ASSOCIATOR IN THE NUCLEUS OF NONASSOCIATIVE RINGS

[1] G KAVITHA, [2] T.RAMESH

[1] Mentor, [2]Lecturer

Department of mathematics

Rgukt rk valley, kadapa,India-516330

Abstract: In this paper we prove that If R is a semiprime non associative ring of char. $\neq 2$ with associators in the nucleus is associative.

Index terms: Nonassociative ring, semiprime ring, characteristic, associator and nucleus

1. Introduction:

The study of nonassociative rings has yields many interesting results in algebra. The results on nonassociative rings in which associators in the nucleus have been scattered throughout the literature. Many mathematicians of recent years studied a class of rings in which the associators have certain special properities. Their investigations throw light on the study of general nonassociative rings.

2. Preliminaries:

A nonassociative ring R is an additive abelian group in which a multiplication is defined, which is distributive over addition on left as well as on the right, that is $(x+y)z = xz + yz,\ z(x+y) = zx + zy$ for all x, y, z in R. The associator (x, y, z) defined by $(x, y, z) = (xy)z - x(yz)$ for all x, y, z in a ring. The nucleus N of a ring R is defined as $N = \{n \in R / (n, R, R) = (R, n, R) = (R, R, n) = 0.$ A ring R is said to be of char. $\neq$ n, if $nx = 0$ implies $x = 0$ for all $x \in R$ and n is a natural number. A ring R is semiprime ring if for any ideal A of R, A^2 =0 implies $A = 0$. These rings are also referred as rings free from trivial ideals.

3. Main Results

Thedy has studied rings in which it is assumed that associators commute with all elements "on rings satisfying $((a,b,c),d) = 0$". In this spirit we consider a class of rings in which all associators are in the nucleus. In this section we prove that if R is a semiprime ring of char. $\neq 2$ with associators in its nucleus, then R is associative. We know that the nucleus N of a ring R consists of all elements n in R such that $(n, r, s) = 0 = (r, s, n) = (n, s, r),$ for all r, s in R … (1)

A ring is semiprime if it has no ideals $A \neq 0$ such that $A^2 = 0.$

Throughout this section R represents a nonassociative ring with all associators in its nucleus. Now we prove the following theorem.

Theorem 3.1. If R is a semi prime ring of char. $\neq 2$ with associators in its nucleus, then R is associative.

Proof: We use an identity called as Teichmuller identity

$$(wx, y, z) - (w, xy, z) + (w, x, yz) = w(x, y,) + (w, x, y)z \quad \dots (2)$$

To verify this identity, we expand the left hand side using the definition of the associator. We now take terms, letting one of the four elements in (2) be in the nucleus. Thus if $w = n$ in (2), we obtain

$$(nx, y, z) - (n, xy, z) + (n, x, yz) = n(x, y, z) + (n, x, y)z$$

Now from (1), we get $(nx, y, z) = n(x, y, z)$ … (3)

Similarly $x = n$ in (2), we obtain

$$(wn, y, z) - (w, ny, z) + (w, n, yz) = w(n, y, z) + (w, n, y)z.$$

So, $(wn, y, z) - (w, ny, z) = 0$

That is $(wn, y, z) = (w, ny, z)$ … (4)

Let $y = n$ in (2). Then

$$(wx, n, z) - (w, xn, z) + (w, x, nz) = w(x, n, z) + (w, x, n)z$$
$$(w, x, nz) - (w, xn, z) = 0$$

Implies $(w, xn, z) = (w, x, nz)$ … (5)

Let $z = n$ in (2). Then

$$(wx, y, n) - (w, xy, n) + (w, x, yn) = w(x, y, n) + (w, x, y)n$$

$$(w, x, yn) - (w, x, y)n = 0$$

$$(w, x, yn) = (w, x, y)n \quad \text{… (6)}$$

For n an element in N and the rest of the elements arbitrary in R. If x is in N in (3), y is in N in (4), z in N in (5), Then N is a subring of R.

All rings R have an ideal A, called the associator ideal. It is defined as the smallest ideal which contains all associators. It actually consists of all finite sums of associators and right multiples of associators. To see that the elements of this form constitutes an ideal we observe that

$$(w, x, y)z.s = ((w, x, y), z, s) + (w, x, y).zs,$$

is again of this form, while

$$z(w, x, y) = (zw, x,) - (, wx, y) + (z, w, xy) - (z, w, x)y,$$

Using (2), is also of the desired form. We have

$$s.(w, x, y)z = -(s, (w, x, y), z) + s(w, x, y).z$$

Using the previous two identities, we see that this is again of the desired form.

The associator ideal is never zero, except when R is associative.

Let $q = (a, b, c)(x, y, z),$ where a, b, c, x, y, z are arbitrary elements of R.

Using (a, b, c) as the nuclear element n in (3) we see that

$$((a, b, c)x, y, z) = (a, b, c)(x, y, z)$$

$$\Rightarrow q = ((a, b, c)x, y, z).$$

At this point (2) implies that

$$(ab, c, x) - (a, bc, x) + (a, b, cx) = a(b, c, x) + (a, b, c)x$$

Thus $(a, b, c)x = -a(b, c, x) + (ab, c, x) - (a, bc, x) + (a, b, cx)$

Inserting this equation on the left hand side of the associator with y and z, as well as using the hypothesis that associators are in the nucleus, we see that

$((a,b,c)x,y,z) = -((a(b,c,x),y,z) + ((ab,c,x),y,z) - ((a,bc,x),y,z) + ((a,b,cx),y,z)$

Then $((a,b,c)x,y,z) = -(a,(b,c,x),y,z)$

By using (3) we obtain

$(a,b,c)(x,y,z) = -(a,(b,c,x),y,z)$

Thus $q = -(a,(b,c,x),y,z)$

Now identify (4) with $n = (b,c,x)$ show that

$(a(b,c,x),y,z) = (a,(b,c,x),y,z)$

$q = -(a,(b,c,x),y,z)$.

Then (2) again implies $(a,(b,c,x)y,z) = (a,b(c,x,y),z)$

$q = (a,b(c,x,y),z)$.

Then (5) gives $q = (a,b(c,x,y),z)$

At this point (2) implies $q = -(a,b,c(x,y,z))$

Consequently (6) gives that $q = -(a,b,c)(x,y,z)$

Since we have consider $q = (a,b,c)(x,y,z)$, now we have $q = -q \Rightarrow 2q = 0$.

Since by hypothesis, R is of char. $\neq 2$, it follows that $q = 0$.

Now we establish that the associator ideal A of R squares to zero. Multiplying (2) on the right by (a,b,c), using the fact that all asociators are in the nucleus by hypothesis and that we have just proved that every product of two associators is zero, we see that

$(w,x,)z(a,b,c) = 0$ … (7)

Then $[(w,x,y)z][(a,b,c)d] = [(w,x,y)z(a,b,c)]d$, since (a,b,c) lies in the nucleus. But then use of (6) gives us

$[(w,x,y)z][(a,b,c)d]$ =0.

Hence $A^2 = 0$. Since R is semiprime. We therefore obtain $A = 0$.

Hence R must be associative

This completes the proof of the theorem.

4. References

[1] Boers, A.H., "The nucleus in the nonassociative ring", Proc. Kon. Ned. Akad. Van wet. Vol 74 et Indag. Math. 33(1971), 464-470.

[2] Kleinfeld, E., "A class of rings which are very nearly associative", Amer. Math. Monthly, 93,720-722,(1986).

[3] Thedy, A., "On rings satisfying $((a,b,c),d) = 0$", Proc.Amer.Math.Soc.Vol.29 (1971), 250-254.

[4] Yen, C.T.," Rings with associators in the nuclei", Chung Yuan J.28 (2000), 7-9.