GENERALIZED $(\in,\in\vee q)$-FUZZY WEAK BI-IDEALS OF NEAR-RINGS

P. Murugadas[1], V. Vetrivel[2], S. Murugambigai[3]
[1,3]Department of Mathematics, Govt. Arts and Science College,
Karur - 639 005, India.
[2]Department of Mathematics, Annamalai University,
Annamalainagar - 608 002, India.

Abstract: In this article, the notion of $(\in,\in\vee q)$-fuzzy weak bi-ideals of near-rings, which is a generalized concept of fuzzy bi-ideals of near-rings is introduced. We also investigate some of its properties with examples.

AMS Subject Classification: 03E72, 18B40, 16D25

Keywords: Near-rings, Bi-ideals, Weak bi-ideals, Fuzzy weak bi-ideals.

1.Introduction

A near-ring satisfies all the axioms of an associative ring expect for commutativity of addition and one of the two distributive laws. After the introduction of fuzzy sets by Zadeh, there have been a number of generalizations of this fundamental concept. Abou-Zaid [1] introduced fuzzy subnear-rings (ideal) and studied some of their related properties in near-rings, see [5,6] for more work in this area. A new type of fuzzy subgroup, the $(\in,\in\vee q)$-fuzzy subgroup, was introduced in an earlier paper of Bhakat and Das [2] by combining belongingness" and quasicoincidence " of fuzzy points and fuzzy sets. In fact $(\in,\in\vee q)$-fuzzy subgroup is an important generalization of ordinary fuzzy subgroup. It is now natural to investigate similar type of generalization of the existing fuzzy subsystems with other algebraic structures, see [3,13,15].

The concept of bi-ideals was applied to near-rings in [11]. The idea of fuzzy ideals of near-rings was first proposed by Kim and Jun [7] and they defined the concept of fuzzy R-subgroups of near-rings. Moreover, Manikantan [8] introduced the notion of fuzzy bi-ideals of near-rings and discussed some of its properties. Yong UK Cho et al. [14] introduced the concept of weak bi-ideals applied it to near-rings. Narayanan and Manikantan [9] have discussed the idea of $(\in,\in\vee q)$-fuzzy subnear-rings and $(\in,\in\vee q)$-fuzzy ideals which is a generalization of fuzzy subnear-rings and fuzzy ideals.

In this paper, we discuss the new notion of generalized fuzzy ideals of near-rings such as $(\in,\in\vee q)$-fuzzy weak bi-ideals which is a generalization of fuzzy bi-ideals of near-rings and establish some properties with examples.

2.Preliminaries

Definition:2.1

A non-empty set N with two binary operations $+$ and $.$ is called a near-ring if

(1) $(N,+)$ is a group,

(2) $(N,.)$ is a semigroup,

(3) $x.(y+z)=x.y+x.z,$ for all $x,y,z\in N$.

We use word 'near-ring' to mean 'left near-ring'. We denote xy instead of $x.y$. Note that $x0=0$ and $x(-y)=-xy$ but in general $0x\neq 0$ for some $x\in N.$

Definition 2.2

An ideal I of a near-ring N is a subset of N such that

(4) $(I,+)$ is a normal subgroup of $(N,+)$,

(5) $NI \subseteq I$,

(6) $((x+i)y - xy) \in I$ for any $i \in I$ and $x, y \in N$

Note that I is a left ideal of N if I satisfies (4) and (5), and I is a right ideal of N if I satisfies (4) and (6).

Definition 2.3

A two sided N-subgroup of a near-ring N is a subset H of N such that

(i) $(H,+)$ is a subgroup of $(N,+)$,

(ii) $NH \subset H$,

(iii) $HN \subset H$.

If H satisfies (i) and (ii) then it is called a left N-subgroup of N. If H satisfies (i) and (iii) then it is called a right N-subgroup of N.

Defintion 2.4

Let N be a near-ring. Given two subsets A and B of N, the product $AB = \{ab \mid a \in A, b \in B\}$. Also we define another operation '*' on the class of subsets of N given by $A * B = \{a(a'+b) - aa' \mid a, a' \in A, b \in B\}$.

Definition 2.5

A subgroup B of $(N,+)$ is said to be a bi-ideal of N if $BNB \cap BN * B \subseteq B$.

Definition 2.6

A subgroup B of $(N,+)$ is said to be a weak bi-ideal of N if $BBB \subseteq B$.

Through out this paper, f_I is the characteristic function of the subset I of N and the characteristic function of N is denoted by N, that means, $N : N \to [0,1]$ mapping every element of N to 1.

Definiton 2.7

A function λ from a nonempty set N to the unit interval $[0,1]$ is called a fuzzy subset of N. Let λ be any fuzzy subset of N, for $t \in [0,1]$ the set $\lambda_t = \{x \in N \mid \lambda(x) \geq t\}$ is called a level subset of λ.

Definition 2.8

Let μ and λ be any two fuzzy subsets of N. Then $\mu \cap \lambda, \mu \cup \lambda, \mu + \lambda, \mu\lambda$ and $\mu * \lambda$ are fuzzy subsets of N defined by: $(\mu \cap \lambda)(x) = min\{\mu(x), \lambda(x)\}$.

$(\mu \cup \lambda)(x) = max\{\mu(x), \lambda(x)\}$.

$$(\mu + \lambda)(x) = \begin{cases} \sup\limits_{x=y+z} \{\min\{\mu(y), \lambda(z)\}\} & \text{ifxisexpressibleasx=y+z,} \\ 0 & \text{otherwise.} \end{cases}$$

$$(\mu\lambda)(x) = \begin{cases} \sup\limits_{x=yz} \{\min\{\mu(y), \lambda(z)\}\} & \text{ifxisexpressibleasx=yz,} \\ 0 & \text{otherwise.} \end{cases}$$

$$(\mu*\lambda)(x)=\begin{cases}\sup_{x=(a+c)b-ab}\{\min\{\mu(y),\lambda(z)\}\} & \text{ifxisexpressibleasx=(a + c)b – ab,}\\ 0 & \text{otherwise.}\end{cases}$$

Definition 2.9

A fuzzy subgroup λ of $(N,+)$ is said to be a fuzzy bi-ideal of N if $\lambda N\lambda\cap\lambda N*\lambda\subseteq\lambda.$

Definition 2.10

A fuzzy subset λ of a group G is said to be a fuzzy subgroup of G if $\forall\, x,y\in G,$

(i) $\lambda(xy)\geq\min\{\lambda(x),\lambda(y)\},$

(ii) $\lambda(x^{-1})=\lambda(x).$

Definitoin 2.11

A fuzzy subset λ of a near-ring N is called a fuzzy N-subgroup of N if

(1) λ is a fuzzy subgroup of $(N,+)$

(2) $\lambda(xy)\geq\lambda(y)$

(3) $\lambda(xy)\geq\lambda(x),$ for all $x,y\in N.$

A fuzzy subset with (1) and (2) is called a fuzzy left N-subgroup of $N,$ whereas a fuzzy subset with (1) and (3) is called a fuzzy right N-subgroup of $N.$

Definiton 2.12

Let λ be a non-empty fuzzy subset of $N.$ λ is a fuzzy ideal of $N,$ if for all $x,y,i\in N$ and

(1) $\lambda(x-y)\geq\min\{\lambda(x),\lambda(y)\}$

(2) $\lambda(x)=\lambda(y+x-y)$

(3) $\lambda(xy)\geq\lambda(x)$

(4) $(\lambda(x(y+i)-xy)\geq\lambda(i))$ for any $x,y,i\in N.$

If λ satisfies (1), (2) and (3), then it is called a fuzzy right ideal of $N.$ If λ satisfies (1), (2) and (4), then it is called a fuzzy left ideal of $N,$ If λ is both fuzzy right as well as fuzzy left ideal of $N,$ then λ is called a fuzzy ideal of $N.$

Definition 2.13

A fuzzy subset λ of a group G is said to be an $(\varepsilon,\varepsilon\vee q)$-fuzzy subgroup of G if for ? $x,y\in G$ and $t,r\in(0,1],$

(i) $x_t,y_t\in\lambda\Rightarrow(xy)_{min\{t,r\}}\varepsilon\vee q\lambda,$

(ii) $x_t\in\lambda\Rightarrow x_t^{-1}\varepsilon\vee q\lambda.$

Definition 2.14

A fuzzy subset λ of a near-ring N is called an $(\varepsilon,\varepsilon\vee q)$-fuzzy two-sided N-subgroup of N if

(i) λ is an $(\varepsilon,\varepsilon\vee q)$- fuzzy subgroup of $(N,+),$

(ii) $\lambda(xy)\geq min\{\lambda(x),0.5\}\,\forall\, x,y\in N,$

(iii) $\lambda(xy)\geq min\{\lambda(y),0.5\}\,\forall\, x,y\in N,$

If λ satisfies (i) and $(ii),$ then λ is called an $(\varepsilon,\varepsilon\vee q)$-fuzzy left N-subgroup of $N.$

Definition 2.15

An $(\varepsilon,\varepsilon\vee q)$-fuzzy subgroup λ of N is called an $(\varepsilon,\varepsilon\vee q)$-fuzzy bi-ideal of N if for all $a\in N,\lambda(a)\geq min\{((\lambda N\lambda)\cap(\lambda N*\lambda))(a),0.5\}$ and

$\lambda(a) \geq min\{(\lambda N \lambda)(a), 0.5\}$, if N is zero symmetric.

Definition 2.16

A fuzzy subset λ is said to be an $(\varepsilon, \varepsilon \vee q)$-fuzzy subnear-ring of N if $\forall\, x, y \in N$.

(i) $\lambda(x+y) \geq min\{\lambda(x), \lambda(y), 0.5\}, \forall\, x, y \in N$.

(ii) $\lambda(-x) \geq min\{\lambda(x), 0.5\}, \forall\, x \in N$.

(iii) $\lambda(xy) \geq min\{\lambda(x), \lambda(y), 0.5\}, \forall\, x, y \in N$.

Definition 2.17

A fuzzy subset of N is an $(\varepsilon, \varepsilon \vee q)$- fuzzy ideal of N if and only if $\forall\, x, y, i \in N$,

(i) $\lambda(x-y) \geq min\{\lambda(x), \lambda(y), 0.5\}$,

(ii) $\lambda(y+x-y) \geq min\{\lambda(x), 0.5\}$,

(iii) $\lambda(xy) \geq min\{\lambda(x), 0.5\}$

(iv) $\lambda(y(x+i)-yx) \geq min\{\lambda(i), 0.5\}$.

3. $(\varepsilon, \varepsilon \vee q)$ Fuzzy weak bi-ideals of near-rings

In this section, we introduce the notion of fuzzy weak bi-ideal of N and discuss some of its properties.

Definition 3.1

A fuzzy subgroup λ of N is called $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of N, if $\lambda(xyz) \geq min\{\lambda(x), \lambda(y), \lambda(z), 0.5\}$.

Example 3.2

Let $N = \{0, a, b, c\}$ be a near-ring with two binary operations $+$ and $\cdot$ is defined as follows:

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

·	0	a	b	c
0	0	a	0	a
a	0	a	0	a
b	0	a	b	c
c	0	a	b	c

Let $\lambda : N \to [0,1]$ be a fuzzy subset defined by $\lambda(0) = 0.7, \lambda(a) = 0.5 = \lambda(b)$ and $\lambda(d) = 0.6$. Then λ is a $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of N, But not a fuzzy weak bi-ideal

Proposition 3.3

Let λ be a fuzzy subgroup of N. Then λ is a $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of N if and only if $\min\{\lambda\lambda\lambda(x), 0.5) \leq \lambda(x)\, \forall\, x \in N$

Proof. Assume that λ is a $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of N. Let $x, y, z, y_1, y_2 \in N$ such that $x = yz$ and $y = y_1 y_2$. Then

$$\begin{aligned}\min\{(\lambda\lambda\lambda)(x), 0.5\} &= \sup_{x=yz} \min\{(\lambda\lambda)(y), \lambda(z), 0.5\} \\ &= \sup_{x=yz} \min\{ \sup_{y=y_1 y_2} \min\{\lambda(y_1), \lambda(y_2)\}, \lambda(z).0.5\} \\ &= \sup_{x=yz} \sup_{y=y_1 y_2} \min\{\min\{\lambda(y_1), \lambda(y_2)\}, \lambda(z), 0.5\}\end{aligned}$$

$$= \sup_{x=y_1y_2z} \min\{\lambda(y_1),\lambda(y_2),\lambda(z),0.5\}$$

is a $(\varepsilon,\varepsilon\vee q)-$ *fuzzy weak bi* $-$ *ideal of* N,

$\lambda(y_1y_2z)\geq\min\{\lambda(y_1),\lambda(y_2),\lambda(z),0.5\}\leq\sup_{x=y_1y_2z}\lambda(y_1y_2z)=\{\lambda(x)\}$. If x can not be expressed as $x=yz$, then $\{(\lambda\lambda\lambda)(x),0.5\}=0\leq\lambda(x)$. In both cases $\{(\lambda\lambda\lambda)(x),0.5\}\leq\lambda(x)$. Conversely, assume that $\{(\lambda\lambda\lambda)(x),0.5\leq\lambda(x)$. For $x',x,y,z\in N$, let x' be such that $x'=xyz$. Then

$$\begin{aligned}\lambda(xyz)=\lambda(x') &\geq(\lambda\lambda\lambda)(x')\\ &=\sup_{x'=pq}\min\{(\lambda\lambda)(p),\lambda(q)\}\\ &=\sup_{x'=pq}\min\{\sup_{p=p_1p_2}\min\{\lambda(p_1),\lambda(p_2)\},\lambda(q)\}\\ &=\sup_{x'=pq}\sup_{p=p_1p_2}\min\{\min\{\lambda(p_1),\lambda(p_2)\},\lambda(q)\}\\ &=\sup_{x'=p_1p_2q}\min\{\lambda(p_1),\lambda(p_2),\lambda(q)\}\\ &\geq\min\{\lambda(x),\lambda(y),\lambda(z)\}\\ &\geq\min\{\lambda(x),\lambda(y),\lambda(z),0.5\}.\end{aligned}$$

Hence $\lambda(xyz)\geq\min\{\lambda(x),\lambda(y),\lambda(z),0.5\}$.

Lemma 3.4

Let μ and λ be $(\varepsilon,\varepsilon\vee q)$-fuzzy weak bi-ideals of N. Then the products $\mu\lambda$ and $\lambda\mu$ are also $(\varepsilon,\varepsilon\vee q)$-fuzzy weak bi-ideals of N.

Proof. Let μ and λ be an $(\varepsilon,\varepsilon\vee q)$-fuzzy weak bi-ideals of N. Then

$$\begin{aligned}(\mu\lambda)(x-y) &=\sup_{x-y=ab}\min\{\mu(a),\lambda(b)\}\\ &\geq\sup_{x-y=a_1b_1-a_2b_2<(a_1-a_2)(b_1-b_2)}\min\{\mu(a_1-a_2),\lambda(b_1-b_2),0.5\}\\ &\geq\sup\min\{\min\{\mu(a_1),\mu(a_2),0.5\},\min\{\lambda(b_1),\lambda(b_2),0.5\},0.5\}\\ &=\sup\min\{\min\{\mu(a_1),\lambda(b_1)\},\min\{\mu(a_2),\lambda(b_2)\},0.5\}\\ &=\min\{\sup_{x=a_1b_1}\min\{\mu(a_1),\lambda(b_1)\},\sup_{y=a_2b_2}\min\{\mu(a_2),\lambda(b_2)\},0.5\}\\ &=\min\{(\mu\lambda)(x),(\mu\lambda)(y),0.5\}.\end{aligned}$$

Similarly we can prove $(\mu\lambda)(\mu\lambda)(\mu\lambda)\subseteq\mu\lambda$

Therefore $\mu\lambda$ is a $(\varepsilon,\varepsilon\vee q)$-fuzzy weak bi-ideal of N. Similarly $\lambda\mu$ is a $(\varepsilon,\varepsilon\vee q)$-fuzzy weak bi-ideal of N. The following theorem is a relation between $(\varepsilon,\varepsilon\vee q)$-fuzzy bi-ideals and $(\varepsilon,\varepsilon\vee q)$-fuzzy weak bi-ideals of near-rings.

Theorem 3.5

Every $(\varepsilon,\varepsilon\vee q)$-fuzzy bi-ideal of N is a $(\varepsilon,\varepsilon\vee q)$-fuzzy weak bi-ideal of N.

Proof. Assume that λ is a $(\varepsilon,\varepsilon\vee q)$-fuzzy bi-ideal of N. Then $\lambda N\lambda\cap\lambda N*\lambda\subseteq\lambda$. We have $\lambda\lambda\lambda\subseteq\lambda N\lambda$ and $\lambda\lambda\lambda\subseteq\lambda N*\lambda$. This implies that $\lambda\lambda\lambda\subseteq\lambda N\lambda\cap\lambda N*\lambda\subseteq\lambda$. Therefore λ is a $(\varepsilon,\varepsilon\vee q)$-fuzzy weak bi-ideal of N. W However the converse of Theorem 3.5, is not true in general which is demonstrated in the following example.

Example 3.6

Let $N = \{0, a, b, c\}$ be a near-ring with two binary operations $+"$ and $\cdot"$ as defined as follows:

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

·	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	a	c	b
c	0	a	b	c

Let $\lambda : N \to [0,1]$ be a fuzzy subset defined by $\lambda(0) = 0.8, \lambda(a) = 0.5, \lambda(b) = 0.2$ and $\lambda(c) = 0.7$. Then λ is a $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of N. But λ is not a $(\varepsilon, \varepsilon \vee q)$-fuzzy bi-ideal of N, since $min\{(\lambda N \lambda)(b), (\lambda N * \lambda)(b)\} = 0.7 \not\leq \lambda(b)$.

Theorem 3.7

Every $(\varepsilon, \varepsilon \vee q)$-fuzzy ideal of N is a $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of N.

Proof. We know that every fuzzy ideal of N is a fuzzy bi-ideal of N. By Theorem 3.14[8], since every fuzzy ideal is $(\varepsilon, \varepsilon \vee q)$-fuzzy ideal, that every $(\varepsilon, \varepsilon \vee q)$-fuzzy ideal of N is $(\varepsilon, \varepsilon \vee q)$-fuzzy bi-ideal of N. By Theorem 3.5, every $(\varepsilon, \varepsilon \vee q)$-fuzzy bi-ideal of N is a $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of N. Therefore λ is a $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of N.

Theorem 3.8

Every fuzzy N-subgroup of N is a $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of N.

Proof. We know that every fuzzy N-subgroup of N is a fuzzy bi-ideal of N By Theorem 3.11[8] and so every $(\varepsilon, \varepsilon \vee q)$-fuzzy N-subgroup is an $(\varepsilon, \varepsilon \vee q)$-fuzzy bi-ideal of N. By Theorem 3.5, every $(\varepsilon, \varepsilon \vee q)$-fuzzy bi-ideal of N is a $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of N. Therefore λ is a $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of N.

The converse of the Theorem 3.7 and 3.8 are not true in general it is shown in the following Example.

Example 3.9

In Example 3.2, λ is an $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of N. But λ is not an $(\varepsilon, \varepsilon \vee q)$-fuzzy ideal of N, because $\lambda(0c) = \lambda(a) = 0.5 \not\geq 0.6 = \lambda(c)$. Also λ is not an $(\varepsilon, \varepsilon \vee q)$-fuzzy N-subgroup of N, because $\lambda(0c) = \lambda(a) = 0.5 \not\geq 0.6 = \lambda(c)$ and $\lambda(ca) = \lambda(a) = 0.5 \not\geq 0.6 = \lambda(c)$.

Example 3.10

In Example 3.6, λ is a $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of N. It can be easily verified that λ is not a $(\varepsilon, \varepsilon \vee q)$-fuzzy ideal of N, λ is not a $(\varepsilon, \varepsilon \vee q)$-fuzzy N-subgroup of N and λ is not a $(\varepsilon, \varepsilon \vee q)$-fuzzy bi-ideal of N. Since

$$\lambda(bc) = \lambda(b) = 0.2 \not\geq \{\lambda(c), 0.5\} = \{0.7, 0.5\} = 0.5,$$
$$\lambda(cb) = \lambda(b) = 0.2 \not\geq \{\lambda(c), 0.5\} = \{0.7, 0.5\} = 0.5$$

Theorem 3.11

Let $\{\lambda_i | i \in \Omega\}$ be family of $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideals of a near-ring N. Then $\bigcap_{i\in\Omega} \lambda_i$ is also a $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of N, where Ω is any index set.

Proof. Let $\{\lambda_i | i \in \Omega\}$ be a family of $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideals of N. Let $x, y, z \in N$ and $\lambda = \bigcap_{i\in\Omega} \lambda_i$. Then, $\lambda(x) = \bigcap_{i\in\Omega} \lambda_i(x) = (\inf_{i\in\Omega} \lambda_i)(x) = \inf_{i\in\Omega} \lambda_i(x)$.

$$\begin{aligned}\lambda(x-y) &= \inf_{i\in\Omega}\{\lambda_i(x-y)\}\\ &\geq \inf_{i\in\Omega}\min\{\lambda_i(x),\lambda_i(y),0.5\}\\ &= \min\{\inf_{i\in\Omega}\lambda_i(x),\inf_{i\in\Omega}\lambda_i(y),0.5\}\\ &= \min\{\bigcap_{i\in\Omega}\lambda_i(x),\bigcap_{i\in\Omega}\lambda_i(y),0.5\}\\ &= \min\{\lambda(x),\lambda(y),0.5\}.\end{aligned}$$

And

$$\begin{aligned}\lambda(xyz) &= \inf_{i\in\Omega}\lambda_i(xyz)\\ &\geq \inf_{i\in\Omega}\min\{\lambda_i(x),\lambda_i(y),\lambda_i(z),0.5\}\\ &= \min\{\inf_{i\in\Omega}\lambda_i(x),\inf_{i\in\Omega}\lambda_i(y),\inf_{i\in\Omega}\lambda_i(z),0.5\}\\ &= \min\{\bigcap_{i\in\Omega}\lambda_i(x),\bigcap_{i\in\Omega}\lambda_i(y),\bigcap_{i\in\Omega}\lambda_i(z),0.5\}\\ &= \min\{\lambda(x),\lambda(y),\lambda(z),0.5\}.\end{aligned}$$

Thus $\lambda=\bigcap_{i\in\Omega}\{\lambda_i,0.5\}$ is a $(\varepsilon,\varepsilon\vee q)$-fuzzy weak bi-ideals of N.

Theorem 3.12

Let λ be an $(\varepsilon,\varepsilon\vee q)$-fuzzy subset of N. Then λ is an $(\varepsilon,\varepsilon\vee q)$-fuzzy weak bi-ideal of N if and only if λ_t is an $(\varepsilon,\varepsilon\vee q)$ weak bi-ideal of N, for all $t\in[0,0.5]$.

Proof. Assume that λ is an $(\varepsilon,\varepsilon\vee q)$-fuzzy weak bi-ideal of N. Let $t\in[0,1]$ and $x,y\in\lambda_t$ be such that $\lambda(x)\geq t$ and $\lambda(y)\geq t$. Then $\lambda(x-y)\geq\min\{\lambda(x),\lambda(y),0.5\}\geq\min\{t,t\}=t$. Thus $x-y\in\lambda_t$. Let $x,y,z\in\lambda_t$ be such that $\lambda(x)\geq t,\lambda(y)\geq t$ and $\lambda(z)\geq t$. This implies that $\lambda(xyz)\geq\min\{\lambda(x),\lambda(y),\lambda(z),0.5\}\geq\min\{t,t,t\}=t$. Therefore $xyz\in\lambda_t$. Hence λ_t is a weak bi-ideal of N.
Conversely, assume that λ_t is a weak bi-ideal of N, for all $t\in[0,1]$. Let $x,y\in N$. Suppose $\lambda(x-y)<\min\{\lambda(x),\lambda(y),0.5\}$. Choose t such that $\lambda(x-y)<t<\min\{\lambda(x),\lambda(y),0.5\}$. This implies that $\lambda(x)>t,\lambda(y)>t$ and $\lambda(x-y)<t$. Then we have $x,y\in\lambda_t$ but $x-y\notin\lambda_t$ a contradiction. Thus $\lambda(x-y)\geq\min\{\lambda(x),\lambda(y),0.5\}$. If possible let there exist $x,y,z\in N$ such that $\lambda(xyz)<\min\{\lambda(x),\lambda(y),\lambda(z),0.5\}$. Choose t such that $\lambda(xyz)<t<\min\{\lambda(x),\lambda(y),\lambda(z),0.5\}$. Then $\lambda(x)>t,\lambda(y)>t,\lambda(z)>t$ and $\lambda(xyz)<t$. So $x,y,z\in\lambda_t$ but $xyz\notin\lambda_t$, which is a contradiction. Hence $\lambda(xyz)\geq\min\{\lambda(x),\lambda(y),\lambda(z),0.5\}$. Therefore λ is a $(\varepsilon,\varepsilon\vee q)$-fuzzy weak bi-ideal of N.

Theorem 3.13

Let B be a nonempty subset of N and λ be an $(\varepsilon,\varepsilon\vee q)$ fuzzy subset of N defined by $\lambda(x)=\{s$ x B $t$otherwise
for some $x\in N,s,t\in[0,0.5]$ and $s>t$. Then B is a weak bi-ideal of N if and only if λ is a $(\varepsilon,\varepsilon\vee q)$-fuzzy weak bi-ideal of N.

Proof. Assume that B is a weak bi-ideal of N. Let $x,y\in N$. We consider four cases:

(1) $x\in B$ and $y\in B$.
(2) $x\in B$ and $y\notin B$.
(3) $x\notin B$ and $y\in B$.
(4) $x\notin B$ and $y\notin B$.

Case (1): If $x\in B$ and $y\in B$, then $\lambda(x)=s=\lambda(y)$. Since B is a weak bi-ideal of N, then $x-y\in B$. Thus, $\lambda(x-y)=s=\min\{s,s,0.5\}=\min\{\lambda(x),\lambda(y),0.5\}$.

Case (2): If $x \in B$ and $y \notin B$, then $\lambda(x) = s$ and $\lambda(y) = t$. So, $\min\{\lambda(x), \lambda(y), 0.5\} = t$. Now, $\lambda(x-y) = s$ or t according as $x - y \in B$ or $x - y \notin B$. By assumption, $s > t$, we have $\lambda(x-y) \geq \min\{\lambda(x), \lambda(y), 0.5\}$. Similarly, we can prove case (3).

Case (4): If $x, y \notin B$, then $\lambda(x) = t = \lambda(y)$. So, $\min\{\lambda(x), \lambda(y), 0.5\} = t \wedge 0.5$. Next, $\lambda(x-y) = s$ or t according as $x - y \in B$ or $x - y \notin B$. So, $\lambda(x-y) \geq \min\{\lambda(x), \lambda(y), 0.5\}$.

Now let $x, y, z \in N$. We have the following eight cases.

(1) $x \in B, y \in B$ and $z \in B$.
(2) $x \notin B, y \in B$ and $z \in B$.
(3) $x \in B, y \notin B$ and $z \in B$.
(4) $x \in B, y \in B$ and $z \notin B$.
(5) $x \notin B, y \notin B$ and $z \in B$.
(6) $x \in B, y \notin B$ and $z \notin B$.
(7) $x \notin B, y \in B$ and $z \notin B$.
(8) $x \notin B, y \notin B$ and $z \notin B$.

These cases can be proved by arguments similar to the fuzzy cases above. Hence, $\lambda(xyz) \geq \min\{\lambda(x), \lambda(y), \lambda(z), 0.5\}$. Therefore λ is a $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of N. Conversely, assume that λ is a $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of N. Let $x, y, z \in B$ be such that $\lambda(x) = \lambda(y) = \lambda(z) = s$ Since λ is a $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of N, we have $\lambda(x-y) \geq \min\{\lambda(x), \lambda(y), 0.5\} = s$ and $\lambda(xyz) \geq \min\{\lambda(x), \lambda(y), \lambda(z), 0.5\} = s$. So, $x - y, xyz \in B$. Hence B is a weak bi-ideal of N.

Theorem 3.14

A nonempty subset B of N is a weak bi-ideal of N if and only if the characteristic function f_B is a $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of N.

Proof. Take $s = 0.5$ and $t = 0$ in Theorem 3.13, we get
$f_B(x) = \{0.5 \text{ x B } 0 \text{otherwise}.$

It follows that B is a weak bi-ideal of N. Conversely, the characteristic function f_B is a $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of R by Theorem 3.13.

Theorem 3.15

Let λ be a $(\varepsilon, \varepsilon \vee q)$- fuzzy weak bi-ideal of N, then the set $N_\lambda = \{x \in N | \lambda(x) = \min\{\lambda(0), 0.5\}\}$ is a weak bi-ideal of N.

Proof. Let λ be a $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of N. Let $x, y \in N_\lambda$. Then $\lambda(x) = \min\{\lambda(0), 0.5\}$, $\lambda(y) = \min\{\lambda(0), 0.5\}$ and $\lambda(x-y) \geq \min\{\lambda(x), \lambda(y), 0.5\} = \min\{\lambda(0), \lambda(0), 0.5\} = \min\{\lambda(0), 0.5\}$. So $\lambda(x-y) = \min\{\lambda(0), 0.5\}$. Thus $x - y \in N_\lambda$. For every $x, y, z \in N_\lambda$, we have $\lambda(xyz) \geq \min\{\lambda(x), \lambda(y), \lambda(z), 0.5\} = \min\{\lambda(0), \lambda(0), \lambda(0), 0.5\} = \min\{\lambda(0), 0.5\}$. Thus $xyz \in N_\lambda$. Hence N_λ is a weak bi-ideal of N.

4.Homomorphism between $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideals of near-rings

In this section, we characterize fuzzy weak bi-ideals using homomorphim.

Definition 4.1

[7] Let f be a mapping from a set N to a set S. Let μ and ν be fuzzy subsets of N and S, respectively. Then $f(\mu)$, the image of μ under f, is a fuzzy subset of S defined by

$$f(\mu)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu(x) \end{cases}$$

f^-1(y) 0otherwise and the pre-image of ν under f is a fuzzy subset of N defined by $f^{-1}(\nu(x)) = \nu(f(x)),$ for all $x \in N$ and $f^{-1}(y) = \{x \in N \mid f(x) = y\}.$

Definition 4.2

[7] Let N and S be near-rings. A map $\theta: N \to S$ is called a (near-ring) homomorphism if $\theta(x+y) = \theta(x) + \theta(y)$ and $\theta(xy) = \theta(x)\theta(y)$ for all $x, y \in N.$

Theorem 4.3

Let $f: N \to S$ be an homomorphism between near-rings N and $S.$ If λ is a $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of $S,$ then $f^{-1}(\lambda)$ is a $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of $N.$

Proof. Let λ be an $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of $S.$ Let $x, y, z \in N.$ Then

$$\begin{aligned} f^{-1}(\lambda)(x-y) &= \lambda(f(x-y)) \\ &= \lambda(f(x) - f(y)) \\ &\geq \min\{\lambda(f(x)), \lambda(f(y)), 0.5\} \\ &= \min\{f^{-1}(\lambda(x)), f^{-1}(\lambda(y)), 0.5\}. \end{aligned}$$

$$\begin{aligned} f^{-1}(\lambda)(xyz) &= \lambda(f(xyz)) \\ &= \lambda(f(x)f(y)f(z)) \\ &\geq \min\{\lambda(f(x)), \lambda(f(y)), \lambda(f(z)), 0.5\} \\ &= \min\{f^{-1}(\lambda(x)), f^{-1}(\lambda(y)), f^{-1}(\lambda(z)), 0.5\}. \end{aligned}$$

Therefore $f^{-1}(\lambda)$ is an $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of $N.$ We now also state the converse of Theorem 4.3, by strengthening the condition on f as follows.

Theorem 4.4

Let $f: N \to S$ be an onto homomorphism of near-rings N and $S.$ Let λ be a fuzzy subset of $S.$ If $f^{-1}(\lambda)$ is an $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of $N,$ then λ is an $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of $S.$

Proof. Let $x, y, z \in S.$ Then $f(a) = x, f(b) = y$ and $f(c) = z$ for some $a, b, c \in N.$ It follows that

$$\begin{aligned} \lambda(x-y) &= \lambda(f(a) - f(b)) \\ &= \lambda(f(a-b)) \\ &= f^{-1}(\lambda)(a-b) \\ &\geq \min\{f^{-1}(\lambda)(a), f^{-1}(\lambda)(b), 0.5\} \\ &= \min\{\lambda(f(a)), \lambda(f(b)), 0.5\} \\ &= \min\{\lambda(x), \lambda(y), 0.5\}. \end{aligned}$$

And

$$\begin{aligned}\lambda(xyz) &= \lambda(f(a)f(b)f(c)) \\ &= \lambda(f(abc)) \\ &= f^{-1}(\lambda)(abc) \\ &\geq \min\{f^{-1}(\lambda)(a), f^{-1}(\lambda)(b), f^{-1}(\lambda)(c), 0.5\} \\ &= \min\{\lambda(f(a)), \lambda(f(b)), \lambda(f(c)), 0.5\} \\ &= \min\{\lambda(x), \lambda(y), \lambda(z), 0.5\}.\end{aligned}$$

Hence λ is a $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of N.

Theorem 4.5

Let $f: N \to S$ be an onto near-ring homomorphism. If λ is a $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of N, then $f(\lambda)$ is a $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of S.

Proof. Let λ be a $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of N. Since $f(\lambda)(x') = \sup_{f(x)=x'} \lambda(x)$, for $x' \in S$ and hence $f(\lambda)$ is nonempty. Let $x', y' \in S$. Then we have

$\{x | x \in f^{-1}(x' - y')\} \supseteq \{x - y | x \in f^{-1}(x')$ and $y \in f^{-1}(y')\}$

and $\{x | x \in f^{-1}(x'y')\} \supseteq \{xy | x \in f^{-1}(x')$ and $y \in f^{-1}(y')\}$

$$\begin{aligned}f(\lambda)(x'-y') &= \sup_{f(z)=x'-y'} \lambda(z) \\ &\geq \sup_{f(x)=x', f(y)=y'} \lambda(x-y) \\ &\geq \sup_{f(x)=x', f(y)=y'} \min\{\lambda(x), \lambda(y), 0.5\} \\ &= \min\{\sup_{f(x)=x'} \lambda(x), \sup_{f(y)=y'} \lambda(y), 0.5\} \\ &= \min\{f(\lambda)(x'), f(\lambda)(y'), 0.5\}.\end{aligned}$$

$$\begin{aligned}f(\lambda)(x'y'z') &= \sup_{f(w)=x'y'z'} \lambda(w) \\ &\geq \sup_{f(x)=x', f(y)=y', f(z)=z'} \lambda(xyz) \\ &\geq \sup_{f(x)=x', f(y)=y', f(z)=z'} \min\{\lambda(x), \lambda(y), \lambda(z), 0.5\} \\ &= \min\{\sup_{f(x)=x'} \lambda(x), \sup_{f(y)=y'} \lambda(y), \sup_{f(z)=z'} \lambda(z), 0.5\} \\ &= \min\{f(\lambda)(x'), f(\lambda)(y'), f(\lambda)(z'), 0.5\}.\end{aligned}$$

Therefore $f(\lambda)$ is a $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideal of S.

Conclusion:

In this article the relation between $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideals and $(\varepsilon, \varepsilon \vee q)$-fuzzy bi-ideals are investigated. Further the homomorphic images and pre-images of $(\varepsilon, \varepsilon \vee q)$-fuzzy weak bi-ideals are analyzed.

References

[1] S. Abou-Zaid, On fuzzy subnear-ring and ideals, Fuzzy sets and Systems, **44** (1991), 139-146.

[2] S.K. Bhakat and P. Das, $(\varepsilon, \varepsilon \vee q)$-fuzzy subgroups, Fuzzy Sets and Systems, **80** (1996), 359-368.

[3] B. Davvaz, $(\varepsilon, \varepsilon \vee q)$-fuzzy subrings and ideals, Soft Computing, **10** (2006), 206-211.

[4] Zhan J, Davvaz B Generalized fuzzy ideals of near-rings, App. Math. J.Chin Univ., **24** (2009), 343-349.

[5] S M Hong, Y B Jun, H S Kim, Fuzzy ideals in near rings, Bull. Korean Math. Soc, **35** (1998), 343-348.

[6] S.D. Kim and H.S. Kim, On fuzzy ideals of near-rings, Bulletin Korean Mathematical Society, **33** (1996), 593-601.

[7] K.H. Kim and Y.B. Jun, On fuzzy R-subgroups of near-rings, Journal of fuzzy Mathematics, **8**(3)(2008), 549-558.

[8] T. Manikantan, Fuzzy bi-ideals of near-rings, Journal of Fuzzy Mathematics, **17** (3) (2009), 659-671.

[9] AL. Narayanan and T. Manikantan, $(\in, \in \vee q)$-fuzzy subnear-rings and $(\in, \in \vee q)$-fuzzy ideals of near-rings, Journal of Applied Mathematics and Computing, **18** (2005), 419-430.

[10] A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl., **35** (1971), 512-517.

[11] T. Tamizh Chelvam and N. Ganesan, On bi-ideals of near-rings, Indian J.pure appl.Math., **18**,(11)(1987), 1002-1005.

[12] Yong UK Cho, T. Tamizh Chelvam and S. Jayalakshmi, Weak bi-ideals of near-rings, J. Korea Soc.Math.Edue. Ser.B. Pure Appl. Math., **14** (3)(2007), 153-159.

[13] XH Yuan, C Zhang, Y H Ren. Generalized fuzzy groups and many valued applications, Fuzzy Sets and Systems, **38** (2003), 205-211.

[14] Yong UK Cho, T. Tamizh Chelvam and S. Jayalakshmi, Weak bi-ideals of near-rings, J. Korea. Soc.Math.Edue.ser.B. pure Appl.Math., **14**(3)(2007), 153-159.

[15] J M Zhan, W A Dudek. Fuzzy h-ideals of hemirings, Inform Sci., **177** (2007), 876-886

[16] L.A. Zadeh, Fuzzy Sets, Information and Control, **8** (1965) 338-353.