On M-projectively ϕ-symmetric (ε)-Lorentzian para Sasakian manifolds

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Abstract: The object of the present paper is to study the M-projective curvature tensor on ϕ-symmetric (ε)-Lorentzian para Sasakian manifolds globally and Locally. Some results are obtained for a 3-dimensional locally M-projective ϕ-symmetric (ε)-Lorentzian para Sasakian manifolds.

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1. INTRODUCTION

In 1969, Takashashi [14] studied Sasakian manifolds equipped with an associated indefinite metric. These indefinite almost contact metric manifolds and indefinite Sasakian manifolds are also called as ε-almost contact metric manifolds and ε-Sasakian manifolds, respectively. The index of a metric plays important roles in differential geometry on it generates variety of vector fields such as space-like, time-like, and light-like fields.

In 1971, Pokhariyal and Mishra [10] defined a tensor field M on a Riemannian manifolds as

$$M(X,Y)Z = R(X,Y)Z - \frac{1}{2(n-1)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \tag{1.1}$$

such a tensor field M is known as M-projective curvature tensor [12,13,14]. So that

$$M(X,Y,Z,U) = g(M(X,Y)Z,U) = M(Z,U,X,Y) \tag{1.2}$$

And Q is the Ricci operator, defined by

$$S(X,Y) = g(QX,Y) \tag{1.3}$$

Where S is the Ricci tensor for arbitrary vector fields X,Y,Z. Such a tensor field M is known as M-projective curvature tensor. Ojha[8,9] defined and studied as M-projective curvature tensor in a Kähler as well as in Sasakian manifolds. This curvature tensor bridges the gap between conformal curvature tensor, conharmonic curvature tensor and concirculer curvature tensor on one side and H-projective curvature tensor on the other.

from (1.1),

$$(\nabla_W M)(X,Y)Z = (\nabla_W R)(X,Y)Z - \frac{1}{2(n-1)}[(\nabla_W S)(Y,Z)X - (\nabla_W S)(X,Z)Y + g(Y,Z)(\nabla_W Q)(X) - g(X,Z)(\nabla_W Q)(Y)] \tag{1.4}$$

The study of Riemannian symmetric manifolds began with the work of Cartan[2]. A Riemannian manifolds (M^n, g) is said to be locally symmetric due to Cartan [2]. if its curvature tensor R satisfies the relation ∇R=0, where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. As a weaker version of local symmetry, the notion of locally ϕ-symmetric Saskian manifolds was introduced by Takahashi[14]. Shaikh and Baishya [11] studied locally ϕ-symmetric LP- Sasakian manifolds in the sense of Takahashi,. The notion of locally ϕ-symmetric manifolds in different structures has been studied by several authors ([5], [11], [12],[13],[2]). An ε −Lorentzian para Sasakian manifolds is said to be ϕ-symmetric if it satisfies;

$$\phi^2((\nabla_W R)(X,Y)Z) = 0 \tag{1.5}$$

for arbitrary vector fields X,Y,Z and W on M.

In particular, if X,Y,Z,W are horizontal vector fields, i.e., orthogonal to ξ, then it is called locally ϕ-symmetric ε −Lorentzian para Sasakian manifolds.

K.L. Duggle and A. Bejancu [1,6] introduced ε −Sasakian manifolds. Also Xufeng and Xiaoli [17] proved that every ε −Sasakian manifolds must be real hypersurface of some indefinite Kahler manifolds. Recently Rajendra Prasad and Vibha Shrivastava defined and study basic properties of ε −Lorentzian para Sasakian manifolds with indefinite metric which also include usual LP Saskian manifolds. In 2007, R. Kumar, R. Rani and R.K. Nagaich studied some interesting properties of ε −Sasakian manifolds[7]. In 2010, Tripathi et al. studied ε-almost para-contact manifolds and in particular ε −para Sasakian manifolds [16].

The present paper is organized as follows:
After introduction in section 2, we introduce the notion of ε −Lorentzian para Sasakian manifold. In section 3, we study globally ϕ-symmetric ε −Lorentzian para Sasakian manifolds. In this section we see that a globally ϕ-symmetric ε −Lorentzian para Sasakian manifolds is an indefinite space form. In section 4, we study the necessary and sufficient condition for 3-dimensional ε −Lorentzian para Sasakian manifolds to be locally symmetric. In section 5, we have some results on globally M-projectively ϕ-symmetric ε −Lorentzian para Sasakian manifolds; In section 6, we obtain, if an ε −Lorentzian para Sasakian manifold is globally M-projectively ϕ-symmetric, then the manifold is an Einstein manifold.

2. ε −LORENTZIAN PARA SASAKIAN MANIFOLDS

A differentiable M manifold of dimension $n = 2m + 1$ is called almost contact manifold, if it admit a $(1,1)$ tensor field ϕ, a contravariant vector field ξ, a 1-form η and Riemannian metric g which satisfy:

$$\phi^2 X = X + \eta(X)\xi; \qquad \eta(\xi) = -1, \qquad \forall X \in \chi(M). \tag{2.1}$$

It follows that

$$\eta(\phi X) = 0; \qquad \phi(\xi) = 0; \qquad rank\phi = n - 1, \tag{2.2}$$

the M is called an almost contact manifold. If there exists a semi-Riemannian metric g satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \varepsilon\eta(X)\eta(Y) \qquad \forall X, Y \in \chi(M), \tag{2.3}$$

with $\varepsilon = \pm 1$ then (ϕ, η, ξ, g) is called an (ε)-almost contact metric structure and M is known an (ε)-almost contact manifold. For an (ε)-almost contact manifold, we also have

$$\eta(X) = \varepsilon g(X, \xi), \qquad \forall X \in \chi(M) \tag{2.4}$$

$$g(\xi, \xi) = -\varepsilon \tag{2.5}$$

Thus ξ is never a light like vector field on M. Here ε is 1 or -1 according as ξ is space like or time like vector field on M, and according to the casual character of ξ, we have two classes of ε −Lorentzian para Sasakian manifold. When $\varepsilon = -1$ and the index of g is an odd number $(\upsilon = 2s + 1)$, then M is time-like ε −Lorentzian para Sasakian manifold and M is a space-like ε −Lorentzian para Sasakian manifold when $\varepsilon = -1$ and $\upsilon = 2s$. For $\varepsilon = 1$ and $\upsilon = 0$, we obtain usual ε −Lorentzian para Sasakian manifold manifold and for $\varepsilon = 1$ and $\upsilon = 1$, M is a Lorentzian para Sasakian manifold.

If

$$d\eta(X, Y) = g(X, \phi Y)$$

for every $X, Y \in \chi(M)$, then M is said to have (ε)-contact metric structure (ϕ, ξ, η, g). An (ε)-almost contact metric structure (ϕ, η, ξ, g) is ε −Lorentzian para Sasakian manifold if only if

$$\begin{aligned}(\nabla_X \phi)Y &= -g(X, \phi Y)\xi - \varepsilon\eta(Y)\phi X, \qquad \forall X, Y \in \chi(X) \\ &= g(X, Y)\xi + \varepsilon\eta(Y)X + 2\varepsilon\eta(X)\eta(Y)\end{aligned} \tag{2.6}$$

Where ∇ denotes the Levi-Citiva connection with respect to g. Also one has

$$\nabla_X \xi = \varepsilon(X - \eta(X)\xi \qquad \forall X \in \chi(X). \tag{2.7}$$

Then for an ε −Lorentzian para Sasakian manifold , we have following relations

$$(\nabla_X \eta)(Y) = g(\phi X, Y) \tag{2.8}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y \tag{2.9}$$

$$S(X, \xi) = \varepsilon(n - 1)\eta(X) \tag{2.10}$$

$$Q(X, \xi) = -\varepsilon(n - 1)\xi \tag{2.11}$$

If an ε −Lorentzian para Sasakian manifold is a space of constant curvature then it is an indefinite space form.

3. Globally ϕ-symmetric (ε)-Lorentzian para Sasakian manifolds

Now we suppose that an (ε)-Lorentzian para Sasakian manifolds is globally ϕ-symmetric. Then by virtue of (2.1) and (1.5) we have

$$(\nabla_W R)(X,Y)Z + \eta((\nabla_W R)(X,Y)Z)\xi = 0. \quad (3.1)$$

which implies

$$(\nabla_W R)(X,Y)Z + \varepsilon g((\nabla_W R)(X,Y)Z,\xi)\xi = 0. \quad (3.2)$$

Now, by using the property of curvature tensor we have

$$g((\nabla_W R)(X,Y)Z,\xi) = g(\nabla_W R)(X,Y)Z,\xi) + g(R(X,Y)\xi,\nabla_W Z) \\ +g(R(\nabla_W X,Y)\xi,Z) + g(R(X,\nabla_W Y)\xi,Z). \quad (3.3)$$

Since ∇ is a metric connection, it follows that

$$g((\nabla_W R)(X,Y)Z,\xi) = g(R(X,Y)\nabla_W \xi,Z) - \nabla_W g(R(X,Y)\xi,Z) \quad (3.4)$$

and

$$\nabla_W g(R(X,Y)\xi,Z) = g(\nabla_W R)(X,Y)\xi,Z) + g(R(X,Y)\xi,\nabla_W Z). \quad (3.5)$$

From (3.4) and (3.5), we get

$$g(\nabla_W R(X,Y)Z,\xi) = -g(\nabla_W R)(X,Y)\xi,Z) - g(R(X,Y)\xi,\nabla_W Z) \\ +g(R(X,Y)\nabla_W \xi,Z). \quad (3.6)$$

Using (3.6) in (3.3), we get

$$g((\nabla_W R)(X,Y)Z,\xi) = -g(\nabla_W R)(X,Y)\xi,Z), \quad (3.7)$$

Using (3.1) we obtain from (3.2) that

$$(\nabla_W R)(X,Y)Z = \varepsilon g\big((\nabla_W R)(X,Y)\xi,Z\big)\xi, \quad (3.8)$$

Using (2.6) and (2.8) in (3.8)

$$(\nabla_W R)(X,Y)\xi = -g(X,W)Y + g(Y,W)X - \varepsilon R(X,Y)W, \quad (3.9)$$

Taking (3.9) in (3.8), we get

$$(\nabla_W R)(X,Y)Z = \{\varepsilon\big(g(Y,W)g(X,Z) - g(X,W)g(Y,Z)\big) - g(R(X,Y)W,Z)\}\xi, \quad (3.10)$$

Again, if (3.10) holds, then (3.7) and (3.9) implies that the manifold is globally ϕ-symmetric. Thus we can state the following.

Theorem 1 *An ε -Lorentzian para Sasakian manifold globally ϕ-symmetric if and only if the relation (3.10) holds for any vector fields X,Y,Z,W tangent to M.*

Next putting $Z = \xi$ in (3.8) and using (3.7) we have

$$(\nabla_W R)(X,Y)\xi = 0, \quad (3.11)$$

for any vector fields X,Y,Z,W on M.From (3.11) and (3.8) it follows that

$$R(X,Y)W = \varepsilon\{g(Y,W)X - g(X,W)Y\}.$$

Thus the manifold is of constant curvature. This leads us to the following:

Theorem 2 *A globally ϕ-symmetric ε -Lorentzian para Sasakian manifold is an indefinite space form.*

4. 3-dimensional locally ϕ-symmetric ε −Lorentzian para Sasakian manifolds

Recall the part of introduction we know 3-dimentional Riemannian manifold, now

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY = S(Y,Z)X - S(Y,Z)X \\ -S(X,Z)Y + \frac{r}{2}[g(X,Z)Y - g(Y,Z)X] \tag{4.1}$$

where Q is the Ricci operator and r is the scalar curvature of manifold. If we putting $Z = \xi$ in (4.1) and use (2.8) we get

$$(\varepsilon - \frac{r}{2})(\eta(Y)X - \eta(X)Y = \eta(Y)QX - \eta(X)QY \tag{4.2}$$

By putting $Y = \xi$ in (4.2) and using (2.9) for $n = 3,$ we obtain

$$QX = (\varepsilon - \frac{r}{2})X - (\varepsilon + \frac{r}{2})\eta(X)\xi \tag{4.3}$$

that is,

$$S(X,Y) = (\varepsilon - \frac{r}{2})g(X,Y) - (\varepsilon + \frac{r}{2})\varepsilon\eta(X)\eta(Y). \tag{4.4}$$

thus from (4.4) and (4.3) in (3.12), we obtain

$$R(X,Y)Z = \left(2\varepsilon - \frac{3r}{2}\right)[g(Y,Z)X - g(X,Z)Y] \ +(\frac{r}{2} + \varepsilon)[g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X) \\ +\varepsilon\eta(X)\eta(Z)Y - \varepsilon\eta(Y)\eta(Z)X].$$

by taking the covariant differentiation of (4.5) we have

$$(\nabla_W R)(X,Y)Z = \frac{-3dr(W)}{2}[g(Y,Z)X - g(X,Z)Y] + \frac{dr(W)}{2}[g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ +\varepsilon\eta(X)\eta(Z)Y - \varepsilon\eta(Y)\eta(Z)X] + (\frac{r}{2} + \varepsilon)[g(X,Z)(\nabla_W\eta)(Y)\xi \\ +g(X,Z)\eta(Y)\nabla_W\xi - g(Y,Z)(\nabla_W\eta)(X)\xi \\ -g(Y,Z)\eta(X)\nabla_W\xi + \varepsilon(\nabla_W\eta)(X)\eta(Z)Y \\ +\varepsilon(\nabla_W\eta)(Z)\eta(X)Y - \varepsilon(\nabla_W\eta)(Y)\eta(Z)X \\ -\varepsilon(\nabla_W\eta)(Z)\eta(Y)X]. \tag{4.6}$$

it we consider X,Y and Z are horizontal vector fields. then

$$(\nabla_W R)(X,Y)Z = \frac{-3dr(W)}{2}[g(Y,Z)X - g(X,Z)Y] + (\frac{r}{2} + \varepsilon)[g(X,Z)(\nabla_W\eta)(Y)\xi \\ -g(Y,Z)(\nabla_W\eta)(X)\xi. \tag{4.7}$$

Taking ϕ^2 on both sides of equation (4.7) then

$$\phi^2((\nabla_W R)(X,Y)Z) = \frac{-3dr(W)}{2}[g(Y,Z)\phi^2X - g(X,Z)\phi^2Y]. \tag{4.8}$$

using (2.1) equation (4.8) gives us

$$\phi^2((\nabla_W R)(X,Y)Z) = \frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y]. \tag{4.9}$$

Let $\phi^2((\nabla_W R)(X,Y)Z) = 0$ for all horizontal vector fields. Then the equation (3.20) implies $dr(W) = 0$. Hence we conclude the following theorem.

Theorem 3 *A 3-dimensional ε −Lorentzian para Sasakian manifold is locally ϕ-symmetric if and only if the scalar curvature r is constant for all horizontal vector fields.*

In particular, by taking $Z = \xi$ in (4.6) we have

$$(\nabla_W R)(X,Y)\xi = (\frac{r}{2} + \varepsilon)[\varepsilon\eta(X)(\nabla_W\eta)(Y)\xi + \varepsilon\eta(X)\eta(Y)\nabla_W\xi$$

$$-\varepsilon\eta(Y)(\nabla_W\eta)(X)\xi - \eta(Y)\eta(X)\nabla_W\xi$$
$$-\varepsilon(\nabla_W\eta)(X)Y + \varepsilon(\nabla_W\eta)(\xi)\eta(X)Y$$
$$+\varepsilon(\nabla_W\eta)(Y)X - \varepsilon(\nabla_W\eta)(\xi)\eta(Y)X]. \quad (4.10)$$

If we assume X, Y, Z are horizontal vector fields, using (2.7) in (4.10) we get

$$(\nabla_W R)(X,Y)\xi = (\frac{r}{2} + \varepsilon)\varepsilon g(X,W)Y - g(Y,W)X]. \quad (4.11)$$

Applying ϕ^2 to the both sides of (4.11) we get

$$\phi^2((\nabla_W R)(X,Y)\xi) = (\frac{r}{2} + \varepsilon)\varepsilon g(X,W)\phi^2 Y - g(Y,W)\phi^2 X]. \quad (4.12)$$

If we take X, Y are orthogonal to ξ in (4.11) and (4.12) we have

$$\phi^2(\nabla_W R)(X,Y)\xi = (\nabla_W R)(X,Y)\xi. \quad (4.13)$$

Now we can state the following:

Theorem 4 *Let M be a 3-dimensional ε −Lorentzian para Sasakian manifold such that*

$$\phi^2(\nabla_W R)(X,Y)\xi = 0$$

for all horizontal vector fields X, Y, W. Then M is a indefinite space form.

5 Globally M-projectively ϕ-symmetric (ε)-Lorentzian para Sasakian manifolds

An (ε)-Lorentzian para Sasakian manifold M is said to be globally M-projectively ϕ-symmetric if the M-projective curvature tensor M satisfies

$$\phi^2(\nabla_W M)(X,Y)Z = 0, \quad (5.1)$$

for all vector fields $X, Y, Z, W \in \chi(M)$.
Let us suppose that M is globally M-projectively ϕ-symmetric. Then by virtue of (5.1) and (2.1), we have

$$(\nabla_W M)(X,Y)Z + \eta((\nabla_W M)(X,Y)Z)\xi = 0. \quad (5.2)$$

From (1.4) it follows that

$$g((\nabla_W R)(X,Y)Z,U) - \frac{1}{2(n-1)}[g(X,U)(\nabla_W S)(Y,Z) \quad (5.3)$$
$$-g(Y,U)(\nabla_W S)(X,Z) + g(Y,Z)g((\nabla_W Q)X,U)$$
$$-g(X,Z)g((\nabla_W Q)Y,U)] + \varepsilon\eta((\nabla_W R)(X,Y)Z)\eta(U)$$
$$-\frac{\varepsilon}{2(n-1)}[(\nabla_W S)(Y,Z)\eta(U) - (\nabla_W S)(X,Z)\eta(U)(Y)$$
$$+g(Y,Z)\eta((\nabla_W Q)X)\eta(U) - g(X,Z)\eta((\nabla_W Q)Y)\eta(U)$$
$$= 0$$

Putting $X = U = e_i$, where $\{e_i\}, i = 1,2,\ldots\ldots,n$, is an orthonormal basis of the tangent space at the manifold, and taking summation over i, we get

$$\frac{n}{2(n-1)}(\nabla_W S)(Y,Z) + \varepsilon\eta((\nabla_W R)(e_i,Y)Z)\eta(e_i)$$
$$-\frac{1}{2(n-1)}[g((\nabla_W Q)e_i,e_i) - \varepsilon\eta(\nabla_W Q)e_i)\eta(e_i)]g(Y,Z)$$
$$+\frac{1}{2(n-1)}[g((\nabla_W Q)Y,Z) + (\nabla_W S)(\xi,Z)\eta(Y) + \varepsilon\eta(\nabla_W Q)Y)\eta Z)]=0 \quad (5.4)$$

Putting $Z = \xi$, we obtain

$$\frac{n}{2(n-1)}(\nabla_W S)(Y,\xi) + \varepsilon\eta((\nabla_W R)(e_i,Y)\xi)\eta(e_i) \quad (5.5)$$
$$-\frac{\varepsilon}{2(n-1)}[dr(W) - \varepsilon\eta((\nabla_W Q)e_i)\eta(e_i) - (\nabla_W S)(\xi,\xi)]\eta(Y)$$
$$= 0.$$

Now

$$\eta((\nabla_W Q)e_i)\eta(e_i) = g((\nabla_W Q)e_i,\xi)g(e_i,\xi) \tag{5.6}$$
$$= g((\nabla_W Q)\xi,\xi)$$
$$= -\varepsilon g(Q(W-\eta(W)\xi),\xi)$$
$$= -\varepsilon S(W,\xi) + \varepsilon\eta(W)S(\xi,\xi) = 0.$$

Which gives

$$\eta((\nabla_W R)(e_i,Y)\xi)\eta(e_i) = g((\nabla_W R)(e_i,Y)\xi,\xi)g(e_i,\xi) \tag{5.7}$$

and

$$g((\nabla_W R)(e_i,Y)\xi,\xi) = g(\nabla_W(e_i,Y)\xi,\xi) - g(R(\nabla_W e_i,Y)\xi,\xi)$$
$$-g(R(e_i,\nabla_W Y)\xi,\xi) - g(R(e_i,Y)\nabla_W\xi,\xi)$$

Since $\{e_i\}$ is an orthonormal basis $\nabla_X e_i = 0$ and using (2.8) we obtain

$$g(R(e_i,\nabla_W Y)\xi,\xi) = \varepsilon\{\eta(e_i)\eta(\nabla_W Y) - \eta(e_i)\eta(\nabla_W Y)\} = 0. \tag{5.8}$$

As $g(R(e_i,Y)\xi,\xi) + g(R(\xi,\xi)Y,e_i) = 0,$ we have

$$g((\nabla_W R)(e_i,Y)\xi,\xi) + g(R(e_i,Y)\xi,\nabla_W\xi) = 0. \tag{5.9}$$

Using this we get

$$g((\nabla_W R)(e_i,Y)\xi,\xi) = 0. \tag{5.10}$$

using (5.6) to (5.10) in (5.5) we get

$$(\nabla_W S)(Y,\xi) = \frac{1}{n}dr(W)\eta(Y). \tag{5.11}$$

Putting $Y=\xi$ in (5.10), we get $dr(W) = 0$. This implies r is constant.
So from (5.10), we have

$$(\nabla_W S)(Y,\xi) = 0 \tag{5.12}$$

Using (2.6), this implies

$$S(Y,W) = -\varepsilon(n-1)g(Y,W). \tag{5.13}$$

Hence we can state the following theorem:

Theorem 5 *A globally M-projectively ϕ-symmetric ε –Lorentzian para Sasakian manifold is an Einstein manifold.*

Now let, $S(X,Y) = \lambda g(X,Y)$,i.e. $QX = \lambda X$. Then from (1.1) we have

$$M(X,Y)Z = R(X,Y)Z - \frac{\lambda}{(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{5.14}$$

which gives us

$$(\nabla_W M)(X,Y)Z = (\nabla_W R)(X,Y)Z. \tag{5.15}$$

Applying ϕ^2 on the both sides of the above equation we have

$$\phi^2(\nabla_W M)(X,Y)Z = \phi^2(\nabla_W R)(X,Y)Z. \tag{5.16}$$

Hence we can state the following theorem:

Theorem 6 *A globally M-projectively ϕ-symmetric ε –Lorentzian para Sasakian manifold is globally ϕ-symmetric.*

6. 3-dimensional locally M-projectively ϕ-symmetric ε −Lorentzian para Sasakian manifold

In a 3-dimensional ε −Lorentzian para Sasakian manifold the curvature tensor R, the Ricci tensor S and the Ricci operator Q are as in (4.5),(4.4) and (4.3), respectively. Now putting (4.3),(4.4) and (4.5) into (1.1) we have

$$M(X,Y)Z = \frac{1}{2}\left(3\varepsilon - \frac{5r}{2}\right)[g(Y,Z)X - g(X,Z)Y] - \left(\frac{r}{2}+\varepsilon\right)[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \varepsilon\eta(Y)\eta(Z)X - \varepsilon\eta(X)\eta(Z)Y]. \tag{6.1}$$

Taking covariant differentiation of (6.1) we have

$$(\nabla_W M)(X,Y)Z = -\frac{5dr(W)}{4}[g(Y,Z)X - g(X,Z)Y] - \frac{dr(W)}{2}[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\xi + \varepsilon\eta(Y)\eta(Z)X - \varepsilon\eta(X)\eta(Z)Y] - \left(\frac{r}{2}+\varepsilon\right)[g(Y,Z)(\nabla_W\eta)(X)\xi + g(Y,Z)\eta(X)\nabla_W\xi - g(X,Z)(\nabla_W\eta)(Y)\xi - g(X,Z)\eta(Y)\nabla_W\xi + \varepsilon(\nabla_W\eta)(Y)\eta(Z) + \varepsilon(\nabla_W\eta)(Z)\eta(Y)X - \varepsilon(\nabla_W\eta)(X)\eta(Z) - \varepsilon(\nabla_W\eta)(Z)\eta(X)Y]. \tag{6.2}$$

If X, Y and Z are horizontal vector fields. then

$$(\nabla_W M)(X,Y)Z = -\frac{5dr(W)}{4}[g(Y,Z)X - g(X,Z)Y] - \left(\frac{r}{2}+\varepsilon\right)[g(Y,Z)(\nabla_W\eta)(X)\xi - g(X,Z)(\nabla_W\eta)(Y)\xi]. \tag{6.3}$$

Applying ϕ^2 on both sides of the equation, we get

$$\phi^2((\nabla_W M)(X,Y)Z) = -\frac{5dr(W)}{2}[g(Y,Z)\phi^2X - g(X,Z)\phi^2Y]. \tag{6.4}$$

Since X, Y and Z are horizontal vector fields, using (2.1) in (6.4) we get

$$\phi^2((\nabla_W R)(X,Y)Z) = -\frac{5dr(W)}{2}[g(Y,Z)X - g(X,Z)Y]. \tag{6.5}$$

Assume that $\phi^2((\nabla_W M)(X,Y)Z) = 0$ with horizontal vector fields. Then the equation (6.5) implies $dr(W) = 0$. Hence we have the following theorem:

Theorem 7: *A 3-dimensional (ε)-Lorentzian para Sasakian manifolds is locally M-projectively ϕ-symmetric if and only if the scalar curvature r is constant for all horizontal vector fields.*

In particular, by taking $Z = \xi$ in (6.2) we have

$$(\nabla_W M)(X,Y)\xi = -\varepsilon\frac{3dr(W)}{4}[\eta(Y)X - \eta(X)Y] - \left(\frac{r}{2}+\varepsilon\right)[\varepsilon\eta(Y)(\nabla_W\eta)(X)\xi + \varepsilon\eta(Y)\eta(X)\nabla_W\xi - \varepsilon\eta(X)(\nabla_W\eta)(Y)\xi - \varepsilon\eta(X)\eta(Y)\nabla_W\xi - \varepsilon(\nabla_W\eta)(Y)X + \varepsilon(\nabla_W\eta)(\xi)\eta(Y)X + \varepsilon(\nabla_W\eta)(X)Y - \varepsilon(\nabla_W\eta)(\xi)\eta(X)Y]. \tag{6.6}$$

using (2.8) in (6.6) we obtain

$$(\nabla_W M)(X,Y)\xi = -\left(\frac{r}{2}+\varepsilon\right)[g(Y,W)X - g(X,W)Y]. \tag{6.7}$$

Applying ϕ^2 on the both sides of (6.7) we get

$$\phi^2(\nabla_W M)(X,Y)\xi = -\left(\frac{r}{2}+\varepsilon\right)[g(Y,W)\phi^2X - g(X,W)\phi^2Y]. \tag{6.8}$$

If we take X, Y, W orthogonal to in ξ (6.7) and (6.8) we have

$$\phi^2(\nabla_W M)(X,Y)\xi = (\nabla_W M)(X,Y)\xi. \tag{6.9}$$

Now we can state the following:

Theorem 8: *Let* M *be a 3-dimensional* (ε)*-Lorentzian para Sasakian such that*

$$\phi^2(\nabla_W M)(X,Y)\xi = 0 \tag{6.10}$$

for all horizontal vector fields X, Y, W. Then M is an indefinite space form.

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