SOLUTION OF HIGHER ORDER DIFFERENTIAL EQUATIONS WITH FUZZY BOUNDARY CONDITIONS BY FINITE DIFFERENCE METHOD

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Abstract

Now a days, The differential equations with Fuzzy boundary conditions(FBCs) is a most popular topic studied by many researchers since it is utilized widely for the purpose of modelling problems in science and engineering.Most of the physical phenomena are model by system of ordinary or partial differential equations.In general to obtain the exact solution of BVPs is difficult, so we have to apply numerical methods.Here we are going to convert differential equations with FBCs to system of linear equations by using Finite Difference Method and we solve that system of linear equations by MATLAB.We are going to find the approximate solution of given differential equations with FBCs by using difference equations. Here we will approximate all types of the differential equations with FBCs by using Finite difference method(FDM).

Keywords: Fuzzy Arithmetic,Fuzzy derivatives,Triangular fuzzy number,Fuzzy Boundary Conditions(FBCs),Finite Difference Method(FDM).

1 Introduction

The differential equation with fuzzy parameters is very much important topic in field of science and engineering to solve dynamic problem. The concept of a fuzzy derivative was first introduced by Chang and Zadeh [56],followed up by Dubois and Prade [17] who used the extension principle in their approach.Other fuzzy derivative concepts were proposed by Puri and Ralescu [45], and Goetschel and Vaxman [26]as an extension of the Hukuhara derivative of multivalued functions.Kandel and Byatt [33] applied the concept of fuzzy differential equation to the analysis of fuzzy dynamical problems.The fuzzy differential equations and fuzzy initial value problems are studied by Kaleva[31, 32]and Seikkala [51]

Two analytical methods for solving an nth-order fuzzy linear differential equation with fuzzy initial conditions presented by Buckley and Feuring [12, 13].Mondal and Roy [42] described the solution procedure for first order linear non-homogeneous ordinary differential equation in fuzzy environment.Existence and uniqueness of fuzzy boundary value problem has been proved by Esfahani et al.[18].Lakshikantham et al. [38] investigated the solution of two point boundary value problems associated with non-linear fuzzy differential equation by using the extension principle.Generalized differentiability concept is used by Bade et al.[11] to investigated first order linear fuzzy differential equations. Based on the idea of collocation method Allahviranloo et al. [5] solved n^{th} order fuzzy linear differential equations.Far and Ghal-Eh [19] proposed an iterative method to solve fuzzy differential equations for the linear system of first order fuzzy differential equation with fuzzy constant coefficient.Variation of constant formula has been handle by Khastan et al.[37] to solve first order fuzzy differential equations. Akin et al.[2] developed an algorithm based on α-cut of fuzzy set for solution of second order fuzzy initial value problems. A new approach has been developed by Gasilov et al.[22] to get the solution of fuzzy initial value problem.

The concept of generalized H-differentiability is studied by Chalco-Cano and Roman Flores [14] to solve fuzzy differential equation.Hasheni et al.[29, 28] studied homotopy analysis method for solution of system of fuzzy differential equations and obtained analytic solution of fuzzy Wave like equations with variable coefficients.As regards, methods to solve n^{th} order fuzzy differential equation are discussed in[5, 25, 30, 35, 48, 55].The Variational iteration method (VIM) was successfully applied by Jafari et al.[30] for solving n^{th} order fuzzy differential equation.A new result on multiple solutions for n^{th} order fuzzy differential equations under generalized differentiability has been proposed by Khastan et al.[35].Based on idea of collocation method allahviranloo et al.[5] solved n^{th} order fuzzy linear differential equations.The integral form of n^{th} order fuzzy differential equations has been developed by Salahshour [48]under generalized differentiability.Mansuri and Ahmady [41] implemented characterization theorem for solving n^{th} order fuzzy differential equations.Also Tapaswini and Chakraverty[53] implemented homotopy perturbation method for the solution of n^{th} order fuzzy linear differential equations. Bade [10] found solutions of fuzzy differential equations based on generalized differentiability.

Paper is organized as In section 2 preliminaries,In section 3 Method of solution of higher order differential equation with FBCs by FDM ,In section 4 Examples based on FDM,In section 5 Result and Discussion,In section 6 Conclusion.

2 Preliminaries

Definition 2.1 Membership function
A fuzzy set $\tilde{A}$ can be defined as a pair of the universal set U and membership function $\mu_{\tilde{A}} : U \to [0,1]$ for each $x \in U$,the number $\mu_{\tilde{A}}$ is called the membership degree of x in $\tilde{A}$.
Definition 2.2 $r - cut$
The $r - cut$ of $\tilde{A}$ is a crisp set and it is defined as

$$A_r = \{x \in U | \mu_{\tilde{A}}(x) \geq r\}$$

For $r = 0$ then $A_0 = closure(supp(\tilde{A}))$
Definition 2.3 Fuzzy Number
A fuzzy number is a fuzzy set like $\mu : R \to I = [0,1]$ which satisfies:

(a) μ is upper semi-continuous,

(b) μ is fuzzy convex i.e $\mu(\lambda x+(1-\lambda)y) \geq min\{\mu(x), \mu(y)\} \forall x, y \in R, \lambda \in [0,1]$,

(c) μ is normal i.e $\exists x_0 \in R$ for which $\mu(x_0) = 1$,

(d) supp $\mu = \{x \in R \mid \mu(x) > 0\}$ is support of u, and its closure cl(supp μ) is compact.

Definition 2.4 Triangular Fuzzy Number
Consider triangular fuzzy number $\tilde{A} = (a, b, c)$ is depicted in Figure 1 The membership function $\mu(x)$ of $\tilde{A}$ will be defined as follows.

$$\mu(x) = \begin{cases} 0 & , x < a \\ \frac{x-a}{b-a} & , a \leq x \leq b \\ \frac{c-x}{c-b} & , b \leq x \leq c \\ 0 & , x > c \end{cases}$$

The triangular fuzzy number $\tilde{A} = (a, b, c)$ can be represented with an order pair of function of r-cut approach i.e.$[\underline{\mu}(r), \overline{\mu}(r)] = [a + (b - a)r, c - (c - b)r]$, where $r \in [0,1]$

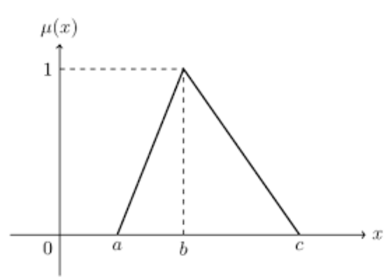

Figure 1: Triangle membership function

A fuzzy number μ in a parametric form is a pair $(\underline{\mu}, \overline{\mu})$ of functions $\underline{\mu}(r), \overline{\mu}(r), 0 \leq r \leq 1$, which satisfies the following requirements:

1. $\underline{\mu}(r)$ is a bounded monotonic increasing left continuous function,
2. $\overline{\mu}(r)$ is a bounded monotonic decreasing right continuous function,
3. $\underline{\mu}(r) \leq \overline{\mu}(r), 0 \leq r \leq 1$.

A crisp number x is simply represented by $(\underline{\mu}(r), \overline{\mu}(r)) = (x, x), 0 \leq r \leq 1$. By appropriate definitions, the fuzzy number space $\{(\underline{\mu}(r), \overline{\mu}(r))\}$ becomes a convex cone E which could be embedded isomorphically and isometrically into a Banach space.

Definition 2.5 Fuzzy arithmetic

For any arbitrary two fuzzy numbers $u = (\underline{u}(r), \overline{u}(r)), v = (\underline{v}(r), \overline{v}(r)), 0 \leq r \leq 1$ and arbitrary $k \in R$.we define addition,subtraction,multiplication,scalar multiplication by k.(see in [21])

$$u + v = (\underline{u}(r) + \underline{u}(r), \overline{v}(r) + \overline{v}(r)),$$
$$u - v = (\underline{u}(r) - \overline{v}(r), \overline{u}(r) - \underline{v}(r)),$$
$$u \cdot v =$$
$$(min\{\underline{u}(r)\overline{v}(r), \underline{u}(r)\underline{v}(r), \overline{u}(r)\overline{v}(r), \overline{u}(r)\underline{v}(r)\}, max\{\underline{u}(r)\overline{v}(r), \underline{u}(r)\underline{v}(r), \overline{u}(r)\overline{v}(r), \overline{u}(r)\underline{v}(r)\})$$

$$ku = \begin{cases} (k\underline{u}(r), k\overline{u}(r)), & k \geq 0 \\ (k\overline{u}(r), k\underline{u}(r)), & k < 0 \end{cases}$$

Definition 2.6 Hukuhara-difference

Let $x, y \in E$.If there exists $z \in E$ such that $x = y + z$,then z is called the Hakuhara-difference of fuzzy numbers x and y,and it is denoted by $z = x \ominus y$. The $\ominus$ sign stands for Hukuhara-difference,and $x \ominus y \neq x + (-1)y$.

Definition 2.7 Hukuhara-differentiability

Let $f : (a, b) \to E$ and $t_0 \in (a, b)$.We say that f is Hukuhara-differential at t_0,if there exists an element $f^{'}(t_0) \in E$ such that for all $h > 0$ sufficiently small, $\exists f(t_0 + h) \ominus f(t_0), f(t_0) \ominus f(t_0 - h)$ and the limits holds(in the metric D)

$$\lim_{h \to 0} \frac{f(t_0 + h) \ominus f(t_0)}{h} = \lim_{h \to 0} \frac{f(t_0) \ominus f(t_0 - h)}{h} = f^{'}(t_0).$$

Theorem 2.1 Lipschits conditios

Suppose the function f in the fuzzy boundary value problem

$$y^{''} = f(x, y, y^{'}), a \leq x \leq b, \tilde{y}(a) = \tilde{\alpha}, \tilde{y}(b) = \tilde{\beta}$$

is continuous on the set

$$D = \{(x, y, y^{'})/a \leq x \leq b, -\infty < y < \infty, -\infty < y^{'} < \infty\}$$

and that the partial derivatives f_y and $f^{'}_y$ are also continuous in D.If

(1)$f_y(x, y, y^{'}) > 0$ for all $(x, y, y^{'}) \in D$

(2)a constant M exists with $|f_{y'}(x, y, y^{'})| \leq M$ for all $(x, y, y^{'}) \in D$

then the fuzzy boundary value problem has unique solution in term of parametric forms of fuzzy number.

3 Finite Difference Equations

The finite difference approximation to the various derivatives are as under:
If $y(x)$ and its derivatives are single valued continuous functions of x then by Taylor's expansion,we get

$$y(x+h) = y(x) + hy^{'}(x) + \frac{h^2}{2}y''(x) + \frac{h^3}{6}y'''(x) + \frac{h^4}{24}y^{(4)}(x)(\zeta^{+}), \quad (1)$$

for some $\zeta^{+} \in (x, x+h)$,and

$$y(x-h) = y(x) - hy^{'}(x) + \frac{h^2}{2}y''(x) - \frac{h^3}{6}y'''(x) + \frac{h^4}{24}y^{(4)}(x)(\zeta^{-}), \quad (2)$$

for some $\zeta^{-} \in (x-h, x)$,and Eq.(1)gives

$$y^{'}(x) = \frac{1}{h}[y(x+h) - y(x)] - \frac{h}{2}y^{''}(x) - ......$$

i.e.

$$y^{'}(x) = \frac{1}{h}[y(x+h) - y(x)] + O(h)$$

which is the forward difference approximation of $y^{'}(x)$ with an error of the order h. Similarly Eq.(2) gives

$$y^{'}(x) = \frac{1}{h}[y(x) - y(x-h)] + \frac{h}{2}y^{''}(x) + ...$$

i.e.

$$y^{'}(x) = \frac{1}{h}[y(x) - y(x-h)] + O(h)$$

which is the backward difference approximation of $y^{'}(x)$ with an error of the order h.

subtracting Eq.(2) from Eq.(1),we get

$$y^{'}(x) = \frac{1}{2h}[y(x+h) - y(x-h)] - \frac{h^2}{6}y^{'''}(\eta),$$

for some $\eta \in (x-h, x+h)$

$$y^{'}(x) = \frac{1}{2h}[y(x+h) - y(x-h)] + O(h^2)$$

which is central-difference approximation of $y^{'}(x)$ with an error of the order h^2. The central difference approximation of $y^{'}(x)$ is better than forward or backward approximation.

Adding Eqs (1) and (2)

$$y^{''}(x) = \frac{1}{h^2}[y(x+h) - 2y(x) + y(x-h)] - \frac{h^2}{24}[y^4(x)(\zeta^+) + y^{(4)}(x)(\zeta^-)]$$

The Intermediate value theorem can be used to simplify this even further.

$$y^{''}(x) = \frac{1}{h^2}[y(x+h) - 2y(x) + y(x-h)] - \frac{h^2}{12}y^{(4)}(x)(\zeta)$$

for some $\zeta \in (x-h, x+h)$

$$y^{''}(x) = \frac{1}{h^2}[y(x+h) - 2y(x) + y(x-h)] + O(h^2)$$

which is the central difference approximation of $y^{''}(x)$ with an error of the order h^2.Similarly we can derive central difference approximations to higher order derivatives.
Hence the working expression for the central difference approximations to first two derivatives of y_i as under:

$$y_i^{'} = \frac{1}{2h}(y_{i+1} - y_{i-1}) \tag{3}$$

$$y_i^{''} = \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1}) \tag{4}$$

Similarly,we can find central difference approximations to higher order derivatives.

$$y_i^{'''} = \frac{1}{2h^3}(y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}) \tag{5}$$

$$y_i^{(4)} = \frac{1}{h^4}(y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}) \tag{6}$$

4 Method of solution of higher order differential equation with FBCs by FDM

Consider the second order differential equation

$$y^{''}(x) + p(x)y'(x) + q(x)y(x) = r(x), x \in [a, b] \tag{7}$$

with the fuzzy boundary conditions

$$y(a) = \tilde{\alpha}, y(b) = \tilde{\beta} \tag{8}$$

where $p(x), q(x), r(x)$ are constants or non-linear continuous functions of x on $[a, b]$.
Here,we divide the interval $I = [a, b]$ into a chosen n numbers of subintervals of equal width. Thus,the step size h,of each of n subintervals is given by $h = \frac{b-a}{n}$
y_i denotes the value of function at i^{th} node of the computational grid.

$y_0 = \tilde{\alpha}, y_n = \tilde{\beta}$.
At any mesh point or node point $x = x_i$,the finite difference representation of the differential equation by fuzzy boundaries can be written as follow:
$(i = 1, 2, ..., n-1)$

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + p(x_i)\frac{y_{i+1} - y_{i-1}}{2h} + q(x_i)y_i$$
$$= r(x_i) + \frac{h^2}{12}[2p(x_i)y^{'''}(\eta_i) + y^{(4)}(\zeta_i)], \tag{9}$$

A Finite difference method with truncation error of order $O(h^2)$ results into the equation below after ignoring truncation error.

$$2(y_{i+1} - 2y_i + y_{i-1}) + hp(x_i)(y_{i+1} - y_{i-1}) + 2h^2q(x_i)y_i = 2h^2r(x_i), \tag{10}$$

The boundary conditions provide solution at the two ends of the grid i.e.at $y_0 = \tilde{\alpha}, y_n = \tilde{\beta}$ The following system is

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ (2h^2q_1 - 4) & (2 + hp_1) & 0 & \cdots & \cdots & 0 \\ (2 - hp_2) & (2h^2q_2 - 4) & (2 + hp_2) & 0 & \cdots & 0 \\ 0 & (2 - hp_3) & (2h^2q_3 - 4) & (2 + hp_3) & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & (2 - hp_{n-2}) & (2h^2q_{n-2} - 4) & (2 + hp_{n-2}) \\ 0 & \cdots & \cdots & 0 & (2 - hp_{n-1}) & (2h^2q_{n-1} - 4) \\ 0 & 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}$$

$$\tilde{X} = \begin{bmatrix} \tilde{y}_0 \\ \tilde{y}_1 \\ \tilde{y}_2 \\ \tilde{y}_3 \\ \vdots \\ \tilde{y}_{n-2} \\ \tilde{y}_{n-1} \\ \tilde{y}_n \end{bmatrix}$$

$$\tilde{B} = \begin{bmatrix} \tilde{\alpha} \\ 2h^2r_1 - (2 - hp_1)\tilde{\alpha} \\ 2h^2r_2 \\ 2h^2r_3 \\ \vdots \\ 2h^2r_{n-2} \\ 2h^2r_{n-1} - (2 + hp_{n-1})\tilde{\beta} \\ \tilde{\beta} \end{bmatrix}$$

The above system of linear equations i.e.$A_{N+1\times N+1}\tilde{X}_{N+1\times 1} = \tilde{B}_{N+1\times 1}$ is a non-singular tridiagonal system and it has unique solution in form of fuzzy parameters.

5 Examples based on Finite Difference Method

Example 5.1 Approximate the solution of differential equation with Fuzzy boundary conditions

$$\begin{cases} y^{''} = x + y, x \in [0,1] \\ y(0) = 0, y(1) = 0 \end{cases}$$

Exact solution is

$$Y(x) = \frac{sinhx}{sinh1} - x \tag{11}$$

Consider differential equation with Fuzzy boundary conditions

$$\begin{cases} y^{''} = x + y, x \in [0,1] \\ y(0) = (-0.5, 0, 0.5), y(1) = (-0.5, 0, 0.5) \end{cases}$$

Put $x_n = x_0 + nh, x_0 = 0, h = 0.1$ and let y_n to be calculated as $y_n = y(x_n)$
Here, we divide the interval $[0,1]$ into ten sub-intervals.

By using central difference approximation:

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = x_i + y_i, i = 1, 2, ..., 10$$

for $i = 1$:

$$100y_0 - 201y_1 + 100y_2 = 0.1$$

for $i = 2$:

$$100y_1 - 201y_2 + 100y_3 = 0.2$$

for $i = 3$:

$$100y_2 - 201y_3 + 100y_4 = 0.3$$

and so on
for $i = 9$:

$$100y_8 - 201y_9 + 100y_{10} = 0.9$$

We set $\underline{y}_0 = \underline{y}_{10} = -0.5 + 0.5r$ and we get system of linear equations with the unknowns $\underline{y}_0, \underline{y}_1, \underline{y}_2, ..., \underline{y}_{10}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -201 & 100 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 100 & -201 & 100 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 100 & -201 & 100 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 100 & -201 & 100 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 100 & -201 & 100 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 100 & -201 & 100 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 100 & -201 & 100 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 100 & -201 & 100 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 100 & -201 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{y}_0 \\ \underline{y}_1 \\ \underline{y}_2 \\ \underline{y}_3 \\ \underline{y}_4 \\ \underline{y}_5 \\ \underline{y}_6 \\ \underline{y}_7 \\ \underline{y}_8 \\ \underline{y}_9 \\ \underline{y}_{10} \end{bmatrix} = \begin{bmatrix} -0.5+0.5r \\ 0.6-0.5r \\ 0.2 \\ 0.3 \\ 0.4 \\ 0.5 \\ 0.6 \\ 0.7 \\ 0.8 \\ 1.4-0.5r \\ -0.5+0.5r \end{bmatrix}$$

The lower bounds for solution are

$$\underline{y}_0(0) = 0.5r - 0.5$$
$$\underline{y}_1(0.1) = 0.00479373r - 0.019549$$
$$\underline{y}_2(0.2) = 0.00463541r - 0.0332936$$
$$\underline{y}_3(0.3) = 0.00452343r - 0.0453711$$
$$\underline{y}_4(0.4) = 0.00445669r - 0.0549022$$
$$\underline{y}_5(0.5) = 0.00443452r - 0.0609824$$
$$\underline{y}_6(0.6) = 0.00445669r - 0.0626725$$
$$\underline{y}_7(0.7) = 0.00452343r - 0.0589892$$
$$\underline{y}_8(0.8) = 0.00463541r - 0.0488959$$
$$\underline{y}_9(0.9) = 0.00479373r - 0.0312915$$
$$\underline{y}_{10}(1) = 0.5r - 0.5$$

We set $\overline{y}_0 = \overline{y}_{10} = -0.5 + 0.5r$ and we get system of linear equations with the unknowns $\overline{y}_0, \overline{y}_1, \overline{y}_2, ..., \overline{y}_{10}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -201 & 100 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 100 & -201 & 100 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 100 & -201 & 100 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 100 & -201 & 100 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 100 & -201 & 100 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 100 & -201 & 100 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 100 & -201 & 100 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 100 & -201 & 100 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 100 & -201 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \overline{y}_0 \\ \overline{y}_1 \\ \overline{y}_2 \\ \overline{y}_3 \\ \overline{y}_4 \\ \overline{y}_5 \\ \overline{y}_6 \\ \overline{y}_7 \\ \overline{y}_8 \\ \overline{y}_9 \\ \overline{y}_{10} \end{bmatrix} = \begin{bmatrix} 0.5-0.5r \\ -0.4+0.5r \\ 0.2 \\ 0.3 \\ 0.4 \\ 0.5 \\ 0.6 \\ 0.7 \\ 0.8 \\ 0.4+0.5r \\ 0.5-0.5r \end{bmatrix}$$

The upper bounds for solution are

$$\overline{y}_0(0) = 0.5 - 0.5r$$

$$\overline{y}_1(0.1) = -0.00479373r - 0.00996158$$

$$\overline{y}_2(0.2) = -0.00463541r - 0.0240228$$

$$\overline{y}_3(0.3) = -0.00452343r - 0.0363242$$

$$\overline{y}_4(0.4) = -0.00445669r - 0.0459889$$

$$\overline{y}_5(0.5) = -0.00443452r - 0.0521134$$

$$\overline{y}_6(0.6) = -0.00445669r - 0.0537591$$

$$\overline{y}_7(0.7) = -0.00452343r - 0.0499424$$

$$\overline{y}_8(0.8) = -0.00463541r - 0.0396251$$

$$\overline{y}_9(0.9) = -0.00479373r - 0.021704$$

$$\overline{y}_{10}(1) = 0.5 - 0.5r$$

Table 1: The approximate values for lower bounds

	x										
r	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	-0.5000	-0.0195	-0.0333	-0.0454	-0.0549	-0.0610	-0.0627	-0.0590	-0.0489	-0.0313	-0.5000
0.1	-0.4500	-0.0191	-0.0328	-0.0449	-0.0545	-0.0605	-0.0622	-0.0585	-0.0484	-0.0308	-0.4500
0.2	-0.4000	-0.0186	-0.0324	-0.0445	-0.0540	-0.0601	-0.0618	-0.0581	-0.0480	-0.0303	-0.4000
0.3	-0.3500	-0.0181	-0.0319	-0.0440	-0.0536	-0.0597	-0.0613	-0.0576	-0.0475	-0.0299	-0.3500
0.4	-0.3000	-0.0176	-0.0314	-0.0436	-0.0531	-0.0592	-0.0609	-0.0572	-0.0470	-0.0294	-0.3000
0.5	-0.2500	-0.0172	-0.0310	-0.0431	-0.0527	-0.0588	-0.0604	-0.0567	-0.0466	-0.0289	-0.2500
0.6	-0.2000	-0.0167	-0.0305	-0.0427	-0.0522	-0.0583	-0.0600	-0.0563	-0.0461	-0.0284	-0.2000
0.7	-0.1500	-0.0162	-0.0300	-0.0422	-0.0518	-0.0579	-0.0596	-0.0558	-0.0457	-0.0279	-0.1500
0.8	-0.1000	-0.0157	-0.0296	-0.0418	-0.0513	-0.0574	-0.0591	-0.0554	-0.0452	-0.0275	-0.1000
0.9	-0.0500	-0.0152	-0.0291	-0.0413	-0.0509	-0.0570	-0.0587	-0.0549	-0.0447	-0.0270	-0.0500
1.0	-0.0000	-0.0148	-0.0287	-0.0408	-0.0504	-0.0562	-0.0582	-0.0545	-0.0443	-0.0265	0.0000

Table 2: The approximate values for upper bounds

	x										
r	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	0.5000	-0.0100	-0.0240	-0.0363	-0.0460	-0.0521	-0.0538	-0.0499	-0.0396	-0.0217	0.5000
0.1	0.4500	-0.0104	-0.0245	-0.0368	-0.0464	-0.0526	-0.0542	-0.0504	-0.0401	-0.0222	0.4500
0.2	0.4000	-0.0109	-0.0249	-0.0372	-0.0469	-0.0530	-0.0547	-0.0508	-0.0406	-0.0227	0.4000
0.3	0.3500	-0.0114	-0.0254	-0.0377	-0.0473	-0.0534	-0.0551	-0.0513	-0.0410	-0.0231	0.3500
0.4	0.3000	-0.0119	-0.0259	-0.0381	-0.0478	-0.0539	-0.0555	-0.0518	-0.0415	-0.0236	0.3000
0.5	0.2500	-0.0124	-0.0263	-0.0386	-0.0482	-0.0543	-0.0560	-0.0522	-0.0419	-0.0241	0.2500
0.6	0.2000	-0.0128	-0.0268	-0.0390	-0.0487	-0.0548	-0.0564	-0.0527	-0.0424	-0.0246	0.2000
0.7	0.1500	-0.0133	-0.0273	-0.0395	-0.0491	-0.0552	-0.0569	-0.0531	-0.0429	-0.0251	0.1500
0.8	0.1000	-0.0138	-0.0277	-0.0399	-0.0496	-0.0557	-0.0573	-0.0536	-0.0433	-0.0255	0.1000
0.9	0.0500	-0.0143	-0.0282	-0.0404	-0.0500	-0.0561	-0.0578	-0.0540	-0.0438	-0.0260	0.0500
1.0	0.0000	-0.0148	-0.0287	-0.0408	-0.0504	-0.0565	-0.0582	-0.0545	-0.0443	-0.0265	0.0000

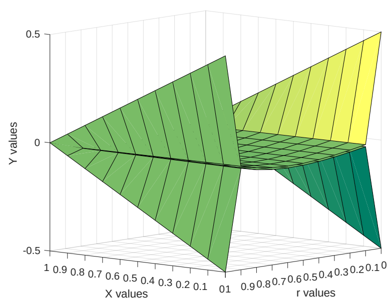

Figure 2: The graph of lower and upper bound of approximate solution

Airy's functions commonly appear in physics, especially in optics, quantum mechanics, electromagnetic, and radioactive transfer. The Airy stress function employed in solid mechanics. In the physical sciences, the Airy function (or Airy function of the first kind) $Ai(x)$ is a special function named after the British astronomer George Biddell Airy (1801–92). The function $Ai(x)$ and the related function $Bi(x)$, are linearly independent solutions to the differential equation.

$$\frac{d^2y}{dx^2} - xy = 0 \tag{12}$$

known as the Airy equation or the Stokes equation. This is the simplest second-order linear differential equation with a turning point (a point where the character of the solutions changes from oscillatory to exponential).

Example 5.2 Approximate the solution of Airy's differential equation with fuzzy boundary conditions

$$\begin{cases} y^{''} - (1 - \frac{x}{5})y = x, x \in [1, 3] \\ y(1) = 2, y(3) = -1 \end{cases}$$

Exact solution is difficult to find.we can Find solution by Mathematica but its to long to write and inter-print.
Consider differential equation with Fuzzy boundary conditions

$$\begin{cases} y^{''} - (1 - \frac{x}{5})y = x, x \in [1,3] \\ y(1) = (1.9, 2, 2.01), y(3) = (-0.5, -1, -1.1) \end{cases}$$

Put $x_n = x_0 + nh, x_0 = 1, h = 0.2$ and let y_n to be calculated as $y_n = y(x_n)$
Here,we divide the interval $[1,3]$ into ten sub-intervals.

By using central difference approximation:

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - (1 - \frac{x_i}{5})y_i = x_i, i = 1, 2, ..., 10$$

This gives

$$y_{i-1} - [2 + (1 - \frac{x_i}{5})h^2]y_i + y_{i+1} = x_i h^2, i = 1, 2, ..., 10$$

for $i = 1$:

$$y_0 - [2 + (1 - \frac{1.2}{5})(0.2)^2]y_1 + y_2 = 1.2(0.2)^2$$

for $i = 2$:

$$y_1 - [2 + (1 - \frac{1.4}{5})(0.2)^2]y_2 + y_3 = 1.4(0.2)^2$$

for $i = 3$:

$$y_2 - [2 + (1 - \frac{1.6}{5})(0.2)^2]y_3 + y_4 = 1.6(0.2)^2$$

and so on
for $i = 9$:

$$y_8 - [2 + (1 - \frac{2.8}{5})(0.2)^2]y_9 + y_{10} = 2.8(0.2)^2$$

We set $\underline{y}_0 = 1.9 + 0.1r, \underline{y}_{10} = -0.5 - 0.5r$ and we get system of linear equations with the unknowns $\underline{y}_0, \underline{y}_1, \underline{y}_2, ...\underline{y}_{10}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2.0304 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2.0288 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2.0272 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2.0256 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2.0240 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2.0224 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2.0208 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2.0192 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2.0176 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{y}_0 \\ \underline{y}_1 \\ \underline{y}_2 \\ \underline{y}_3 \\ \underline{y}_4 \\ \underline{y}_5 \\ \underline{y}_6 \\ \underline{y}_7 \\ \underline{y}_8 \\ \underline{y}_9 \\ \underline{y}_{10} \end{bmatrix}$$

$$= \begin{bmatrix} 1.9+0.1r \\ -1.852-0.1r \\ 0.56 \\ 0.064 \\ 0.072 \\ 0.08 \\ 0.088 \\ 0.096 \\ 0.1040 \\ 1.612+0.5r \\ -0.5-0.5r \end{bmatrix}$$

The lower bounds for solution are

$$\underline{y}_0(1) = 1.9 + 0.1r$$
$$\underline{y}_1(1.2) = 0.048771r + 1.30194$$
$$\underline{y}_2(1.4) = 0.791465 - 0.00097538r$$
$$\underline{y}_3(1.6) = 0.359782 - 0.0507498r$$
$$\underline{y}_4(1.8) = 0.00188402 - 0.101905r$$
$$\underline{y}_5(2.0) = -0.213168r - 0.49663$$
$$\underline{y}_6(2.2) = -0.275443r - 0.624419$$
$$\underline{y}_7(2.4) = -0.343446r - 0.677196$$
$$\underline{y}_8(2.6) = 0.359782 - 0.0507498r$$
$$\underline{y}_9(2.8) = -0.418044r - 0.638975$$
$$\underline{y}_{10}(3) = -0.5r - 0.5$$

We set $\overline{y}_0 = 2.01 - 0.01r$,$\overline{y}_4 = -1.1 + 0.1r$ and we get system of linear equations with the unknowns $\overline{y}_0, \overline{y}_1, \overline{y}_2, ..., \overline{y}_{10}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2.0304 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2.0288 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2.0272 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2.0256 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2.0240 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2.0224 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2.0208 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2.0192 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2.0176 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \overline{y}_0 \\ \overline{y}_1 \\ \overline{y}_2 \\ \overline{y}_3 \\ \overline{y}_4 \\ \overline{y}_5 \\ \overline{y}_6 \\ \overline{y}_7 \\ \overline{y}_8 \\ \overline{y}_9 \\ \overline{y}_{10} \end{bmatrix}$$

$$= \begin{bmatrix} 2.01 - 0.01r \\ -1.962 + 0.01r \\ 0.56 \\ 0.064 \\ 0.072 \\ 0.08 \\ 0.088 \\ 0.096 \\ 0.1040 \\ 1.122 - 0.01r \\ -1.01 + 0.01r \end{bmatrix}$$

The upper bounds for solution are

$$\overline{y}_0(1) = 2.01 - 0.01r$$
$$\overline{y}_1(1.2) = 1.35836 - 0.00764766r$$
$$\overline{y}_2(1.4) = 0.796018 - 0.00552781r$$
$$\overline{y}_3(1.6) = 0.312599 - 0.00356715r$$
$$\overline{y}_4(1.8) = -0.00170353r - 0.0983172$$
$$\overline{y}_5(2.0) = 0.000116485r - 0.43975$$
$$\overline{y}_6(2.2) = 0.0019393r - 0.711737$$
$$\overline{y}_7(2.4) = 0.00380555r - 0.903667$$
$$\underline{y}_8(2.6) = 0.00575095r - 1.02639$$
$$\overline{y}_9(2.8) = 0.00780678r - 1.06483$$
$$\overline{y}_{10}(3) = 0.01r - 1.01$$

Table 3: The approximate values for lower bounds

	x										
r	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	1.9000	1.3019	0.7915	0.3598	0.0019	-0.2840	-0.4966	-0.6244	-0.6772	-0.6390	-0.5000
1.2	1.9100	1.3068	0.7914	0.3547	-0.0083	-0.2995	-0.5179	-0.6520	-0.7115	-0.6808	-0.5500
1.4	1.9200	1.3117	0.7913	0.3496	-0.0185	-0.3151	-0.5393	-0.6795	-0.7459	-0.7226	-0.6000
0.6	1.9300	1.3166	0.7912	0.3446	-0.0287	-0.3307	-0.5606	-0.7071	-0.7802	-0.7644	-0.6500
0.8	1.9400	1.3215	0.7911	0.3395	-0.0389	-0.3462	-0.5819	-0.7346	-0.8146	-0.8062	-0.7000
2.0	1.9500	1.3263	0.7910	0.3344	-0.0491	-0.3618	-0.6032	-0.7621	-0.8489	-0.8480	-0.7500
2.2	1.9600	1.3312	0.7909	0.3243	-0.0694	-0.3774	-0.6245	-0.7897	-0.8833	-0.8898	-0.8000
2.4	1.9700	1.3361	0.7908	0.3293	-0.0518	-0.3929	-0.6458	-0.8172	-0.9176	-0.9316	-0.8500
2.6	1.9800	1.3410	0.7907	0.3192	-0.0796	-0.4085	-0.6672	-0.8448	-0.9520	-0.9734	-0.9000
2.8	1.9900	1.3458	0.7906	0.3141	-0.0898	-0.4241	-0.6885	-0.8723	-0.9863	-1.0152	-0.9500
3.0	2.0000	1.3507	0.7905	0.3090	-0.1000	-0.4396	-0.7098	-0.8999	-1.0206	-1.0570	-1.0000

Table 4: The approximate values for upper bounds

	x										
r	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	2.0100	1.3584	0.7960	0.3126	-0.0983	-04398	-0.7117	-0.9037	-1.0264	-1.0648	-1.0100
1.2	2.0090	1.3576	0.7955	0.3122	-0.0985	-04397	-0.7115	-0.9033	-1.0264	-1.0640	-1.0090
1.4	2.0080	1.3568	0.7949	0.3119	-0.0987	-04397	-0.7113	-0.9029	-1.0252	-1.0633	-1.0080
1.6	2.0070	1.3561	0.7944	0.3115	-0.0988	-04397	-0.7112	-0.9025	-1.0247	-1.0625	-1.0070
1.8	2.0060	1.3553	0.7938	0.3112	-0.0990	-04397	-0.7110	-0.0921	-1.0241	-1.0617	-1.0060
2.0	2.0050	1.3545	0.7933	0.3108	-0.0992	-04397	-0.7108	-0.9018	-1.0235	-1.0609	-1.0050
2.2	2.0040	1.3538	0.7927	0.3105	-0.0993	-04397	-0.7106	-0.9014	-1.0229	-1.0601	-1.0040
2.4	2.0030	1.3530	0.7921	0.3101	-0.0995	-04397	-0.7104	-0.9010	-1.0224	-1.0594	-1.0030
2.6	2.0020	1.3522	0.7916	0.3097	-0.0997	-04397	-0.7102	-0.9006	-1.0218	-1.0586	-1.0020
2.8	2.0010	1.3515	0.7910	0.3094	-0.0999	-04396	-0.7100	-0.9002	-1.0212	-1.0578	-1.0010
3.0	2.0000	1.3507	0.7905	0.3090	-1.0000	-04396	-0.7098	-0.8999	-1.0206	-1.0570	-1.0000

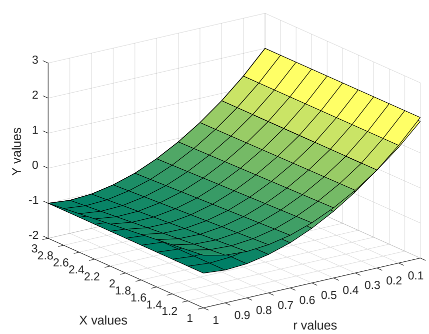

Figure 3: The graph of lower and upper bound of approximate solution

6 Result and Discussion

Tables 1,2 shows the values of lower and upper bound for approximation solution of Example No.1 respectively. Figure 2 shows the graphical representation of lower and upper bound for approximation solution of Example No.1. Table 3,4 shows the values of lower and upper bound for approximation solution of Example No.2 respectively. Figure 3 shows the graphical representation of lower and upper bound for approximation solution of Example No.2 in which Airy's function is involved. Here we consider boundary conditions fuzzy so except mid point system has fuzzy parameters on right hand side then after solving system by MATLAB we got all y parameters fuzzy. Here we have solved higher order differential equations with fuzzy boundaries which cannot be solved by using standard method, that can be solved by using the Finite difference Method.

7 Conclusion

The differential equation with fuzzy boundary conditions is investigated as a solution of difference equation. Here the Finite difference method is applied for second order differential equation with fuzzy boundary conditions for sack of simplicity but it is also applicable to solve n^{th} order differential equation with fuzzy boundary conditions. Here we solved all type of higher Order differential equations with fuzzy boundary conditions by using Finite difference method. The approximate and the exact solutions of given examples are nearly consistent to each-other. So, Our method is practical and applicable to solve n^{th} order differential equations with fuzzy boundary conditions.

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