Some Common Fixed Point Theorems in Fuzzy-2 Metric Space by using (CLRg) property

KAMAL WADHWA And ASHLEKHA DUBEY*
Govt. P.G. College Pipariya M.P. India,
***Sant Hirdaram Girls College Bhopal,**

Abstract : The aim of this paper is to prove some common fixed point theorems for a pair of weakly compatible mappings in fuzzy -2 metric space by using (CLRg) property.

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I. Introduction

The fundamental work for the fuzzy theory was first given by Zadeh [7] in 1965, who introduced the concept of fuzzy set. Kramosil and Michalek [6] developed the fuzzy metric space and later George and Veeramani [3] modified the notion of fuzzy metric spaces by introducing the concept of continuous t–norm.

Many researchers have extremely developed the theory by defining different concepts and amalgamation of many properties. Fuzzy set theory has its significance in various fields such as communication, gaming, signal processing, modeling theory, image processing, etc.

In 2002, Aamri and EI Moutawakil [1] defined the property (E.A.) requires the containment and closedness of ranges for the existence of fixed points. In 2009, Abbas et al. [2] introduced the notion of common property (E.A.). Later on, Sintunavarat and Kumam [5] give the idea of "Common limit in the range property "which does not require the closeness' of the subspaces for the existence of fixed point for a pair of mappings.

The purpose of this work is to prove some common fixed point theorems for a pair of weakly compatible mappings in fuzzy-2 metric spaces by using (CLRg) property.

1. Preliminaries

Definition 2.1 [6]: An operation $* : [0,1]^3 \to [0,1]$ is called a t-norm of $\{[0,1],*\}$ is an abelian topological monoid with unit 1 such that $a_1 * b_1 * c_1 \le a_2 * b_2 * c_2$ whenever $a_1 \le a_2$, $b_1 \le b_2$, $c_1 \le c_2$ for all $a_1, b_1, c_1, a_2, b_2, c_2 \in [0,1]$.

Definition 2.2 [6]: A 3-tuple $(X, M, *)$ is said to be a fuzzy 2- metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^3 \times [0, \infty)$ such that the following axioms holds:

a) $M(x, y, z, 0) = 0$;
b) $M(x, y, z, t) = 1$ for all $t > 0$ if and only if at least two of three points are equal ;
c) $M(x, y, z, t) = M(y, x, z, t) = M(z, x, y, t)$ symmetry about three variables;
d) $M(x, y, u, t_1) * M(x, u, z, t_2) * M(u, y, z, t_3) \le M(x, y, z, t_1 + t_2 + t_3)$, for all $x, y, z, u \in X$ and $t_1, t_2, t_3 > 0$;
e) $M(x, y, z, .): [0, \infty) \to [0,1]$ is left continuous;
f) $\lim_{t\to\infty} M(x, y, z, t) = 1$.

The function value $M(x, y, z, t)$ may be interpreted as the probability that the area of triangle is less than t.

Definition 2.3 [6]: Let f and g be two self-maps of a fuzzy metric space $(X, M, *)$ then they are said to be weakly compatible if they commute at their coincidence points i.e $fu = gu$ for some $\in X$, then $fgu = gfu$.

Definition 2.4 [6]: A pair of self-mappings $\{f, g\}$ of a fuzzy 2- metric spaces $(X, M, *)$ is said to satisfy E.A. property if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} M(fx_n, gx_n, a, t) = 1 \qquad \text{for some } t \in X.$$

Definition 2.5: [5]: Suppose that $(X, M, *)$ is a fuzzy metric space $f, g : X \to X$, two mappings f and g are said to satisfy the common limit in the range of g property if

$$\lim_{n\to\infty} f x_n = \lim_{n\to\infty} g x_n = gx \qquad \text{for some } x \in X.$$

The common limit in the range property will be denoted by the (CLRg) property.

Now common limit in the range property in fuzzy 2-metric spaces defines as follows:

Definition 2.6: Suppose that $(X, M,*)$ is a fuzzy 2- metric space $f, g : X \to X$, two mappings f and g are said to satisfy the common limit in the range of g property if

$$\lim_{n\to\infty} M(fx_n, gx, a, t) = \lim_{n\to\infty} M(gx_n, gx, a, t) = 1 \text{ for some } x \in X.$$

Lemma 2.7 [6]: Let $(X, M,*)$ be a fuzzy 2- metric space. If there exists a number $k \in (0,1)$, $M(x, y, z, kt) \geq M(x, y, z, t)$ for all $x, y, z \in X$ & $t > 0$ with $z \neq x, z \neq y$ and $t > 0$, then $x = y$.

Definition 2.8: Let Φ be the set of all real continuous functions $H: [0,1]^4 \to [0,1]$ non decreasing in each coordinate variable and such that $H(t, 1, t, t) \geq t$, $H(1, t, t, t) \geq t$, for all $t \in [0,1]$.

2. Main Results

Theorem 3.1 Let $(X, M,*)$ be a fuzzy-2 metric space and let f, g be weakly compatible self mappings of X such that ,

$$M(fx, fy, a, kt) \geq Min\left\{\begin{array}{c} M(fy, gy, a, t), \frac{1}{2}[M(gx, gy, a, t) + M(fx, gy, a, t)], \\ M(fy, gy, a, t) * M(gx, gy, a, t), \\ M(fy, gy, a, t) * \left(\frac{(M(gx, gy, a, t) + M(fx, gy, a, t))}{2}\right) \end{array}\right\} \quad \mathbf{(3.1.1)}$$

for all $x, y \in X$, where $t > 0$. If f and g satisfy (CLRg) property then f and g have a unique common fixed point.

Proof: It follows f and g satisfy (CLRg) property then we can find a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx \quad \text{for some } x \in X. \quad (3.1.2)$$

Let t be a continuity point of $(X, M,*)$ then

$$M(fx_n, fx, a, kt) \geq Min\left\{\begin{array}{c} M(fx, gx, a, t), \frac{1}{2}[M(gx_n, gx, a, t) + M(fx_n, gx, a, t)], \\ M(fx, gx, a, t) * M(gx_n, gx, a, t), \\ M(fx, gx, a, t) * \left(\frac{(M(gx_n, gx, a, t) + M(fx_n, gx, a, t))}{2}\right) \end{array}\right\} \quad (3.1.3)$$

for all $n \in N$.

By taking the limit n tends to infinity in (3.1.3), we have

$$M(gx, fx, a, kt) \geq Min\left\{\begin{array}{c} M(fx, gx, a, t), \frac{1}{2}[M(gx, gx, a, t) + M(gx, gx, a, t)], \\ M(fx, gx, a, t) * M(gx, gx, a, t), \\ M(fx, gx, a, t) * \left(\frac{(M(gx, gx, a, t) + M(gx, gx, a, t))}{2}\right) \end{array}\right\}$$

$$= Min\{M(fx, gx, a, t), 1, M(fx, gx, a, t), M(fx, gx, a, t)\}$$

$$= M(fx, gx, a, t)$$

Hence from the lemma (2.7) we have $fx = gx$

Next, we let $fx = gx$ =z (say).

Since f and g are weakly compatible mappings $fgx = gfx$ which implies that $fz =$

$$fgx = gfx = gz \quad (3.1.4)$$

We claim that $fz = z$, assume not, then it follows from condition (3.1.1) that

$$M(fz, z, a, kt) = M(fz, fx, a, kt) \geq$$

$$Min\left\{\begin{array}{c} M(fx, gx, a, t), \frac{1}{2}[M(gz, gx, a, t) + M(fz, gx, a, t)], \\ M(fx, gx, a, t) * M(gz, gx, a, t), \\ M(fx, gx, a, t) * \left(\frac{(M(gz, gx, a, t) + M(fz, gx, a, t))}{2}\right) \end{array}\right\}$$

$$= Min\{1, M(fz, gx, a, t), M(fz, gx, a, t), M(fz, gx, a, t)\}$$

$= M(fz, gx, a, t)$

Hence from the lemma (2.7) we have $fz = z$.

That is $z = fz = gz$.

Therefore z is a common fixed point of f and g .

For uniqueness of a common fixed point, we suppose that w is another common fixed point in which $w \neq z$. From condition (3.1.1) we have,

$$M(z,w,a,kt) = M(fz,fw,a,kt) \geq Min\left\{\begin{matrix} M(fw,gw,a,t), \frac{1}{2}[M(gz,gw,a,t) + M(fz,gw,a,t)], \\ M(fw,gw,a,t) * M(gz,gw,a,t), \\ M(fw,gw,a,t) * \left(\frac{(M(gz,gw,a,t) + M(fz,gw,a,t))}{2}\right) \end{matrix}\right\}$$

$$\geq Min\left\{\begin{matrix} M(w,w,a,t), \frac{1}{2}[M(z,w,a,t) + M(z,w,a,t)], \\ M(w,w,a,t) * M(z,w,a,t), \\ M(w,w,a,t) * \left(\frac{(M(z,w,a,t) + M(z,w,a,t))}{2}\right) \end{matrix}\right\}$$

$\geq Min\{1, M(z,w,a,t), 1 * M(z,w,a,t), 1 * M(z,w,a,t)\}$

$\geq Min\{1, M(z,w,a,t), M(z,w,a,t), M(z,w,a,t)\}$

$= M(z,w,a,t)$

Hence from the lemma (2.7) we have $z = w$, which implies that f and g have a unique a common fixed point.

Theorem 3.2 Let $(X, M,*)$ be a fuzzy- 2 metric space and let f, g be weakly compatible self mappings of X satisfying the following condition:

For some $H \in \Phi$ there exists a constant $k \in (0,1)$ such that,

$$M(fx,fy,a,kt) \geq H\left\{\begin{matrix} M(fy,gy,a,t), \frac{1}{2}[M(gx,gy,a,t) + M(fx,gy,a,t)], \\ M(fy,gy,a,t) * M(gx,gy,a,t), \\ M(fy,gy,a,t) * \left(\frac{(M(gx,gy,a,t) + M(fx,gy,a,t))}{2}\right) \end{matrix}\right\} \quad \mathbf{(3.2.1)}$$

for all $x, y \in X$, where $t > 0$. If f and g satisfy (CLRg) property then f and g have a unique common fixed point.

Proof: It follows f and g satisfy (CLRg) property then we can find a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx \quad \text{for some } x \in X. \quad (3.2.2)$$

Let t be a continuity point of $(X, M,*)$ then $M(fx_n, fx, a, kt) \geq$

$$H\left\{\begin{matrix} M(fx,gx,a,t), \frac{1}{2}[M(gx_n,gx,a,t) + M(fx_n,gx,a,t)], \\ M(fx,gx,a,t) * M(gx_n,gx,a,t), \\ M(fx,gx,a,t) * \left(\frac{(M(gx_n,gx,a,t)+M(fx_n,gx,a,t))}{2}\right) \end{matrix}\right\} \quad (3.2.3)$$

for all $n \in N$.

By taking the limit n tends to infinity in (3.2.3), we have

$$M(gx,fx,a,kt) \geq H\left\{\begin{matrix} M(fx,gx,a,t), \frac{1}{2}[M(gx,gx,a,t) + M(gx,gx,a,t)], \\ M(fx,gx,a,t) * M(gx,gx,a,t), \\ M(fx,gx,a,t) * \left(\frac{(M(gx,gx,a,t) + M(gx,gx,a,t))}{2}\right) \end{matrix}\right\}$$

$= H\{M(fx,gx,a,t), 1, M(fx,gx,a,t), M(fx,gx,a,t)\}$

Using 2.8 we have,

$$M(gx,fx,a,kt) \geq M(fx,gx,a,t)$$

Hence from the lemma (2.7) we have $fx = gx$

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We claim that $fz = z$ assume not, then it follows from condition (3.1.1) that

$$M(fz,z,a,kt) = M(fz,fx,a,kt) \geq$$

$$H\left\{\begin{matrix} M(fx,gx,a,t), \frac{1}{2}[M(gz,gx,a,t) + M(fz,gx,a,t)], \\ M(fx,gx,a,t) * M(gz,gx,a,t), \\ M(fx,gx,a,t) * \left(\frac{(M(gz,gx,a,t) + M(fz,gx,a,t))}{2}\right) \end{matrix}\right\}$$

$$= H\{1, M(fz,gx,a,t), M(fz,gx,a,t), M(fz,gx,a,t)\}$$

Using 2.8 we have,

$$M(fz,z,a,kt) \geq M(fz,gx,a,t)$$

Hence from the lemma (2.7) we have $fz = z$.

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$$\geq H\left\{\begin{matrix} M(w,w,a,t), \frac{1}{2}[M(z,w,a,t) + M(z,w,a,t)], \\ M(w,w,a,t) * M(z,w,a,t), \\ M(w,w,a,t) * \left(\frac{(M(z,w,a,t) + M(z,w,a,t))}{2}\right) \end{matrix}\right\}$$

$$\geq H\{1, M(z,w,a,t), 1 * M(z,w,a,t), 1 * M(z,w,a,t)\}$$

$$= H\{1, M(z,w,a,t), M(z,w,a,t), M(z,w,a,t)\}$$

Using 2.8 we have,

$$M(z,w,a,kt) \geq M(z,w,a,t)$$

Hence from the lemma (2.7) we have $z = w$, which implies that f and g have a unique a common fixed point.

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