# Robust Finite-time stability analysis for neural networks with time-varying delays

[1] G. Mahendrakumar, [2] N. Shyamsundarraj, [3] R. Samidurai
[1]Research Scholar, [2] Research Scholar, [3]Assitant Professor
[1] Department of Mathematics, Thiruvalluvar University, Vellore-632 115, India

***Abstract:*** This paper is concerned with the problem of robust finite-time stability analysis for neural networks with time-varying delays. We construct a new Lyapunov-Krasovskii function with suitable activation function condition and then utilizing Jensen's inequality technique. A novel set of sufficient conditions are derived in terms of linear matrix inequalities (LMIs). Finally, a numerical example is given to demonstrate the usefulness and effectiveness of the proposed results.
**Keywords:** Robust Finite-time; Stability; Linear matrix inequality; Lyapunov-krasovskii function.

## I. INTRODUCTION

Neural networks are generally recognized as one of the simplified models of neural processing in the human brain, which can provide good performance and strong capability of information processing. In recent years, much attention has been put on neural networks dynamics due to their many successful applications in areas of pattern recognition, image and signal processing, associative memories, optimization problems and even mechanics of structures and materials [1, 2]. It should be noted that time-delay inevitably exists, due to the finite switching speed of involved electronics and the inherent communication time among neurons. Exactly, time delay is a main factor that can cause performance degradation and/or instability of neural networks. Therefore, stability analysis of neural networks with time delay has attracted considerable attention of many researchers in the last few decades and a number of excellent results have emerged.

The existing stability criteria can be grouped into delay-independent and delay-dependent ones. In general, delay-dependent criteria are less conservative than the delay independent ones. For delay-dependent stability criteria, the maximum delay bound is an important index for evaluating the conservatism of the criteria. In turn, rather significant research efforts [3]-[6] have been devoted to reducing the conservativeness of delay dependent stability criteria for neural networks with time delay. Thus, effort needs to study the stability of NNs with time delays. Consequently, several approaches based on various tools to evaluate the stability analysis of NNs with time delay components, and a lot of related results have been reported in the sense of Lyapunov stability method via LMIs approach (see [12]-[30]) and reference therein.

Sometimes people pay more attention to the performance requirements of the dynamic system in finite time. Since Dorato P. and Weiss L. had proposed the concept of finite-time stability for the first time [7, 8] finite-time boundedness, stability and stabilization have been widely studied [9]-[12] in linear or nonlinear systems. In [9] investigated the problem of finite time stability of continuous autonomous system. Input-output finite- time stability of linear systems have been studied in [10]. The problem finite-time boundedness and stabilization of uncertain switched neural networks with time-varying delay is discussed in [11]. In [12] authors investigated the problem of finite-time non-fragile passivity control for neural networks with time-varying delay. Which is the main motivation of this paper. Since it is not sensible to associate them together. This attention motivates us to make known to multiple time delays into the stability of the neural networks. Thus, the concept of two time-varying delays have been received an extensive attention in the field of stability analysis problems [14]-[18] many interesting results about stability and passivity of NNs have been obtained with additive time delays [19]-[23].

Inspired by the discussion above, in this paper, we investigated finite-time stability problem of neural networks with time-varying delays. A suitable Lyapunov-Krasovskii function (LKF) with triple and four integral terms are constructed and a tighter upper bound for the derivative of LKF is derived. By utilizing Jensen's lemma and Wirtinger-type inequality technique. All the obtained criteria are expressed in terms of LMIs that can be solved by using Matlab Toolbox. Finally, a numerical example is given to demonstrate that the proposed condition.

**Notations:** Throughout this paper, the following notions are used. Let $\mathbb{C}^n$,$\mathbb{R}^{m\times n}$ and $\mathbb{R}^{m\times n}$ denote, respectively, the n-dimensional Euclidean space and the set of all $m \times n$ real and complex matrices. The subscript $*$ and $T$ denote matrix complex conjugation and transposition and matrix transposition. The notation $X \geq Y$ (respectively, $X > Y$) means that $X - Y$ is positive semidefinite (respectively, positive definite). The shorthand notation $diag[M1, M2, \ldots, Mn]$ denotes a block diagonal matrix with diagonal blocks being the matrices $M1, M2, \ldots, Mn$. $I$ is the identity matrix with appropriate dimension.

## 2. Problem formulation and preliminaries

In this paper, we consider the following neural networks with discrete and distributed time-varying delays described by

$$\dot{y}(\mathrm{t}) = -Cy(t) + Af(y(t)) + Bf(y(t - \tau(t))) + D\int_{t-d(t)}^{t} f(y(s))ds$$
$$y(t) = \varphi(t), t \in [-\delta, 0],\ \delta = max[\tau, d] \tag{1}$$

where $y(t) \in R^n$ is the state vector, $f(y(t)) \in \mathbb{R}^n$denotes the activation function. $\mathbb{C} = diag\{c1, c2, \ldots, c3\} > 0$ is a diagonal matrix and $A, B, C$ are the connection weight matrix, discrete delayed connection weight matrix and distributed delayed connection weight matrix respectively.

($\boldsymbol{A_1}$)The time-varying delay $\tau(t)$ and $d(t)$ are satisfying the following condition:

$$0 \leq \tau(t) \leq \tau, \dot{\tau}(t) \leq \mu, 0 \leq d(t) \leq d. \tag{2}$$

where $\tau$, $d$ and $\mu$ are known constants. The initial function $\varphi(t)$ is continuous defined on [-$\delta$, 0]

($\boldsymbol{A_2}$) The each neuron activation function in the neural network (1) is assumed to satisfy

$$h_j^- \leq \frac{f_i(x) - f_i(y)}{x - y} \leq h_j^+, \forall x, y \in \mathbb{R}, x \neq y, i = 1,2, \ldots\mathrm{n} \tag{3}$$

Where $\boldsymbol{h_j^-}$ **and** $\boldsymbol{h_j^+}$ are some known constants.

**Definition 2.1** *(Finite-time stability)* For a given time constant Tc, neural network (1) is said to be finite-time stable with respect to $c_1, c_2, T_c, L$ if $sup_{-\delta\leq t_0 \leq 0}\{y^T(t_0)Ly^T(t_0), \dot{y}(t_0)L\dot{y}(t_0)\} \leq c_1 \Rightarrow y^T(t)Ly^T(t) \leq c_2$ for $t \in [0, T_c]$, where $0 < c_1 < c_2$, L is positive definite matrix.

**Lemma 2.2** *[13]* For a positive matrix M, scalars $h_U > h_L > 0$, such that the following integrations are well defined, it holds that

$$-(h_U - h_L)\int_{t-h_U}^{t-h_L} x^T(s)Mx(s)ds \leq -\left(\int_{t-h_U}^{t-h_L} x(s)\,ds\right)^T M\left(\int_{t-h_U}^{t-h_L} x(s)\,ds\right),$$

$$-\frac{h_U^2 - h_L^2}{2}\int_{t-h_U}^{t-h_L}\int_s^t x^T(u)Mx(u)duds \leq -\left(\int_{t-h_U}^{t-h_L}\int_s^t x(u)duds\right)^T M\left(\int_{t-h_U}^{t-h_L}\int_s^t x(u)duds\right),$$

$$-\frac{h_U^3 - h_L^3}{6}\int_{t-h_U}^{t-h_L}\int_s^t\int_u^t x^T(v)Mx(v)dvduds \leq -\left(\int_{t-h_U}^{t-h_L}\int_s^t\int_u^t x(v)dvduds\right)^T M\left(\int_{t-h_U}^{t-h_L}\int_s^t\int_u^t x(v)dvduds\right).$$

**Lemma 2.3** *[14]* For given symmetric positive definite matrices $R > 0$ and for any differentiable function $\omega(\cdot) \in [a, b] \to \mathbb{R}^n$ the following inequality holds:

$$\int_a^b \dot{\omega}^T(s)R\dot{\omega}(s)ds \geq \frac{1}{b-a}\begin{bmatrix}\omega(b)\\ \omega(a)\\ \upsilon\end{bmatrix}^T W_2(R)\begin{bmatrix}\omega(b)\\ \omega(a)\\ \upsilon\end{bmatrix}$$

Where

$$\upsilon = \frac{1}{b-a}\int_a^b \omega(s)ds,\ W_2(R) = \begin{bmatrix}R & -R & 0\\ * & R & 0\\ 0 & 0 & 0\end{bmatrix} + \frac{\pi^2}{4}\begin{bmatrix}R & R & -2R\\ * & R & -2R\\ 0 & 0 & 4R\end{bmatrix}.$$

**Lemma 2.4** *[15]* Let $H, E$ *and* $F(t)$ be real matrices of appropriate dimensions with $F(t)$ satisfying $F^T(t)F(t) \leq I$. Then, for any scalar $\epsilon > 0$.

$$HF(t)E + (HF(t)E)^T \leq \epsilon^{-1}HH^T + \epsilon E^T.$$

## 3. Main results

This section will focus on the problem of finite-time stability and finite-time robust stability criteria with discrete and distributed time-varying delays.

### 3.1. Finite-time stability criteria

**Theorem 3.1** *Assume that the conditions* **(A1)** *and* **(A2)** *hold. For given scalars* $\tau, \mu, d$ *and* $\delta$ *then system* **(1)** *is finite-time stable with respect to* $c_1, c_2, T_c, L$, *if there exists positive definite matrices* $P, Q_1, Q_2, R_1, R_2, S, T, U$ *and positive diagonal matrices* $U_1, U_2$ *such that the following LMIs hold:*

$$\Theta = \begin{bmatrix}
\Theta_{(1,1)} & 0 & \Theta_{(1,3)} & \Theta_{(1,4)} & \Theta_{(1,5)} & \Theta_{(1,6)} & \Theta_{(1,7)} & \Theta_{(1,8)} & \Theta_{(1,9)}\\
* & \Theta_{(2,2)} & 0 & 0 & \Theta_{(2,5)} & \Theta_{(2,6)} & 0 & 0 & 0\\
* & * & \Theta_{(3,3)} & 0 & 0 & 0 & \Theta_{(3,7)} & 0 & 0\\
* & * & * & \Theta_{(4,4)} & \Theta_{(4,5)} & \Theta_{(4,6)} & 0 & 0 & \Theta_{(4,9)}\\
* & * & * & * & \Theta_{(5,5)} & 0 & 0 & 0 & 0\\
* & * & * & * & * & \Theta_{(6,6)} & 0 & 0 & 0\\
* & * & * & * & * & * & \Theta_{(7,7)} & 0 & 0\\
* & * & * & * & * & * & 0 & \Theta_{(8,8)} & 0\\
* & * & * & * & * & * & 0 & 0 & \Theta_{(9,9)}
\end{bmatrix} < 0, \quad (4)$$

$$\lambda_1 I \leq \hat{P} \leq \lambda_2 I, 0 \leq \widehat{Q_1} \leq \lambda_3 I, 0 \leq \widehat{Q_2} \leq \lambda_4 I, 0 \leq \widehat{R_1} \leq \lambda_5 I, 0 \leq \widehat{R_2} \leq \lambda_6 I,$$
$$0 \leq \hat{S} \leq \lambda_7 I, 0 \leq \hat{T} \leq \lambda_8 I, 0 \leq \hat{U} \leq \lambda_9 I, \quad (5)$$

$$c_1\Lambda < c_2\lambda_1 e^{-\delta T_c}, \quad (6)$$

Where

$$\Theta_{(1,1)} = Q_1 + Q_2 + \tau R_2 - \frac{1}{\tau}\left(R_1 + \frac{\pi^2}{4}R_1\right) - S - S^T - \frac{3\tau}{2}T - F_1\Lambda_1 - G_1C - CG_1^T,$$

$$\Theta_{(1,3)} = -\frac{1}{\tau}\left(-R_1 + \frac{\pi^2}{4}R_1\right), \Theta_{(1,4)} = P - G_1 - AG_2^T, \Theta_{(1,5)} = F_2\Lambda_1 + F_2\Lambda_3 + G_1W_0,$$

$$\Theta_{(1,6)} = -F_2\Lambda_3 + G_1W_1, \Theta_{(1,7)} = \frac{\pi^2}{2\tau^2}R_1 + \frac{2}{\tau}S, \Theta_{(1,8)} = \frac{3}{\tau}T, \Theta_{(1,9)} = G_1B\Theta,$$

$$\Theta_{(2,2)} = -(1-\mu)Q_2 - F_1\Lambda_2 - F_1\Lambda_3, \Theta_{(2,5)} = -\Lambda_3 F_2^T, \Theta_{(2,6)} = F_2\Lambda_2 + F_2\Lambda_3,$$

$$\Theta_{(3,3)} = -Q_1 - \frac{1}{\tau}\left(R_1 + \frac{\pi^2}{4}R_1\right), \Theta_{(3,7)} = \frac{\pi^2}{2\tau^2}R_1, \Theta_{(4,4)} = \tau R_1 + \frac{\tau^2}{2}S + \frac{\tau^3}{6}T - G_2 - G_2^T,$$

$$\Theta_{(4,5)} = G_2W_0, \Theta_{(4,6)} = G_2W_1, \Theta_{(4,9)} = G_2B\Theta, \Theta_{(5,5)} = -\Lambda_1 - \Lambda_3, \Theta_{(5,6)} = \Lambda_3,$$

$$\Theta_{(6,6)} = -\Lambda_2 - \Lambda_3, \Theta_{(7,7)} = -\frac{\pi^2}{\tau^3}R_1 - \frac{1}{\tau}R_2 - \frac{2}{\tau^2}S, \Theta_{(8,8)} = -\frac{6}{\tau^3}T, \Theta_{(9,9)} = -\delta I,$$
$$\lambda_1 = \lambda_{min}(\hat{P}), \lambda_2 = \lambda_{max}(\hat{P}), \lambda_3 = \lambda_{max}(\widehat{Q_1}), \lambda_4 = \lambda_{max}(\widehat{Q_2}), \lambda_5 = \lambda_{max}(\widehat{R_1}),$$
$$\lambda_6 = \lambda_{max}(\widehat{R_2}), \lambda_7 = \lambda_{max}(\hat{S}), \lambda_8 = \lambda_{max}(\hat{T}), \lambda_9 = \lambda_{max}(\hat{U}).$$

**Proof.** We choose the following Lyapunov-Krasovskii function:

$$V(y(t)) = \sum_{1}^{6} V_i(y(t)) \tag{7}$$

Where

$$V_1(y(t)) = y^T P y(t),$$
$$V_2(y(t)) = \int_{t-\tau}^{t} y^T(s)Q_1 y(s)ds + \int_{t-\tau(t)}^{t} y^T(s)Q_2 y(s)ds,$$
$$V_3(y(t)) = \int_{-\tau}^{0}\int_{t+\beta}^{t} \dot{y}^T(s)R_1\dot{y}(s)duds + \int_{-\tau}^{0}\int_{t+\beta}^{t} y^T(s)R_2 y(s)duds,$$
$$V_4(y(t)) = \int_{-\tau}^{0}\int_{\theta}^{0}\int_{t+\beta}^{t} \dot{y}^T(s)S\dot{y}(s)dsd\beta d\theta,$$
$$V_5(y(t)) = \int_{-\tau}^{0}\int_{\lambda}^{0}\int_{\theta}^{0}\int_{t+\beta}^{t} \dot{y}^T(s)T\dot{y}(s)dsd\beta d\theta d\lambda,$$
$$V_6(y(t)) = \int_{-d(t)}^{0}\int_{t+\beta}^{t} f^T(y(s))Uf(y(s))duds.$$

Taking the time-derivative of $V(y(t))$ along any trajectory of system (1), it yields that

$$\dot{V}(y(t)) = \sum_{i=1}^{6} \dot{V}_i(y(t)), \tag{8}$$

Where

$$\dot{V}_1(y(t)) = 2y^T(t)P\dot{y}(t), \tag{9}$$
$$\dot{V}_2(y(t)) \le y^T(t)Q_1 y(t) - y^T(t-\tau)Q_1 y(t-\tau) + y^T(t)Q_2 y(t)$$
$$-(1-\mu)y^T(t-\tau(t))Q_1 y(t-\tau(t)), \tag{10}$$
$$\dot{V}_3(y(t)) = \tau\dot{y}^T(t)R_1\dot{y}(t) + \tau y^T(t)R_2 y(t) - \int_{t-\tau}^{t} \dot{y}^T(s)R_1\dot{y}(s)\,ds - \int_{t-\tau}^{t} y^T(s)R_2 y(s)\,ds, \tag{11}$$
$$\dot{V}_4(y(t)) = \frac{\tau^2}{2}\dot{y}^T(t)S\dot{y}(t) - \int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{y}^T(s)S\dot{y}(s)\,dsd\theta, \tag{12}$$
$$\dot{V}_5(y(t)) = \frac{\tau^2}{6}\dot{y}^T(t)T\dot{y}(t) - \int_{-\tau}^{0}\int_{\lambda}^{0}\int_{t+\theta}^{t} \dot{y}^T(s)T\dot{y}(s)\,dsd\theta\,d\lambda, \tag{13}$$
$$\dot{V}_6(y(t)) = df^T(y(t))Uf(y(t)) - \int_{t-d(t)}^{t} f^T(y(s))Uf(y(s)). \tag{14}$$

By applying Lemma 2.2, the first integral in (11) we can obtain

$$-\int_{t-\tau}^{t} \dot{y}^T(s)R_1\dot{y}(s)\,ds \le -\frac{1}{\tau}\begin{bmatrix} y(t) \\ y(t-\tau) \\ v_1 \end{bmatrix}^T W_1(R_1)\begin{bmatrix} y(t) \\ y(t-\tau) \\ v_1 \end{bmatrix} \quad (15)$$

Where

$$v_1 = \frac{1}{\tau}\int_{t-\tau}^{t} y(s)\,ds, W_2(R_1) = W_0(R_1) + \frac{\pi^2}{4}\begin{bmatrix} R_1 & R_1 & -2R_1 \\ * & R_1 & -2R_1 \\ * & * & 4R_1 \end{bmatrix}, W_0(R_1) = \begin{bmatrix} R_1 & -R_1 & 0 \\ * & R_1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$-\int_{t-\tau}^{t} \dot{y}^T(s)R_1\dot{y}(s)\,ds$$

$$\le \frac{1}{\tau}\Bigg[y^T(t)\left(R_1 + \frac{\pi^2}{4}R_1\right)y(t) + 2y^T(t)\left(-R_1 + \frac{\pi^2}{4}R_1\right)y(t-\tau)$$

$$+ y^T(t-\tau)\left(R_1 + \frac{\pi^2}{4}R_1\right)y(t-\tau) + 2y^T(t)\left(-\frac{\pi^2}{2\tau}R_1\right)\int_{t-\tau}^{t} y(s)\,ds$$

$$+ 2y^T(t-\tau)\left(-\frac{\pi^2}{2\tau}R_1\right)\int_{t-\tau}^{t} y(s)\,ds + \int_{t-\tau}^{t} y^T(s)\,ds\left(\frac{\pi^2}{\tau^2}R_1\right)\int_{t-\tau}^{t} y(s)\,ds\Bigg]. \quad (16)$$

Applying Lemma 2.1, integral term in equation (11)-(14), as follows

$$-\int_{t-\tau}^{t} y^T(s)R_2 y(s)\,ds = -\frac{1}{\tau}\left(\int_{t-\tau}^{t} y^T(s)\,ds\right)R_2\left(\int_{t-\tau}^{t} y(s)\,ds\right) \quad (17)$$

$$-\int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{y}^T(s)S\dot{y}(s)\,dsd\theta \le -\frac{2}{\tau^2}\left(\int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{y}^T(s)\,dsd\theta\right)S\left(\int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{y}(s)\,dsd\theta\right)$$

$$= -\frac{2}{\tau^2}\left[\tau y^T(t) - \int_{t-\tau}^{t} y^T(s)\,ds\right]S\left[\tau y(t) - \int_{t-\tau}^{t} y(s)\,ds\right] \quad (18)$$

$$-\int_{-\tau}^{0}\int_{\lambda}^{0}\int_{t+\theta}^{t} \dot{y}^T(s)T\dot{y}(s)\,dsd\theta\,d\lambda \le -\frac{6}{\tau^3}\left(\int_{-\tau}^{0}\int_{\lambda}^{0}\int_{t+\theta}^{t} \dot{y}^T(s)\,dsd\theta\,d\lambda\right)T\left(\int_{-\tau}^{0}\int_{\lambda}^{0}\int_{t+\theta}^{t} \dot{y}(s)\,dsd\theta\,d\lambda\right)$$

$$= -\frac{6}{\tau^3}\left[\frac{\tau^2}{2}y^T(t) - \int_{-\tau}^{0}\int_{t+\lambda}^{t} y^T(s)\,dsd\lambda\right]T\left[\frac{\tau^2}{2}y(t) - \int_{-\tau}^{0}\int_{t+\lambda}^{t} y(s)\,dsd\lambda\right] \quad (19)$$

$$-\int_{t-d(t)}^{t} f^T(y(s))Uf(y(s)) \le -\frac{1}{d}\left(\int_{t-d(t)}^{t} f^T(y(s))\,ds\right)U\left(\int_{t-d(t)}^{t} f(y(s))\,ds\right) \quad (20)$$

From Assumption **(*A*2)**, we have,

$$[f_p(y(t)) - h_p^-(y(t))][h_p^+(y(t)) - f_p(y(t))] \ge 0, p = 1,2, \dots, n$$

which is equivalent to,

$$\begin{bmatrix} y^T(t) \\ f^T(y(t)) \end{bmatrix}\begin{bmatrix} -h_p^- h_p^+ e_p e_p^T & \frac{h_p^+ + h_p^-}{2} e_p e_p^T \\ * & -e_p e_p^T \end{bmatrix}\begin{bmatrix} y(t) \\ f(y(t)) \end{bmatrix} \ge 0$$

with $e_p$ denotes the unit column vector with the element 1 on its $p^{th}$ row and zero elsewhere. Then, for any positive diagonal matrix $U_1$, it follows that

$$\begin{bmatrix} y^T(t) \\ f^T(y(t)) \end{bmatrix} \begin{bmatrix} -H_1U_1 & H_2U_1 \\ * & -U_1 \end{bmatrix} \begin{bmatrix} y(t) \\ f(y(t)) \end{bmatrix} \geq 0 \tag{21}$$

By similar analysis, we can also get that the following inequality holds for any positive diagonal matrix $U_2$

$$\begin{bmatrix} y^T(t-\tau(t)) \\ f^T(y(t-\tau(t))) \end{bmatrix} \begin{bmatrix} -H_1U_2 & H_2U_2 \\ * & -U_2 \end{bmatrix} \begin{bmatrix} y(t-\tau(t)) \\ f(y(t-\tau(t))) \end{bmatrix} \geq 0 \tag{22}$$

On the other hand, for any matrices $G_1$and $G_2$ with appropriate dimensions, it is true that,

$$0 = 2[y^T(t)\, G_1 + \dot{y}^T(t)\, G_2]\left[-\dot{y}(t) - Cy(t) + Af(y(t)) + Bf\left(y(t-\tau(t))\right) + D\int_{t-d(t)}^{t} f(y(s))\, ds\right] \tag{23}$$

From (9)-(23) it can be deduced that

$$\dot{V}(y(t)) - \delta V(y(t)) \leq \Omega^T(t)\Theta\Omega(t)$$

Where

$$\Omega(t) = \left[y^T(t)\ \ y^T(t-\tau(t))\ y^T(t)\ \ \dot{y}(t)\ f^T(y(t))\ f^T(y(t-\tau(t)))\ \int_{t-\tau}^{t} y^T(s)\, ds\ \ \int_{t-\tau}^{t} f^T(y(s))ds\ \int_{-\tau}^{0}\int_{t+\lambda}^{t} y^T(s)\, ds d\lambda\right]^T$$

and $\Theta$ is shown in (4).

$$\dot{V}(y(t)) < \delta V(y(t)), \tag{25}$$

Multiplying (25) by $\mathrm{e}^{-\delta t}$, we can obtain

$$\frac{d}{dt}(\mathrm{e}^{-\delta t}V) < 0, \tag{26}$$

Integrating the aforementioned inequality between 0 to $t$, with $t \in [0, T_c]$ it follows that:

$$\mathrm{e}^{-\delta t}V(y(t)) < V(y(0)), \tag{27}$$

Then

$$V(y(t)) < \mathrm{e}^{\delta t}V(y(0)). \tag{28}$$

Letting $\hat{P} = L^{-\frac{1}{2}}PL^{-\frac{1}{2}}$, $\widehat{Q_1} = L^{-\frac{1}{2}}Q_1L^{-\frac{1}{2}}$, $\widehat{Q_2} = L^{-\frac{1}{2}}Q_2L^{-\frac{1}{2}}$, $\widehat{R_1} = L^{-\frac{1}{2}}R_1L^{-\frac{1}{2}}$, $\widehat{R_2} = L^{-\frac{1}{2}}R_2L^{-\frac{1}{2}}$, $\hat{S} = L^{-\frac{1}{2}}SL^{-\frac{1}{2}}$, $\hat{T} = L^{-\frac{1}{2}}TL^{-\frac{1}{2}}, \hat{U} = L^{-\frac{1}{2}}UL^{-\frac{1}{2}}$,

We obtain

$$V\big(y(0)\big) = e^{\delta t}\Bigg[y^T(0)L^{-\frac{1}{2}}\hat{P}L^{-\frac{1}{2}}y(0) + \int_{-\tau}^{0} y^T(s)\,L^{-\frac{1}{2}}\widehat{Q_1}L^{-\frac{1}{2}}y(s)ds + \int_{-\tau(t)}^{0} y^T(s)\,L^{-\frac{1}{2}}\widehat{Q_2}L^{-\frac{1}{2}}y(s)ds$$
$$+ \int_{-\tau}^{0}\int_{\beta}^{0} \dot{y}^T(s)\,L^{-\frac{1}{2}}\widehat{R_1}L^{-\frac{1}{2}}\dot{y}(s)duds + \int_{-\tau}^{0}\int_{\beta}^{0} y^T(s)\,L^{-\frac{1}{2}}\widehat{R_1}L^{-\frac{1}{2}}y(s)duds$$
$$+ \int_{-\tau}^{0}\int_{\theta}^{0}\int_{\beta}^{0} \dot{y}^T(s)\,L^{-\frac{1}{2}}\hat{S}L^{-\frac{1}{2}}\dot{y}(s)ds\,d\beta d\theta + \int_{-\tau}^{0}\int_{\lambda}^{0}\int_{\theta}^{0}\int_{\beta}^{0} \dot{y}^T(s)\,L^{-\frac{1}{2}}\hat{T}L^{-\frac{1}{2}}\dot{y}(s)ds\,d\beta d\theta\,d\lambda$$
$$+ \int_{-d}^{0}\int_{\beta}^{0} f^T\big(y(s)\big)\,L^{-\frac{1}{2}}\hat{U}L^{-\frac{1}{2}}f\big(y(s)\big)duds\Bigg],$$
$$\leq e^{\delta t}\Big\{\lambda_{max}\big(\hat{P}\big)y^T(0)Ly(0) + \tau\lambda_{max}\big(\widehat{Q_1}\big) + \tau\lambda_{max}\big(\widehat{Q_2}\big) + \frac{\tau^2}{2}\lambda_{max}\big(\widehat{R_1}\big) + \frac{\tau^2}{2}\lambda_{max}\big(\widehat{R_2}\big) + \frac{\tau^3}{6}\lambda_{max}\big(\hat{S}\big)$$
$$+ \frac{\tau^4}{24}\lambda_{max}\big(\hat{T}\big) + \frac{d^2}{2}\lambda_{max}\big(\hat{U}\big)\Big\} sup_{-\tau\leq\theta\leq 0}\{y^T(\theta)Ly(\theta), \dot{y}^T(t)L\dot{y}(t)\},$$

$$V\big(y(0)\big) \leq c_1\Lambda. \tag{29}$$

Where $\Lambda = \lambda_2 + \tau[\lambda_3 + \lambda_4] + \frac{\tau^2}{2}[\lambda_5 + \lambda_6] + \frac{\tau^3}{6}\lambda_7 + \frac{\tau^4}{24}\lambda_8 + \frac{d^2}{2}\lambda_9$.

On the other hand, if follows from (7) that:

$$V\big(y(t)\big) \geq y^T(t)Py(t) \geq \lambda_{min}\big(\hat{P}\big)y^T(t)Ly(t) = \lambda_1 y^T(t)Ly(t), \tag{30}$$

Combining the inequalities (28)-(30), we get

$$c_2 \leq \frac{c_1\Lambda}{\lambda_1}e^{\delta T_c}. \tag{31}$$

Then from condition (6), we arrive at $y^T(t)Ly(t) < c_2$. From definition 2.1, the system (1), is finite-time stable with respect to $c_1, c_2, T_c, L$. This completes the proof.

## 3.2. Finite-time robust stability criteria:

In this subsection, based on Theorem 3.1, we are now ready to develop stability criterion for the neural networks with time-varying parameters uncertainties. Now, we consider the following uncertain neural networks as:

$$\dot{y}^T(t) = \big(C + \Delta C(t)\big)y(t) + \big(A + \Delta A(t)\big)f\big(y(t)\big) + \big(B + \Delta B(t)\big)f\left(y\big(t - \tau(t)\big)\right)$$
$$+ \big(D + \Delta D(t)\big)\int_{t-d(t)}^{t} f(y(s))ds, \tag{32}$$

Where $\Delta C(t), \Delta A(t), \Delta B(t)$ and $\Delta D(t)$ are the time-varying parameters uncertainties. Which are assumed to be of the form $\Delta C(t), \Delta A(t), \Delta B(t), \Delta D(t) = HF(t)[E_1, E_2, E_3, E_4]$.

**Theorem 3.2** Assume that the conditions **(A1)** and **(A2)** hold. For given scalars $\tau, \mu, d$ and $\delta$ then system (32) is finite-time stable with respect to $c_1, c_2, T_c, L$, if there exists positive definite matrices $P, Q_1, Q_2, R_1, R_2, S, T, U$ and positive diagonal matrices $U_1, U_2$ such that LMIs (4), (5) and (6).

$$\begin{bmatrix} \Theta + \epsilon\xi_2^T\xi_2 & \xi_1^T \\ \xi_1 & -\epsilon I \end{bmatrix} < 0, \qquad (33)$$

Where

$$\xi_1 = [[G_1\,H\;0\;0\;0\;0\;0\;0\;0\;G_2H\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;]]^T,$$
$$\xi_2 = [-E_1\;\;0\;0\;0\;0\;0\;0\;0\;E_2\;0\;0\;E_3\;0\;0\;0\;0\;0\;0\;0\;0\;0\;E_4\;0\;0\;0\;]^T.$$

and $\Theta$ is defined in Theorem 3.1

**Proof:** Replacing $C, A, B, D$ in LMI (4) with $C + HF(t)E_1,\; A + HF(t)E_2,\; B + HF(t)E_3,\; D + HF(t)E_4$ yields,

$$\Theta + \xi_1^T F(t)\xi_2 + \xi_2^T F(t)\xi_1 < 0, \qquad (34)$$

Applying Lemma 2.3, it can be deduced that for $\epsilon > 0$,

$$\Theta + \epsilon^{-1}\xi_1^T\xi_1 + \epsilon\xi_2^T\xi_2 < 0. \qquad (35)$$

which is equivalent to (33) in the sense of the Schur complement [16]. The proof is completed.

## 4. Numerical Examples.

In this section, we will give a example showing the effectiveness of the results given here.

**Example** Consider system (1) with the following parameters:

$$C = \begin{bmatrix} 1.8 & 0 \\ 0 & 1.3 \end{bmatrix}, \quad A = \begin{bmatrix} 0.9 & -0.1 \\ 0.2 & 0.4 \end{bmatrix}, \quad B = \begin{bmatrix} 0.6 & 0.4 \\ -0.3 & -0.2 \end{bmatrix},, \quad D = \begin{bmatrix} 0.15 & 0.3 \\ 0.3 & 0.1 \end{bmatrix}.$$

The activation function is described by $h(x) = [\tanh(1.3y_1)\;\tanh(1.2y_2)]^T$, then it is easy to see that, $H_1 = diag\{0,0\}$ and $H_2 = diag\{0.70,0.45\}$. Let the time-varying delay satisfying $\tau(t) = 2 + 0.3\sin(t), d(t) = 3 + 0.1\sin(t)$ and for given scalars $\mu = 0.6,\; c_1 = 0.3,\; T_c = 35,\; \delta = 0.6$, and matrix $L = I$, by by solving the LMIs (4)-(6) in Theorem 3.1, using Matlab LMI toolbox, we can obtain the feasible solutions for optimal minimum value of $c_2 = 4.661$, some of the obtained decision variable are

$$P = \begin{bmatrix} 1.7730 & 0.1333 \\ 0.1333 & 1.5546 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 2.7989 & 0.1375 \\ 0.1375 & 2.0325 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 1.2570 & 0.7673 \\ 0.7673 & 1.1177 \end{bmatrix},$$

$$R_1 = \begin{bmatrix} 0.6173 & 0.1141 \\ 0.1141 & 0.9452 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 4.9929 & 1.1368 \\ 1.1368 & 4.4056 \end{bmatrix}, \quad S = \begin{bmatrix} 6.9524 & 0.9036 \\ 0.9036 & 5.9067 \end{bmatrix},$$

$$T = \begin{bmatrix} 4.4056 & -0.6468 \\ -0.6468 & 2.8762 \end{bmatrix}, \quad U_1 = \begin{bmatrix} 4.5428 & 0 \\ 0 & 3.1345 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 3.3728 & 0 \\ 0 & 3.0749 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} 1.6686 & 0.2584 \\ 0.2584 & 1.9764 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0.1537 & 0.9199 \\ 0.9199 & 0.9191 \end{bmatrix}.$$

The above results shows that all the conditions stated in Theorem 3.1, have been satisfied. Hence, it can be concluded that the considered system is finite-time stable with respect to $(c_1, c_2, T_c, L)$.

## 5. Conclusion.

This paper has investigated the problem of robust finite-time stability analysis for neural networks with time-varying delays. Based on a suitable Lyapunov-Krasovskii function with triple and four integral terms have been constructed. A novel set of sufficient conditions are derived in terms of linear matrix inequalities. Finally, a numerical example is given to demonstrate the usefulness and effectiveness of the proposed results.

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