Some Equivalent Conditions on Secondary J-Normal Matrices

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ABSTRACT

In this paper we introduce the concept of secondary J-unitarily equivalent matrices (J_S-unitarily equivalent matrices) and extend the results of [3] in the content of J_S -normal matrices.

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1. Introduction

The concept of J_S - normal matrices was introduced in [1]. In this paper, our intention is to define J_S-unitarily equivalent matrices and prove some equivalent conditions on J_S - normal matrices.

Definition 1.1 [1].

(i) A matrix $A \in C_{n\times n}$ is said to be Secondary J- normal (J_S -normal) if $A^{\#}A = AA^{\#}$.

(ii) A matrix $A \in C_{n\times n}$ is said to be J_S - hermitian matrix $A^{\#} = A$.

2. Equivalent Conditions on J_S -normal Matrices

Definition 2.1. A matrix $A \in C_{n\times n}$ is said to be Secondary J-unitary (J_S -unitary) if $A^{\#}A = AA^{\#} = I$. For example, A=$\begin{pmatrix} -2i & -i \\ i & 0 \end{pmatrix}$ is an J_S -unitary matrix.

Definition 2.2. Let $A, B \in C_{n\times n}$. The matrix B is said to be Secondary J- unitarily equivalent (J_S -unitarily equivalent) to A if there exists an J_S -unitary matrix U such that $B = U^{\#}AU$.

Example 2.3. Let A=$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and B=$\begin{pmatrix} -1 & -1 \\ 3 & 3 \end{pmatrix}$. Then if we take U=$\begin{pmatrix} -2i & -i \\ i & 0 \end{pmatrix}$, it can be verified that $U^{\#}U = UU^{\#} = I$ and $B = U^{\#}AU$. Hence B is Secondary J- unitarily equivalent to A.

Theorem 2.4. Let $A \in C_{n\times n}$. If A is J_S -unitarily equivalent to a diagonal matrix, then A is J_S -normal.

Proof. Let $A \in C_{n\times n}$. If A is J_S -unitarily equivalent to a diagonal matrix D, then there exists an J_S -unitary matrix P such that $P^{\#}AP = D$ which implies that $A = PDP^{\#}$ as $P^{\#}P = I$. Now $AA^{\#} = PDP^{\#}PD^{\#}P^{\#} = PDD^{\#}P^{\#}$. Also, $A^{\#}A = PD^{\#}P^{\#}PDP^{\#} = PD^{\#}DP^{\#}$. Since D and D$^{\#}$ are each diagonal, $DD^{\#} = D^{\#}D$ and hence $A^{\#}A = AA^{\#}$ so that A is J_S -normal.

Remark 2.5. It can be shown that A is J_S -normal $\Leftrightarrow A^{-1}A^{\#}$ is J_S -unitary.

Theorem 2.6. Let $H, N \in C_{n\times n}$ be invertible. If $B = HNH$, where H is J_S - hermitian and N is J_S - normal, then $B^{-1}B^{\#}$ is similar to an J_S -unitary matrix.

Proof. Let $H, N \in C_{n\times n}$ be invertible. If $B = HNH$, then

$$B^{-1}B^{\#}=H^{-1}N^{-1}H^{-1}H^{\#}N^{\#}H^{\#}=H^{-1}N^{-1}H^{-1}HN^{\#}H$$

as $H^{\#}=H$ and hence $B^{-1}B^{\#}=H^{-1}N^{-1}N^{\#}H$. Since N is J_S-normal, from Remark 2.5, $N^{-1}N^{\#}$ is J_S-unitary and hence the result follows.

Theorem 2.7. If A is J_S-normal and $AB=0$, then $A^{\#}B=0$.

Theorem 2.8. If X is an eigenvector of an J_S-normal matrix A corresponding to an eigenvalue λ, then X is also an eigenvector of $A^{\#}$ corresponding to the eigenvalue $\bar{\lambda}$.

Proof. Let $A\in C_{n\times n}$ be J_S-normal. Since X is an eigenvector of A corresponding to an eigenvalue λ, $AX=\lambda X$. Since A is normal, it can be easily seen that $A-\lambda I$ and $(A-\lambda I)^{\#}$ commute and hence $A-\lambda I$ is J_S-normal. Now $AX=\lambda X\Rightarrow(A-\lambda I)X=0$. Since $A-\lambda I$ is J_S- normal, by Theorem 3.4, $(A-\lambda I)^{\#}X=0$ which implies that $\left(A^{\#}-\bar{\lambda}I\right)X=0$ and hence which leads to the result.

Theorem 2.9. If $A\in C_{n\times n}$ is J_S-unitary and if λ is an eigenvalue of A, then $|\lambda|=1$.

Proof. Since $A\in C_{n\times n}$ is J_S-unitary, A is J_S-normal. Since λ is an eigenvalue of A, there exists an eigenvector $V\neq 0$ such that $AV=\lambda V$ which implies $A^{\#}V=\bar{\lambda}V$ as A is J_S- normal. Now $V=IV=A^{\#}AV$ which leads to $V\left(1-\lambda\bar{\lambda}\right)=0$. Since $V\neq 0$, $1-\lambda\bar{\lambda}=0$ which implies that $|\lambda|=1$.

Theorem 2.10. Let $A\in C_{n\times n}$. Assume that $AV=VP$, where V is J_S- unitary and P is nonsingular and J_S-hermitian such that if P^2 commutes with V, then P also commutes with V. Then the following conditions are equivalent.

i. A is J_S-normal
ii. $VP=PV$
iii. $AV=VA$
iv. $AP=PA$

Proof. Let $AV=VP$. Since V is J_S-unitary $VV^{\#}=V^{\#}V=I$ and since P is J_S- hermitian, $P^{\#}=P$.

$(i)\Leftrightarrow(ii)$: If A is J_S- normal, then $A^{\#}A=AA^{\#}$. Since A=VP, $(VP)^{\#}(VP)=(VP)(VP)^{\#}$ which implies that $VP^2V^{\theta}=P$. Post multiply by V, we have $VP^2=P^2V$ and hence $VP=PV$ by our assumption.

Conversely, if $VP=PV$, then $P^{\#}V^{\#}=V^{\#}P^{\#}$. Now $AA^{\#}=VPP^{\#}V^{\#}=VPV^{\#}P^{\#}=VP^{\#}V^{\#}P$ as $P^{\#}=P$. Therefore

$$AA^{\#}=VV^{\#}P^{\#}P=V^{\#}VPP=V^{\#}PVP=V^{\#}P^{\#}VP=(PV)^{\#}(VP)=(VP)^{\#}(VP)=A^{\#}A$$

and hence A is J_S - normal.

$(i)\Leftrightarrow(iii)$: If A is J_S- normal, then by (ii), $VP=PV$. Now $AV=(VP)V=V(VP)=VA$. Conversely, if $AV=VA$, then $(VP)V=V(VP)$, pre multiply by $V^{\#}$, $V^{\#}V(PV)=V^{\#}V(VP)$ which implies $PV=VP$ and hence A is J_S-normal.

$(i)\Leftrightarrow(iv)$: If A is J_S- normal, then $AP=(VP)P=PVP=PA$. Conversely, if $AP=PA$, then $(VP)P=P(VP)$. Post multiply by P^{-1}, we have $VP=PV$ and so A is J_S- normal.

Theorem 2.11. Let $A\in C_{n\times n}$. Assume that $A=VP$, where V is J_S- unitary and P is nonsingular and J_S-hermitian such that if P^2 commutes with V, then P also commutes with V. Then the following conditions are equivalent.

i. A is J_S- normal.
ii. Any eigenvector of V is an eigenvector of P (as long as V has distinct eigen values)
iii. Any eigenvector of P is an eigenvector of V (as long as P has distinct eigen values)
iv. Any eigenvector of V is an eigenvector of A (as long as V has distinct eigen values)
v. Any eigenvector of A is an eigenvector of V (as long as A has distinct eigen values)
vi. Any eigenvector of P is an eigenvector of A (as long as P has distinct eigen values)
vii. Any eigenvector of A is an eigenvector of P (as long as A has distinct eigen values)

Proof`. $(i)\Leftrightarrow(ii)$: Let V have distinct eigenvalues. If we prove $VP=PV\Leftrightarrow$ any eigenvector of V is an eigenvector of P, then $(i)\Leftrightarrow(ii)$ follows by Theorem 2.10. Assume that any eigenvector of V is an eigenvector of P. If X is an eigenvector of V, then X is also an eigenvector of P. Therefore there exist eigen values λ and μ such that $VX = \lambda X$ and $PX = \mu X$. Now $VX = \lambda X$ implies $PVX = P\lambda X = \lambda\mu X$. Similarly $PX = \mu X$ implies $VPX = \lambda\mu X$. Therefore $PVX = VPX \Rightarrow (PV - VP)X = 0$ which implies $PV = VP$ as $X \neq 0$.

Conversely, assume that $PV = VP$. If X is an eigenvector of V, then there exists an eigenvalue λ such that $VX = \lambda X$. Let μ be an eigenvalue of V such that $VX = \mu X$ therefore $\lambda \neq \mu$. Now $PV = VP$ implies $(VP - PV)X = 0$ which shows that $VPX = \lambda PX$. Similarly $VX = \mu X$ implies $VPX = \mu PX$. Therefore $\lambda PX = \mu PX \Rightarrow (\lambda - \mu)PX = 0 \Rightarrow PX = 0$ as $\lambda - \mu \neq 0$. Therefore $PX = 0X$ and hence X is an eigenvector of P corresponding to the eigenvalue 0. In general, if μ is any eigenvalue of V, then we can prove that X is also an eigenvector of P. Therefore any eigenvector of V is also an eigenvector of P.

Similarly proof holds for other equivalent conditions.

References

[1] K. Jaikumar, K. Sindhu, On Secondary J-Normal Matrices, International Journal of Scientific Research in Mathematical and Statistical Sciences, 6(2)(2019), 65-72.

[2] K . Jaikumar, K. Sindhu, Some Conditions on J_s - Normal Matrices – Accepted (International Journal for Research in Engineering Application & Management).

[3] S. Krishnamoorthy, R. Vijayakumar, Some Equivalent Conditions on s-Normal Matrices, Int.J.Contemp.Math.Sciences, 4(29)(2009), 1449-1454.

[4] S. Krishnamoorthy, R. Vijayakumar, On s-normal matrices, Journal of Analysis and computation, 2(2009).

[5] Anna Lee, Secondary Symmetric, Secondary skew symmetric and Secondary orthogonal matrices, Period math. Hungary 7(1976), 63-76.