ON (H,1)(C,1) SUMMABILITY OF FOURIER SERIES AND ITS CONJUGATE SERIES

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Abstract :*In this Paper, introduce the Concept of (H,1)(C,1) Product Operators and establishes two new theorems on (H,1)(C,1) Product Summability of Fourier Series and its Conjugate Series. The result obtained in the Paper further extend Several known result on linear operators*

Degree of Approximation, (C, 1) Summability,(H,1) Summability, (H,1)(C,1) Product Summability, Fourier Series, Lebesgue integral

I. INTRODUCTION

In this field of Summability of Fourier Series & its allied a Series, Several researchers like Jadia[1], Singh and Singh[2], Pandey[3],Singh[4], Khare[5], Mittal and kumar[6] have Studied $(N, P_n), (N, P, q)$, almost (N, P, q) and matrix Summability methods of Fourier series and its Conjugate series using different conditions. But nothing Seems to have been done so far to study (H,1)(C,1) Product Summability of Fourier series and its Conjugate Series . Therefore, in this Paper, two theorems on (H,1)(C,1) Summability of Fourier series and its Conjugate Series under a general Condition have been Proved.

II. DEFINITION AND NOTATION

Let$\sum_{n=0}^{\infty} u_n$ be a given infinite series with Sequence of its n^{th}Partial sum of $\{S_n\}$. The (C,1) transform is defined as the n^{th}Partial sum of The (C,1) Summability and is given by

$$t_n = \frac{S_0+S_1+S_2+\_\_\_\_\_\_+S_n}{n+1} = \frac{1}{n+1}\sum_{k=0}^{n} S_n \to S \text{ as } n \to \infty \tag{1.1}$$

Then the infinite series $\sum_{n=0}^{\infty} u_n$ is summable to the definite number S by (C, 1) method.
If

$$(\text{H}, 1) = H_n{}^1 = t_k(n) = \frac{1}{logn}\sum_{k=0}^{n} \frac{S_{n-k}}{k+1} \text{ as n} \to \infty \tag{1.2}$$

Then the infinite series $\sum_{n=0}^{\infty} u_n$ is said to be Summable (H,1) to a definite number S (Hardy[8]). The (H,1) transform of (C,1) transform defines (H,1)(C,1) transform and we denote it by $(HC)_n^1$. Thus if

$$(HC)_n^1 = \frac{1}{logn}\sum_{k=0}^{n}\left(\frac{S_{n-k}}{k+1}\right) C_k^1 \to S \text{ as } n \to \infty \tag{1.3}$$

Then the series $\sum_{n=0}^{\infty} u_n$ is said to be Summable by (H,1)(C,1) means or Summable (H,1)(C,1) to a definite number S. Therefore, we can write

$$(HC)_n^1 \to S \text{ as } n \to \infty$$

The method (H,1)(C,1) is regular and this case is Supposed throughout this Paper.

Let f(x) be a 2π- periodic function of x and integrable over [-π, π] in the sense of Lebesgue.. The Fourier series f(x) is given by

$$\text{f(x)} \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \equiv \sum_{n=1}^{\infty} A_n(x) \tag{1.4}$$

The conjugate series of Fourier series (1.4) is given by

$$\sum_{n=1}^{\infty}(a_n \cos nx - b_n \sin nx) \equiv \sum_{n=1}^{\infty} B_n(\mathbf{x}) \tag{1.5}$$

We shall use the following notations:

$$\Phi(t) = f(x+t) + f(x-t) - 2f(x)$$

$$\Psi(t) = f(x+t) + f(x-t)$$

$$K_n(t) = \frac{1}{2\pi . \log n}\sum_{k=0}^{n}\left\{\frac{1}{(K+1)^2}\sum_{v=0}^{k}\frac{\sin\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}$$

$$\widetilde{K_n}(t) = \frac{1}{2\pi \log n}\sum_{k=0}^{n}\left\{\frac{1}{(K+1)^2}\sum_{v=0}^{k}\frac{\cos\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}$$

And $\tau = \left[\frac{1}{t}\right]$, where τ denotes the greatest integer not greater than $\frac{1}{t}$

2. MAIN THEOREMS

We will prove the following theorems,

2.1 Theorem 1. Let $\{p_n\}$ be a positive, monotonic, non-increasing sequence of real constants such that

$$p_n = \sum_{v=0}^{n} p_v \to \infty \text{ as } n \to \infty$$

if,

$$\emptyset(t) = \int_0^t |\emptyset(u)| du = o\left[\frac{t}{\alpha(\frac{1}{t}), p_\tau}\right], \text{as } t \to +0 \qquad (2.1)$$

Where α (t) is positive, monotonic and non-increasing function of t and

$$\log n = O[\{\alpha(n)\}. p_n], \text{as } n \to \infty \qquad (2.2)$$

Then the Fourier series (1.4) is summable (H, 1)(C,1) to f(x).

2.2 Theorem 2. Let $\{p_n\}$ be a positive, monotonic, non-increasing sequence of real constants such that

$$p_n = \sum_{v=0}^{n} p_v \to \infty \text{ as } n \to \infty$$

If,

$$\psi(t) = \int_0^t |\psi(u)| du = o\left[\frac{t}{\alpha(\frac{1}{t}), p_\tau}\right], \text{as } t \to +0 \qquad (2.3)$$

where α(t) is a positive, monotonic and non-increasing function of t and condition (2.2), then the conjugate series (1.5) is summable to (H,1)(C,1) to

$$\tilde{f}(x) = \frac{-1}{2\pi} \int_0^{2\pi} \psi(t) \cot\left(\frac{t}{2}\right) dt$$

At any point where this integral exists.

3. LEMMAS

For the Proof of our theorems, following Lemmas are required.

Lemma 1. $|k_n(t)| = O(n),\quad \text{for } 0 \le t \le \frac{1}{n}$

Proof: For $0 \le t \le \frac{1}{n}$; $\sin nt \le n \sin t$; $|\cos nt| \le 1$

$$|k_n(t)| \le \frac{1}{2\pi . \log n}\left|\sum_{k=0}^{n}\left[\frac{1}{(K+1)^2}\sum_{v=0}^{k}\frac{\sin\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right]\right|$$

$$\le \frac{1}{2\pi . \log\, n}\left|\sum_{k=0}^{n}\left[\frac{1}{(K+1)^2}\sum_{v=0}^{k}\frac{(2v+1)\sin\frac{t}{2}}{\sin\frac{t}{2}}\right]\right|$$

$$\le \frac{1}{2\pi . \log n}\left|\sum_{k=0}^{n}\left[\frac{1}{(K+1)^2}(2k+1)\right]\right|$$

$$= \frac{1}{2\pi . \log n}(2n+1)\sum_{k=0}^{n}\frac{1}{(K+1)^2}$$

$$= \frac{2n+1}{2\pi . \log n}$$

$$= O(n)$$

Lemma 2. $|K_n(t)| = o\left(\frac{1}{tn}\right), \text{for } \frac{1}{n} \le t \le \pi$

Proof:- For $\frac{1}{n} \le t \le \pi$; $\sin(t/2) \ge t/\pi$ and $\sin nt \le 1$

$$|\mathbf{k_n(t)}| \le \frac{\mathbf{1}}{2\pi . \log n}\left|\sum_{\mathbf{k=0}}^{\mathbf{n}}\left[\frac{1}{(K+1)^2}\sum_{v=0}^{k}\frac{\sin\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right]\right|$$

$$\le \frac{\mathbf{1}}{2\pi . \log n}\left|\sum_{\mathbf{k=0}}^{\mathbf{n}}\left[\frac{1}{(K+1)^2}\sum_{\mathbf{v=0}}^{\mathbf{k}} . \frac{\mathbf{1}}{\left(\frac{\mathbf{t}}{\boldsymbol{\pi}}\right)}\right]\right|$$

$$= \frac{\mathbf{1}}{2t . \log n}\left|\sum_{\mathbf{k=0}}^{\mathbf{n}}\left[\frac{1}{(K+1)^2}\sum_{\mathbf{v=0}}^{\mathbf{k}}(\mathbf{1})\right]\right|$$

$$= \frac{\mathbf{1}}{2t . \log n}\left[\sum_{\mathbf{k=0}}^{\mathbf{n}}\frac{1}{(K+1)^2}\right]$$

$$= \frac{\mathbf{1}}{2t . \log n}$$

$$= O\left(\frac{1}{tn}\right)$$

Lemma 3. $\widetilde{k_n}(t) = O\left(\frac{1}{tn}\right)$, for $0 \le t \le \frac{1}{n}$

Proof:- For $0 \le t \le \frac{1}{n}$; $\sin(t/2) \ge t/\pi$ and $|\cos nt| \le 1$

$$|\widetilde{k_n}(t)| \le \frac{1}{2\pi.\log n}\left|\sum_{k=0}^{n}\left[\frac{1}{(K+1)^2}\sum_{v=0}^{k}\frac{\cos\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right]\right|$$

$$\le \frac{1}{2\pi.\log n}\sum_{k=0}^{n}\left[\frac{1}{(K+1)^2}\sum_{v=0}^{k}\left|\frac{\cos\left(v+\frac{1}{2}\right)t}{\sin t/2}\right|\right]$$

=

$$= \frac{1}{2t.\log n}\left[\sum_{k=0}^{n}\frac{1}{(K+1)^2}\right]$$

$$= \frac{1}{2t.\log n}$$

$$= O\left(\frac{1}{tn}\right)$$

Lemmas 4. For $0 \le a \le b \le \infty$, $0 \le t \le \pi$ and any n, we have

$$|\widetilde{k_n}(t)| = O\left(\frac{1}{tn}\right)$$

Proof:- For $\frac{1}{n} \le t \le \pi, \sin(t/2) \ge t/\pi$

$$|\widetilde{k_n}(t)| \le \frac{1}{2\pi.\log n}\left|\sum_{k=0}^{n}\left\{\frac{1}{(K+1)^2}\sum_{v=0}^{k}\frac{\cos\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right|$$

$$\le \frac{1}{2t.\log n}\left|\sum_{k=0}^{n}\left[\frac{1}{(K+1)^2}\mathrm{Re}\left\{\sum_{v=0}^{k}e^{i\left(v+\frac{1}{2}\right)t}\right\}\right]\right|$$

$$\le \frac{1}{2t.\log n}\left|\sum_{k=0}^{n}\left[\frac{1}{(K+1)^2}\mathrm{Re}\left\{\sum_{v=0}^{k}e^{ivt}\right\}\right]\right|\left|e^{it/2}\right|$$

$$\le \frac{1}{2t.\log n}\left|\sum_{k=0}^{n}\left[\frac{1}{(K+1)^2}\mathrm{Re}\left\{\sum_{v=0}^{k}e^{ivt}\right\}\right]\right|$$

$$\le \frac{1}{2t.\log n}\left|\sum_{k=0}^{\tau-1}\left[\frac{1}{(K+1)^2}\mathrm{Re}\left\{\sum_{v=0}^{k}e^{ivt}\right\}\right]\right| +$$

$$\frac{1}{2t.\log n}\left|\sum_{k=\tau}^{n}\left[\frac{1}{(K+1)^2}\mathrm{Re}\{\sum_{v=0}^{k}e^{ivt}\}\right]\right| \qquad (3.1)$$

$$= k_1 + k_2$$

Now condition first term of (3.1)

$$|k_1| \le \frac{1}{2t.\log n}\left|\sum_{k=0}^{\tau-1}\left[\frac{1}{(K+1)^2}\mathrm{Re}\left\{\sum_{v=0}^{k}e^{ivt}\right\}\right]\right|$$

$$\le \frac{1}{2t.\log n}\left|\sum_{k=0}^{\tau-1}\left[\frac{1}{(K+1)^2}\sum_{v=0}^{k}1\right]\right|\left|e^{ivt}\right|$$

$$\le \frac{1}{2t.\log n}\sum_{k=0}^{\tau-1}\frac{1}{(K+1)^2}$$

$$= \frac{1}{2t.\log n}$$

$$= O\left(\frac{1}{tn}\right) \qquad (3.2)$$

And

$$|k_2| \le \frac{1}{2t.\log n}\left|\sum_{k=\tau}^{n}\left[\frac{1}{(K+1)^2}\mathrm{Re}\left\{\sum_{v=0}^{k}e^{ivt}\right\}\right]\right|$$

$$\leq \frac{1}{2t.\log n}\sum_{k=\tau}^{n}\frac{1}{(K+1)^2}\max_{0\leq m\leq k}\left|\sum_{v=0}^{k}e^{ivt}\right|$$

$$\leq \frac{1}{2t.\log n}\sum_{k=\tau}^{n}\frac{1}{(K+1)^2}(1+k)$$

$$= \frac{1}{2t.\log n}\sum_{k=\tau}^{n}\frac{1}{(k+1)}$$

$$= O\left(\frac{1}{tn}\right) \qquad (3.3)$$

Combining (3.1) (3.2) and (3.3), we get

$$|\widetilde{k_n}(t)| \leq O\left(\frac{1}{tn}\right)$$

4. PROOF OF MAIN THEOREMS

Proof of theorem 1.

Following Titchmarsh [9] and using Riemann-Lebesgue theorem, $S_n(f;x)$ of the series (1.4) is given by

$$S_n(f;x) - f(x) = \frac{1}{2\pi}\int_0^{\pi}\emptyset(t)\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt$$

Therefore using (1.1), the (C, 1), transform C_n^1 of $S_n(f;x)$ is given by

$$C_n^1 - f(x) = \frac{1}{2\pi(n+1)}\int_0^{\pi}\emptyset(t)\left\{\sum_{v=0}^{k}\frac{\sin\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}dt$$

Now, denoting (H, 1)(C,1) transform of $S_n(f;x)$ by $(HC)_n^1$,we write

$$(HC)_n^1 - f(x) = \frac{1}{2\pi.\log n}\sum_{k=0}^{n}\left[\frac{1}{(K+1)}\int_0^{\pi}\frac{\emptyset(t)}{\sin\frac{t}{2}}\left(\frac{1}{K+1}\right)\left\{\sum_{v=0}^{k}\sin\left(v+\frac{1}{2}\right)t\right\}dt\right]$$

$$= \int_0^{\pi}\emptyset(t)k_n(t)dt \qquad (4.1)$$

we have to show that, under the hypothesis of theorem

$$\int_0^{\pi}\emptyset(t)k_n(t)dt = o(1), \text{as } n\to\infty$$

For $0<\delta<\pi$, We have

$$\int_0^{\pi}\emptyset(t)k_n(t)dt = \left[\int_0^{1/n}\emptyset(t) + \int_{1/n}^{\delta}\emptyset(t) + \int_{\delta}^{\pi}\emptyset(t)\right]k_n(t)dt$$

$$= I_1 + I_2 + I_3 \text{ (say)} \qquad (4.2)$$

We consider,

$$|I_1| \leq \int_0^{1/n}|\emptyset(t)||k_n(t)|dt$$

$$=O(n)\left[\int_0^{1/n}|\emptyset(t)|dt\right] \text{ (using Lemma 1)}$$

$$= O(n)\left[o\left\{\frac{1}{n\alpha(n).p_n}\right\}\right] \text{ by (2.1}$$

$$= o\left\{\frac{1}{\alpha(n).p_n}\right\}$$

$$= o\left\{\frac{1}{\log n}\right\} \text{ using(2.2)}$$

$$= o(1), \text{as } n\to\infty \qquad (4.3)$$

Now we consider

$$|I_2| \leq \int_{1/n}^{\delta}|\emptyset(t)||k_n(t)|dt$$

$$= O\left[\int_{1/n}^{\delta}|\emptyset(t)|\left(\frac{1}{tn}\right)dt\right] \text{ (using Lemma 2)}$$

$$= O\left(\frac{1}{n}\right)\left[\left\{\frac{1}{t}\emptyset(t)\right\}_{1/n}^{\delta} + \int_{1/n}^{\delta}\frac{1}{t^2}\emptyset(t)\,dt\right]$$

$$= O\left(\frac{1}{n}\right)\left[o\left\{\frac{1}{\alpha\left(\frac{1}{t}\right).p_t}\right\}_{1/n}^{\delta} + \int_{1/n}^{\delta}o\left(\frac{1}{t\alpha\left(\frac{1}{t}\right).p_t}\right)dt\right] \text{ by (2.1)}$$

Putting $\frac{1}{t}=u$ in second term,

$$= O\left(\frac{1}{n}\right)\left[o\left\{\frac{1}{\alpha(n).p_n}\right\} + \int_{1/\delta}^{n} o\left(\frac{1}{u\alpha(u).p_u}\right) du\right]$$
$$= o\left\{\frac{1}{\alpha(n).p_n}\right\} + o\left\{\frac{1}{n\alpha(n).p_n}\right\}\int_{1/\delta}^{n} 1.du$$
$$= o\left\{\frac{1}{\log n}\right\} + o\left\{\frac{1}{\log n}\right\} \text{ by (2.2)}$$

Using second mean value theorem for the integral in the second term as α (n) is monotonic

$$= o(1) + o(1) \text{ as}, n \to \infty$$
$$= o(1), \text{as } n \to \infty \tag{4.4}$$

Now by Riemann-Lebesgue theorem and by regularity condition of the method of Summabilty, we have

$$|I_3| \le \int_{\delta}^{\pi} |\emptyset(t)||k_n(t)|dt$$
$$= o(1), \text{as } n \to \infty \tag{4.5}$$

Combining (4.3), (4.4) and (4.5) we have

$$(HC)_n^1 - f(x) = o(1), \text{as } n \to \infty$$

This completes the proof of theorem 1.

Proof of Theorem 2. Let $\tilde{s_n}(f;x)$ denotes the partial sum of series (1.5).

Then following Lal [7] and using Riemann-Lebesgue Theorem ,$\tilde{s_n}(f;x)$ of series (1.5) is given by

$$\tilde{s_n}(f;x) - \tilde{f}(x) = \frac{1}{2\pi}\int_0^{\pi} \Psi(t)\frac{\cos\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt$$

Therefore, using (1.5), the (C, 1) transform C_n^1 of $\tilde{s_n}(f;x)$ is given by

$$\tilde{C}_n^1 - \tilde{f}(x) = \frac{1}{2\pi(n+1)}\int_0^{\pi}\psi(t)\left\{\sum_{v=0}^{k}\frac{\cos\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}dt$$

Now denoting $\overline{(H,1)(C,1)}$ transform of $\tilde{s_n}(f;x)$ by $(\overline{HC})_n^1$, we write

$$(\overline{HC})_n^1 - \tilde{f}(x) = \frac{1}{2\pi.\log n}\sum_{k=0}^{n}\left[\frac{1}{(K+1)}\int_0^{\pi}\frac{\psi(t)}{\sin\frac{t}{2}}\left(\frac{1}{K+1}\right)\left\{\sum_{v=0}^{k}\cos\left(v+\frac{1}{2}\right)t\right\}dt\right]$$
$$= \int_0^{\pi}\psi(t)\widetilde{k_n}(t)dt \tag{4.6}$$

In order to prove the Theorem, we have to show that, under the hypothesis of theorem

$$\int_0^{\pi}\psi(t)\widetilde{k_n}(t)dt = o(1) \quad \text{as } n\to\infty$$

For0< δ < π, we have

$$\int_0^{\pi}\psi(t)\widetilde{k_n}(t)dt = \left[\int_0^{1/n}\psi(t) + \int_{1/n}^{\delta}\psi(t) + \int_{\delta}^{\pi}\psi(t)\right]\widetilde{k_n}(t)dt$$
$$= J_1 + J_2 + J_3 \quad \text{(Say)} \tag{4.7}$$

Now We consider,

$$|J_1 \le |\int_0^{1/n} |\psi(t)|\,|\widetilde{k_n(t)}|dt$$
$$= O\left[\int_0^{1/n}\left(\frac{1}{tn}\right)|\psi(t)|dt\right] \text{(using Lemma 3)}$$
$$= O\left(\frac{1}{n}\right)\left[\int_0^{1/n}\frac{1}{t}|\psi(t)|\,dt\right]$$
$$= O(n)\left(\frac{1}{n}\right)\left[o\left\{\frac{1}{n\alpha(n).p_n}\right\}\right] \text{ by(2.3)}$$
$$= o\left\{\frac{1}{\alpha(n).p_n}\right\}$$
$$= o\left\{\frac{1}{\log n}\right\} \quad \text{using (2.2)}$$
$$= o(1), \text{ as } n \to \infty \tag{4.8}$$

Now, $$|J_2| \le \int_{1/n}^{\delta}|\psi(t)|\,|\widetilde{k_n}(t)|dt$$
$$= O\left[\int_{1/n}^{\delta}\left(\frac{1}{tn}\right)|\psi(t)|dt\right] \text{ by lemma 4}$$
$$= O\left(\frac{1}{n}\right)\left[\int_{1/n}^{\delta}\frac{1}{t}|\psi(t)|dt\right]$$
$$= O\left(\frac{1}{n}\right)\left[\left\{\frac{1}{t}\psi(t)\right\}_{1/n}^{\delta} + \int_{1/n}^{\delta}\frac{1}{t^2}\psi(t)dt\right]$$
$$= O\left(\frac{1}{n}\right)\left[o\left\{\frac{1}{\alpha\left(\frac{1}{t}\right)p_t}\right\}_{1/n}^{\delta} + \int_{1/n}^{\delta} o\left(\frac{1}{t\alpha\left(\frac{1}{t}\right).p_t}\right)dt\right] \text{ by (2.3)}$$

Putting $\frac{1}{t}$ = u, in second term,

$$= O\left(\frac{1}{n}\right)\left[o\left\{\frac{1}{\alpha(n).p_n}\right\} + \int_{1/\delta}^{n} o\left(\frac{1}{u\alpha(u).p_t}\right) du\right]$$

$$= o\left\{\frac{1}{\alpha(n).p_n}\right\} + o\left\{\frac{1}{n\alpha(n).p_n}\right\}\int_{1/\delta}^{n} 1.\, du$$

$$= o\left\{\frac{1}{\log n}\right\} + o\left\{\frac{1}{\log n}\right\} \text{ by } (2.2)$$

Using second -mean value theorem for the integral in the second term as $\alpha(n)$ is monotonic

$$= o(1) + o(1) \text{ , as } n \to \infty$$

$$= o(1) \text{ , as } n \to \infty \qquad (4.9)$$

Now by Riemann – Lebesgue theorem and by regularity condition of the method of Summability, we have

$$| J_3 | \leq \int_{\delta}^{\pi} |\psi(t)| \left|\widetilde{k_n}(t)\right| dt$$

$$= o(1) \text{ , as } n \to \infty \qquad (4.10)$$

Combining (4.8), (4.9) and (4.10) we get,

$$(\overline{HC})_n^1 - \tilde{f}(x) = o(1), \text{ as } n \to \infty$$

This completes the proof of theorem 2.

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