CERTAIN PROPERTIES OF THE NORMALIZED WRIGHT FUNCTION

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Abstract:-In this paper, we establish some properties for the normalized Wright function, to be included in the class of k-parabolic starlike respectively k-parabolic uniformly convex functions of order γ and also obtain some properties for the class $k - S_p(\lambda, \gamma)$,

$k - UCV(\lambda, \gamma)$.

Key words:-Wright function, k-parabolic starlike function, convex function, k- uniformly convex function.

1. Introduction & Definitions:-

Let $U = \{z \in \mathbb{C}: |z| < 1\}$ be an open unit disk in complex plane $\mathbb{C}$, centered at zero, A be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which is analytic in the open unit disk U.

We denote the class of starlike functions f with respect to origin is S^* [2], where f is starlike in U if $f: U \to \mathbb{C}$ is univalent and f(U) is starlike domain in $\mathbb{C}$ with respect to origin. It is well known $f \in A$ is starlike in U if and only if

$$f \in S^* \Leftrightarrow \Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \forall z \in U.$$

We denote the class of convex functions $f \in A$ by $\mathcal{K}$ [2], $f \in A$ is convex in U if $f: U \to \mathbb{C}$ is univalent and f(U) is convex domain in $\mathbb{C}$. The function $f \in A$ is convex in U if and only if

$$f \in \mathcal{K} \Leftrightarrow \Re\left(\frac{zf''(z)}{f'(z)} + 1\right) > 0, \forall z \in U.$$

A function $f \in A$ is said to be uniformly convex in U if f is a univalent convex function along with the property that, for every circular arc γ contained in U, with center ζ, also in U, the arc f(γ) is convex. [1]The class of uniformly convex functions is denoted by UCV

$$f \in UCV \Leftrightarrow \Re\left(\frac{zf''(z)}{f'(z)} + 1\right) > \left|\frac{zf''(z)}{f'(z)}\right|, \forall z \in U.$$

Definition 1.1 In [7], RØnning defined the class of parabolic starlike function

$$S_p = \{F \in S^* | F(z) = zf'(z), f \in UCV\}.$$

The class S_p of parabolic starlike functions consists of functions $f \in A$ satisfying

$$f \in S_p \Leftrightarrow \Re\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right|, \forall z \in U. \quad (1)$$

Definition 1.2 In [3, 4], A function $f \in A$ is said to be in the class, of k-parabolic starlike functions of order γ, denoted by $k - S_p(\gamma)$ if

$$f \in \mathrm{k} - S_p(\gamma) \Leftrightarrow \Re\left(\frac{zf'(z)}{f(z)} - \gamma\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right|, \forall\, z \in U \tag{2}$$

and $k - S_p = \{F \in S^* | F(z) = zf'(z), f \in k - UCV\}$, where -1 < γ ≤ 1 and k ≥ 0.

Definition 1.3 In [4, 5], A function $f \in A$ is said to be in the class, of k-uniformly convex functions of order γ, denoted by $k - UCV(\gamma)$ if

$$f \in \mathrm{k} - UCV(\gamma) \Leftrightarrow \Re\left(1 + \frac{zf''(z)}{f'(z)} - \gamma\right) > k\left|\frac{zf''(z)}{f'(z)}\right|, \forall\, z \in U \tag{3}$$

where -1 < γ ≤ 1 and k ≥ 0.

Definition 1.4 In [2], the authors generalized the class of k-parabolic starlike, k- uniformly convex functions of order γ for $0 \leq \gamma < 1, 0 \leq \lambda < 1$ and k ≥ 0, the function $f \in A$ belonges to the class $k - S_p(\lambda, \gamma)$if

$$f \in \mathrm{k} - S_p(\lambda, \gamma) \Leftrightarrow \Re\left(\frac{zf'(z)}{(1-\lambda)f(z) + \lambda zf'(z)} - \gamma\right) > k\left|\frac{zf'(z)}{(1-\lambda)f(z) + \lambda zf'(z)} - 1\right| \tag{4}$$

$$\forall\, z \in U$$

For the above same condition

$$f \in \mathrm{k} - UCV(\lambda, \gamma) \Leftrightarrow \Re\left(\frac{f'(z) + zf''(z)}{f'(z) + \lambda zf''(z)} - \gamma\right) > k\left|\frac{f'(z) + zf''(z)}{f'(z) + \lambda zf''(z)} - 1\right| \tag{5}$$

If we put 1) $\lambda = 0$ in (4), (5)$\Longrightarrow \mathrm{k} - S_p(0, \gamma) = \mathrm{k} - S_p(\gamma)$

$$\Longrightarrow \mathrm{k} - UCV(0, \gamma) = \mathrm{k} - UCV(\gamma)$$

2) $\lambda = 0, \gamma = 0$ In (4), (5)$\Longrightarrow \mathrm{k} - S_p(0,0) = \mathrm{k} - S_p$

$$\Longrightarrow \mathrm{k} - UCV(0,0) = \mathrm{k} - UCV$$

where $0 \leq \gamma < 1, 0 \leq \lambda < 1$.

The Wright function is defined as

$$W_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{1}{n!\,\Gamma(\alpha n + \beta)} z^n \tag{6}$$

where $\alpha > -1, \beta \in \mathbb{C}, z \in U$ was introduced by Wright [9]. This series is absolutely convergent in $\mathbb{C}$ for $\alpha > -1$ and absolutely convergent in open unit disk U for $\alpha = -1$.

The Wright function is defined as also in [6]

$$W_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\beta)}{n!\,\Gamma(\alpha n+\beta)} z^n$$

$$\alpha > -1, \beta > 0, z \in U, \alpha+\beta > 0.$$

The pochhammer symbol, defined in terms of Euler's gamma function is given as

$$(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)} = \left\{ \begin{matrix} 1, if\ (n=0) \\ z(z+1)\ldots.(z+n-1)\,, if\ (n \in \mathbb{N}) \end{matrix} \right\}.$$

Wright function can be represented in the terms of generalized hypergeometric function, Bessel function, Fox H-function, Meijer G-function [8] etc.

Normalization form of the Wright function [6]

$$zW_{\alpha,\beta}(z) = \mathcal{W}_{\alpha,\beta}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta)}{(n-1)!\,\Gamma[\alpha(n-1)+\beta]} z^n \tag{7}$$

In [6], the authors have proved that if

i. $\alpha \geq 1, \beta > \{(2-\alpha)+\sqrt{5\alpha^2-16\alpha+12}\}/2(1-\alpha)$ then $\mathcal{W}_{\alpha,\beta}$ is starlike.

ii. $\alpha \geq 1, \beta > \{(4-\alpha)+\sqrt{5\alpha^2-28\alpha+32}\}/2(1-\alpha)$ then $\mathcal{W}_{\alpha,\beta}$ is convex.

2. Main results:-

In this article, we find some sufficient conditions of the normalized Wright function $\mathcal{W}_{\alpha,\beta}$ to be in the class $S^*, \mathcal{K}, UCV, S_p, \mathrm{k}-S_p(\gamma), \mathrm{k}-UCV(\gamma), k-S_p(\lambda,\gamma)$ and $k-UCV(\lambda,\gamma)$.

Theorem 2.1 $\alpha \geq 0, \beta > 0, k \geq 0$ and $0 \leq \gamma < 1$ if

$$\sum_{n=2}^{\infty} \frac{[(n-1)(k+1)+(1-\gamma)]}{\Gamma(n)\Gamma[\alpha(n-1)+\beta]} \leq \frac{(1-\gamma)}{\Gamma(\beta)}$$

then $\mathcal{W}_{\alpha,\beta} \in \mathrm{k}-S_p(\gamma)$.

Proof:-It is sufficient to show that

$$k\left|\frac{z\mathcal{W}'_{\alpha,\beta}(z)}{\mathcal{W}_{\alpha,\beta}(z)}-1\right| - \Re\left(\frac{z\mathcal{W}'_{\alpha,\beta}(z)}{\mathcal{W}_{\alpha,\beta}(z)}-1\right) \leq (1-\gamma)$$

Now, we have

$$k\left|\frac{z\mathcal{W}'_{\alpha,\beta}(z)}{\mathcal{W}_{\alpha,\beta}(z)}-1\right| - \Re\left(\frac{z\mathcal{W}'_{\alpha,\beta}(z)}{\mathcal{W}_{\alpha,\beta}(z)}-1\right) \leq (1+k)\left|\frac{z\mathcal{W}'_{\alpha,\beta}(z)}{\mathcal{W}_{\alpha,\beta}(z)}-1\right| \leq (1-\gamma)$$

The above inequality is equivalent to

$$\left|\frac{z\mathcal{W}'_{\alpha,\beta}(z)}{\mathcal{W}_{\alpha,\beta}(z)}-1\right| \leq \frac{(1-\gamma)}{(1+k)}$$

$$\left|\frac{z\left\{1+\sum_{n=2}^{\infty}\frac{\mathrm{n}\Gamma(\beta)}{\Gamma(n)\Gamma[\alpha(n-1)+\beta]}z^{n-1}\right\}}{z+\sum_{n=2}^{\infty}\frac{\Gamma(\beta)}{\Gamma(n)\Gamma[\alpha(n-1)+\beta]}z^{n}}-1\right|\leq\frac{(1-\gamma)}{(1+k)}$$

$$\left|\frac{z+\sum_{n=2}^{\infty}\frac{\mathrm{n}\Gamma(\beta)}{\Gamma(n)\Gamma[\alpha(n-1)+\beta]}z^{n}-z-\sum_{n=2}^{\infty}\frac{\Gamma(\beta)}{\Gamma(n)\Gamma[\alpha(n-1)+\beta]}z^{n}}{z+\sum_{n=2}^{\infty}\frac{\Gamma(\beta)}{\Gamma(n)\Gamma[\alpha(n-1)+\beta]}z^{n}}\right|\leq\frac{(1-\gamma)}{(1+k)}$$

$$\left|\frac{\sum_{n=2}^{\infty}\frac{(\mathrm{n}-1)\Gamma(\beta)}{\Gamma(n)\Gamma[\alpha(n-1)+\beta]}z^{n}}{z+\sum_{n=2}^{\infty}\frac{\Gamma(\beta)}{\Gamma(n)\Gamma[\alpha(n-1)+\beta]}z^{n}}-1\right|\leq\frac{\sum_{n=2}^{\infty}\frac{(\mathrm{n}-1)\Gamma(\beta)}{\Gamma(n)\Gamma[\alpha(n-1)+\beta]}|z|^{n}}{1-\sum_{n=2}^{\infty}\frac{\Gamma(\beta)}{\Gamma(n)\Gamma[\alpha(n-1)+\beta]}|z|^{n}}\leq\frac{(1-\gamma)}{(1+k)}$$

Considering $z\to 1$ along to the real axis

$$\sum_{n=2}^{\infty}\frac{(\mathrm{n}-1)\Gamma(\beta)}{\Gamma(n)\Gamma[\alpha(n-1)+\beta]}\leq\frac{(1-\gamma)}{(1+k)}\left(1-\sum_{n=2}^{\infty}\frac{\Gamma(\beta)}{\Gamma(n)\Gamma[\alpha(n-1)+\beta]}\right)$$

$$\sum_{n=2}^{\infty}\frac{(n-1)\Gamma(\beta)(k+1)+(1-\gamma)\Gamma(\beta)}{\Gamma(n)\Gamma[\alpha(n-1)+\beta]}\leq(1-\gamma)$$

$$\sum_{n=2}^{\infty}\frac{[(n-1)(k+1)+(1-\gamma)]}{\Gamma(n)\Gamma[\alpha(n-1)+\beta]}\leq\frac{(1-\gamma)}{\Gamma(\beta)}$$

Finally, we obtain the condition (8) so $\mathcal{W}_{\alpha,\beta}\in \mathrm{k}-S_p(\gamma)$.

Theorem 2.2 $\alpha\geq 0,\beta>0,k\geq 0$ and $0\leq\gamma<1$ if

$$\sum_{n=2}^{\infty}\frac{n[(n-1)(k+1)+(1-\gamma)]}{\Gamma(n)\Gamma[\alpha(n-1)+\beta]}\leq\frac{(1-\gamma)}{\Gamma(\beta)} \tag{9}$$

then $\mathcal{W}_{\alpha,\beta}\in \mathrm{k}-UCV(\gamma)$.

Proof:-It is sufficient to show that

$$k\left|\frac{z\mathcal{W}''_{\alpha,\beta}(z)}{\mathcal{W}'_{\alpha,\beta}(z)}\right|-\Re\left(\frac{z\mathcal{W}''_{\alpha,\beta}(z)}{\mathcal{W}'_{\alpha,\beta}(z)}\right)\leq(1-\gamma)$$

Now, we have

$$k\left|\frac{z\mathcal{W}''_{\alpha,\beta}(z)}{\mathcal{W}'_{\alpha,\beta}(z)}\right|-\Re\left(\frac{z\mathcal{W}''_{\alpha,\beta}(z)}{\mathcal{W}'_{\alpha,\beta}(z)}\right)\leq(1+k)\left|\frac{z\mathcal{W}''_{\alpha,\beta}(z)}{\mathcal{W}'_{\alpha,\beta}(z)}\right|\leq(1-\gamma)$$

The above inequality is equivalent to

$$\left|\frac{z\mathcal{W}''_{\alpha,\beta}(z)}{\mathcal{W}'_{\alpha,\beta}(z)}\right|\leq\frac{(1-\gamma)}{(1+k)}$$

Considering the first and second derivative of the Wright function, then obtain

$$\left|\frac{\sum_{n=2}^{\infty}\frac{n(n-1)\Gamma(\beta)}{\Gamma(n)\Gamma[\alpha(n-1)+\beta]}z^n}{1+\sum_{n=2}^{\infty}\frac{\Gamma(\beta)}{\Gamma(n)\Gamma[\alpha(n-1)+\beta]}z^n}\right| \le \frac{\sum_{n=2}^{\infty}\frac{n(n-1)\Gamma(\beta)}{\Gamma(n)\Gamma[\alpha(n-1)+\beta]}|z|^n}{1-\sum_{n=2}^{\infty}\frac{\Gamma(\beta)}{\Gamma(n)\Gamma[\alpha(n-1)+\beta]}|z|^n} \le \frac{(1-\gamma)}{(1+k)}$$

Letting $z \to 1$ along to the real axis, we obtain

$$\sum_{n=2}^{\infty}\frac{n(n-1)\Gamma(\beta)}{\Gamma(n)\Gamma[\alpha(n-1)+\beta]} \le \frac{(1-\gamma)}{(1+k)}\left(1-\sum_{n=2}^{\infty}\frac{\Gamma(\beta)}{\Gamma(n)\Gamma[\alpha(n-1)+\beta]}\right)$$

$$\sum_{n=2}^{\infty}\frac{n(n-1)\Gamma(\beta)(k+1)+n(1-\gamma)\Gamma(\beta)}{\Gamma(n)\Gamma[\alpha(n-1)+\beta]} \le (1-\gamma)$$

$$\sum_{n=2}^{\infty}\frac{n[(n-1)(k+1)+(1-\gamma)]}{\Gamma(n)\Gamma[\alpha(n-1)+\beta]} \le \frac{(1-\gamma)}{\Gamma(\beta)}$$

Finally, we obtain the required result (9).

For k=1 and γ=0, we obtain the following properties for the classes S_p and UCV.

Corollary 2.1 Let α, β >0, k=1 and γ =0 then

$$\sum_{n=2}^{\infty}\frac{[2n-1]}{\Gamma(n)\Gamma[\alpha(n-1)+\beta]} \le \frac{1}{\Gamma(\beta)} \tag{10}$$

then $\mathcal{W}_{\alpha,\beta} \in \mathrm{k}-S_p$.

Corollary 2.2 Let α, β >0, k=1 and γ =0 then

$$\sum_{n=2}^{\infty}\frac{[n(2n-1)]}{\Gamma(n)\Gamma[\alpha(n-1)+\beta]} \le \frac{1}{\Gamma(\beta)} \tag{11}$$

then $\mathcal{W}_{\alpha,\beta} \in \mathrm{k}-\mathrm{UCV}$.

Theorem 2.3 $\alpha \ge 0, \beta > 0, k \ge 0, 0 \le \gamma < 1$ and $0 \le \lambda < 1$ if

$$\sum_{n=2}^{\infty}\frac{(n-1)[1-\lambda\gamma+k(1-\lambda)]+(1-\gamma)}{\Gamma(n)\Gamma[\alpha(n-1)+\beta]} \le \frac{(1-\gamma)}{\Gamma(\beta)} \tag{12}$$

then $\mathcal{W}_{\alpha,\beta} \in \mathrm{k}-S_p(\lambda,\gamma)$.

Proof:-By using the definition (4)

$$\Re\left(\frac{z\mathcal{W}'_{\alpha,\beta}}{(1-\lambda)\mathcal{W}_{\alpha,\beta}+\lambda z\mathcal{W}'_{\alpha,\beta}}-\gamma\right) > k\left|\frac{z\mathcal{W}'_{\alpha,\beta}}{(1-\lambda)\mathcal{W}_{\alpha,\beta}+\lambda z\mathcal{W}'_{\alpha,\beta}}-1\right|$$

The above inequality is equivalent to

$$(1+k)\left|\frac{z\mathcal{W}'_{\alpha,\beta}}{(1-\lambda)\mathcal{W}_{\alpha,\beta}+\lambda z\mathcal{W}'_{\alpha,\beta}}-1\right|\leq(1-\gamma)$$

$$(1+k)\left|\frac{z\mathcal{W}'_{\alpha,\beta}-(1-\lambda)\mathcal{W}_{\alpha,\beta}-\lambda z\mathcal{W}'_{\alpha,\beta}}{(1-\lambda)\mathcal{W}_{\alpha,\beta}+\lambda z\mathcal{W}'_{\alpha,\beta}}\right|\leq(1-\gamma)$$

$$(1-\lambda)(1+k)\left|\frac{z\mathcal{W}'_{\alpha,\beta}-\mathcal{W}_{\alpha,\beta}}{(1-\lambda)\mathcal{W}_{\alpha,\beta}+\lambda z\mathcal{W}'_{\alpha,\beta}}\right|\leq(1-\gamma)$$

$$(1-\lambda)(1+k)\left|\frac{\sum_{n=2}^{\infty}\frac{(\mathrm{n}-1)\Gamma(\beta)}{\Gamma(n)\Gamma[\alpha(n-1)+\beta]}z^n}{1+\sum_{n=2}^{\infty}\frac{\{\lambda(\mathrm{n}-1)+1\}\Gamma(\beta)}{\Gamma(n)\Gamma[\alpha(n-1)+\beta]}z^n}-1\right|$$

$$\leq(1-\lambda)(1+k)\frac{\sum_{n=2}^{\infty}\frac{(\mathrm{n}-1)\Gamma(\beta)}{\Gamma(n)\Gamma[\alpha(n-1)+\beta]}|z|^n}{1-\sum_{n=2}^{\infty}\frac{\{\lambda(\mathrm{n}-1)+1\}\Gamma(\beta)}{\Gamma(n)\Gamma[\alpha(n-1)+\beta]}|z|^n}\leq(1-\gamma)$$

Considering $z\to 1$ along to the real axis

$$(1-\lambda)(1+k)\sum_{n=2}^{\infty}\frac{(\mathrm{n}-1)\Gamma(\beta)}{\Gamma(n)\Gamma[\alpha(n-1)+\beta]}\leq(1-\gamma)\left(1-\sum_{n=2}^{\infty}\frac{\{\lambda(\mathrm{n}-1)+1\}\Gamma(\beta)}{\Gamma(n)\Gamma[\alpha(n-1)+\beta]}\right)$$

which is equivalent to

$$\Gamma(\beta)\sum_{n=2}^{\infty}\frac{(n-1)[1-\lambda\gamma+k(1-\lambda)]+(1-\gamma)}{\Gamma(n)\Gamma[\alpha(n-1)+\beta]}\leq(1-\gamma)$$

$$\sum_{n=2}^{\infty}\frac{(n-1)[1-\lambda\gamma+k(1-\lambda)]+(1-\gamma)}{\Gamma(n)\Gamma[\alpha(n-1)+\beta]}\leq\frac{(1-\gamma)}{\Gamma(\beta)}$$

Finally, we obtain the condition (12) so $\mathcal{W}_{\alpha,\beta}\in \mathrm{k}-S_p(\lambda,\gamma)$.

Theorem 2.4 $\alpha\geq 0,\beta>0,k\geq 0, 0\leq\gamma<1$ and $0\leq\lambda<1$ if

$$\sum_{n=2}^{\infty}\frac{n(n-1)[1-\lambda\gamma+k(1-\lambda)]+(1-\gamma)}{\Gamma(n)\Gamma[\alpha(n-1)+\beta]}\leq\frac{(1-\gamma)}{\Gamma(\beta)} \tag{13}$$

then $\mathcal{W}_{\alpha,\beta}\in \mathrm{k}-\mathrm{UCV}(\lambda,\gamma)$.

Solution is same as above theorem 2.3, then proof the condition (13).

Now we put $k=\lambda=\gamma=0$ in eq. (12), (13) then got the analytical criteria for the class S^* and $\mathcal{K}$ respectively.

Corollary 2.3 Let α, β >0, $k = \lambda = \gamma = 0$ then

$$\sum_{n=2}^{\infty} \frac{[n-1]}{\Gamma(n)\Gamma[\alpha(n-1)+\beta]} \leq \frac{1}{\Gamma(\beta)} \tag{14}$$

then $\mathcal{W}_{\alpha,\beta} \in S^*$.

Corollary 2.4 Let α, β >0, $k = \lambda = \gamma = 0$ then

$$\sum_{n=2}^{\infty} \frac{n^2}{\Gamma(n)\Gamma[\alpha(n-1)+\beta]} \leq \frac{1}{\Gamma(\beta)} \tag{15}$$

then $\mathcal{W}_{\alpha,\beta} \in \mathcal{K}$.

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