{2}$$

In the case $q_2 = 1$, the existence of three such equilibrium points is possible. [2] The first, named L_1, exists only for $q_1 > 0$ and belongs to the interval ($-m_2$, m_1). The second, named L_2, exists for any value of q_1 and belongs to the interval (m_1, $-m_2$). The third, named L_3, also exists for $q_1 > 0$ and belongs to the interval ($-m_1$, $-m_2$). The characteristic roots of these points are

$$\lambda_{12} = \pm\sqrt{-R}, \quad \lambda_{3456} = \pm\sqrt{\frac{1}{2}\left[R - 2 \pm \sqrt{R(9R - 8)}\right]}, \tag{3}$$

where $R = \dfrac{Q_1}{\left|X_{L_j} + m_2\right|^3} + \dfrac{Q_2}{\left|X_{L_j} - m_2\right|^3}$,

and X_{L_j} is the position of any of the L_j, $j = 1, 2, 3$, on the OX axis. In our case ($q_1 = 0.5$, $q_2 = 1$) these equilibrium points are unstable.

3. SECOND ORDER APPROXIMATION OF PERIODIC SOLUTIONS

Let L be any of the collinear equilibrium points L_j, j = 1, 2, 3. If a new coordinate system is defined with L as its origin and Lx, Ly, Lz its axes, parallel to OX, OY and OZ, respectively, the proper transformation between the two systems is given by the relations:

$$X = X_E + x, \quad Y = y, \quad Z = z. \tag{4}$$

To study planar motion we put $z(0)=\dot{z}(0)=0$ and we find that

$$z(t) = \dot{z}(t)=\ddot{z}(t)=0, \quad \text{for any } t \in R. \tag{5}$$

Then the first and second of Equations (1) are transformed through (4) in the Lxy coordinate system and the second members of the equations obtained are expanded into Taylor series up to second order terms giving

$$\ddot{x}-2\dot{y}=A_1x+A_2x^2+A_3y^2, \quad \ddot{y}+2\dot{x}=B_1y+B_2xy, \tag{6}$$

where

$$A_1=1+2R,$$

$$A_2=-3\left[\frac{Q_1}{\left|X_L+m_2\right|^5}\left(X_L+m_2\right)-\frac{Q_2}{\left|X_L+m_1\right|^5}\left(X_L+m_1\right)\right],$$

$$A_2=\frac{A_2}{2}, \qquad B_1=1-R, \qquad B_2=-A_2.$$

The conditions for convergence of the series mentioned above are:

$$\frac{\left|2\left(X_L+m_2\right)x+x^2+y^2\right|}{\left|X_L+m_2\right|^2}<1, \qquad \frac{\left|2\left(X_L+m_1\right)x+x^2+y^2\right|}{\left|X_L+m_1\right|^2}<1.$$

We search for periodic solutions in the form of second order expansions in powers of a parameter ε:

$$x(\tau)=X_1(\tau)\varepsilon+x_2(\tau)\varepsilon^2, \quad y(\tau)=y_1(\tau)\varepsilon+y_2(\tau)\varepsilon^2, \tag{7a}$$

while time is expanded by the expression

$$t=\left(1+t_1\varepsilon+t_2\varepsilon^2\right)\tau, \tag{7b}$$

in order to erase any secular term in the future analysis. We substitute the relations (7) into (6). Retaining terms of powers in ε not greater than two and denoting by primes (') the τ-derivatives, we have to solve the system

$$\begin{aligned}&[x_1''-2y_1'-A_1x_1]\varepsilon+[x_2''-2y_2'-A_1x_2]\varepsilon^2=g_1(\tau)\varepsilon^2,\\&[y_1''-2x_1'-B_1y_1]\varepsilon+[y_2''-2x_2'-B_1y_2]\varepsilon^2=g_2(\tau)\varepsilon^2,\end{aligned} \tag{8}$$

where $g_1(\tau)=A_2x_1^2+A_3y_1^2+2A_1t_1x_1+2t_1y_1'$, $g_2(\tau)=B_2x_1y_1+2B_1t_1y_1-2t_1x_1'$.

Since these equations hold for any τ, we can equate the coefficients of the same powers of ε. Defining the differential operator

$$F(D)=\begin{pmatrix}D^2-A_1 & -2D\\ 2D & D^2-B_1\end{pmatrix},$$

we have to solve successively:

The First Order System

$$F(D)=\begin{pmatrix}x_1\\y_1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}. \tag{9}$$

The general solution of (9) is:

$$x_1(\tau)=\sum_{i=3}^{6}a_i\exp(\lambda_i\tau), \quad y_1(\tau)=\sum_{i=3}^{6}b_i\exp(\lambda_i\tau), \tag{10}$$

where λ_i, i = 3, ..., 6 are characteristic roots of the System (9), given by Equation (3). Periodic orbits can be obtained if at least one pair of imaginary roots exists. By a suitable choice of the coefficients of the exponential terms of (10), we may have a special periodic solution which contains only the frequency corresponding to a specific imaginary part. We denote this frequency by ω. Equations (9) admit the periodic solution

$$x_1(\tau) = A\cos(\omega\tau) + B\sin(\omega\tau), \quad y_1(\tau) = A^*\cos(\omega\tau) + B^*\sin(\omega\tau),$$

where the coefficients A, B, A^*, B^* are connected by the relations

$$A^*=\frac{2\omega}{B_1+\omega^2}B, \quad B^*=\frac{2\omega}{B_1+\omega^2}A. \tag{11}$$

Without any loss of generality, we put $y_i(0)$ = 0. Then A^* = 0 and, consequently, B = 0. This means that $x_1'(0)=0$ too. Finally, the above solution becomes:

$$x_1(\tau) = A\cos(\omega\tau), \quad y_1(\tau) = B^*\sin(\omega\tau). \tag{12}$$

The Second Order System

$$F(D)=\begin{pmatrix}x_2\\y_2\end{pmatrix}=\begin{pmatrix}g_1(\tau)\\g_2(\tau)\end{pmatrix}. \tag{13}$$

Substituting (11) into Equations (12), functions g_i, i = 1, 2, become

$$g_1(\tau)=K_0+K_1\cos(\omega\tau)+K_2\cos(2\omega\tau), \quad g_2(\tau)=A_1\sin(\omega\tau)+A_2\sin(2\omega\tau), \tag{14}$$

where

$$K_0 = \frac{1}{2}\left[A_2 A^2 + A_3 B^{*2}\right], \quad K_1 = 2t_1\left[B^* \omega + A_1 A\right], \quad K_2 = \frac{1}{2}\left[A_2 A^2 - A_3 B^{*2}\right],$$

$$A_1 = 2t_1\left[A\omega + B_1 B^*\right], \quad A_2 = \frac{1}{2}\left[B_2 A B^*\right].$$

A periodic solution of System (12) is

$x_2(\tau) = M_0 + M_1 \cos(\omega\tau)$,

$y_2(\tau) = N_1 \sin(2\omega\tau)$,

where

$$M_0 = \frac{K_0}{A_1}, \quad M_1 = \frac{1}{\psi}\left[-4K_2\omega^2 + 4\mathbf{v}_2\,\omega + B_1 K_2\right],$$

$$N_1 = \frac{1}{\psi}\left[-4\mathbf{v}_2\,\omega^2 + 4K_2\omega + B_1\,\mathbf{v}_2\right],$$

$$\psi = \left|F(2\omega i)\right| = 16\omega^4 - 4(4 - A_1 - B_1)\omega^2 + A_1 B_1.$$

To avoid secular terms we have substituted

$t_1 = 0$. (15)

Finally, a second order approximation of periodic solutions around the collinear equilibrium points, as a function of parameter, ε, is obtained from Equations (11), (14) and (15):

$$x(t;\varepsilon) = [A\cos(\omega\tau)]\varepsilon + [M_0 + M_1\cos(2\omega\tau)]\varepsilon^2,$$
$$y(t;\varepsilon) = [B^*\sin(\omega\tau)]\varepsilon + [N_1\sin(2\omega\tau)]\varepsilon^2. \quad (16)$$

It is expected to be successful for values of ε that are small enough. For example, it has been tested by use of the Jacobian integral that for $\varepsilon \in [0, 0.05]$ the integral value conservation was 7 digits.

The period of this solution is $T = \frac{2\pi}{\omega}$ and it is symmetric w.r.t. the *OX* axis, since $(x, y)(-t) = (x, -y)(t)$.

4. CONCLUSION

We have shown a second order approximation of periodic solutions around the collinear equilibrium points, as a function of parameter, ε, is expected to be successful for values of ε that are small enough.

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