Benefits of Positive Convexity in Mathematical Finance and Portfolio Management

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Abstract : In this paper, the concept of convexity has been explained in a financial model. The benefits of positive convexity are given with reference to mathematical finance and portfolio management. It is proved that the higher the rating or credibility of the issuer, the less the convexity and the less the gain from risk-return game or strategies; less convexity means less price-volatility or risk; less risk means less return.

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Introduction

In mathematical finance, convexity refers to non-linearities in a financial model. In other words, if the price of an underlying variable changes, the price of an output does not change linearly, but depends on the second derivative (or, loosely speaking, higher-order terms) of the modeling function. Geometrically, the model is no longer flat but curved, and the degree of curvature is called the convexity. Strictly speaking, convexity refers to the second derivative of output price with respect to an input price. In derivative pricing, this is referred to as Gamma (Γ), one of the Greeks. In practice the most significant of these is bond convexity, the second derivative of bond price with respect to interest rates. As the second derivative is the first non-linear term, and thus often the most significant, "convexity" is also used loosely to refer to non-linearities generally, including higher-order terms. Refining a model to account for non-linearities is referred to as a convexity correction. Formally, the convexity adjustment arises from the Jensen inequality in probability theory: the expected value of a convex function is greater than or equal to the function of the expected value.

Geometrically, if the model price curves up on both sides of the present value (the payoff function is convex up, and is above a tangent line at that point), then if the price of the underlying changes, the price of the output is greater than is modeled using only the first derivative. Conversely, if the model price curves down (the convexity is, the payoff function is below the tangent line), the price of the output is lower than is modeled using only the first derivative.

The precise convexity adjustment depends on the model of future price movements of the underlying (the probability distribution) and on the model of the price, though it is linear in the convexity (second derivative of the price function). The convexity can be used to interpret derivative pricing: mathematically, convexity is optionality – the price of an option (the value of optionality) corresponds to the convexity of the underlying payout.

The value of an option is due to the convexity of the ultimate payout: one has the option to buy an asset or not (in a call; for a put it is an option to sell), and the ultimate payout function is convex – "optionality" corresponds to convexity in the payout. Thus, if one purchases a call option, the expected value of the option is higher than simply taking the expected future value of the underlying and inputting it into the option payout function: the expected value of a convex function is higher than the function of the expected value (Jensen inequality). The price of the option – the value of the optionality – thus reflects the convexity of the payoff function.

This value is isolated via a straddle – purchasing an at-the-money straddle (whose value increases if the price of the underlying increases or decreases) has (initially) no delta: one is simply purchasing convexity (optionality), without taking a position on the underlying asset – one benefits from the degree of movement, not the direction.

From the point of view of risk management, being long convexity (having positive Gamma and hence (ignoring interest rates and Delta) negative Theta) means that one benefits from volatility (positive Gamma), but loses money over time (negative Theta) – one net profits if prices move more than expected, and net lose if prices move less than expected.

Convexity Adjustments

From a modeling perspective, convexity adjustments arise every time the underlying financial variables modeled are not martingale under the pricing measure. Applying Girsanov theorem allows expressing the dynamics of the modeled financial variables under the pricing measure and therefore estimating this convexity adjustment.[1] Typical examples of convexity adjustments include:

- Quanto options: the underlying is denominated in a currency different from the payment currency. If the discounted underlying is martingale under its domestic risk neutral measure, it is not any more under the payment currency risk neutral measure
- Constant Maturity Swap (CMS) instruments (swaps, caps/floors) [2]
- Option-adjusted spread (OAS) analysis for mortgage-backed securities or other callable bonds

Positive Convexity

Property of some bonds that when market interest rates rise their price depreciates at a rate slower than the rate at which their price appreciates when the interest rates fall. On a graph it is seen as a bulging (convex) price/yield curve in which the bond's price at very high and very low yields is greater than the price shown as a straight (tangent) line. All other things being equal, positive convexity increases bond return, therefore investors prefer to hold such bonds. All non-callable (option-free) bonds have positive convexity, and so do most fixed interest rate and maturity date bonds. Putable bonds have greater positive convexity than all other bonds. Mortgage-backed bonds have negative convexity and so do the callable bonds. A pass-through bond can exhibit both positive and negative convexity depending on the current mortgage interest rate compared with the interest rate on the underlying mortgages. Positive convexity and negative convexity, however, positive convexity is just convexity, and negative convexity is just concavity.

Benefits of Positive Convexity in Portfolio Management

Many clients are concerned about duration when looking at purchasing investments, but few realize the importance of convexity. As we all know, duration is a measure of the sensitivity of a bond’s price to a change in interest rates. However, it is important to remember that duration is an approximation that assumes a linear relationship exists between bond yields and prices. Therefore, a portfolio manager must take convexity into account when managing interest-rate risk. A security that exhibits positive convexity will have greater price appreciation when interest rates fall but not all convexity is beneficial to a portfolio. Many asset classes have embedded call options which cause securities to exhibit negative convexity. In a falling interest-rate environment, the call option effectively places a cap on the security’s price, as the bond should not trade at a negative yield-to-call.

Adding positive convexity to the portfolio can be a great strategy in this low interest-rate environment. Many clients have a large portion of the portfolio in callable agencies. While these securities are important for pledging purposes and agencies generate good monthly cash flow, diversifying away from these asset classes can help improve a portfolio’s performance. As an alternative, corporate and municipal bonds tend to offer some much needed positive convexity. Many investment grade corporate bonds are issued without embedded call options, and municipal bonds that do have a call option often have five years of call protection.

One strategy which have been implemented for many clients involves adding positive convexity to the portfolio to hedge against further rate declines. This strategy can be specifically tailored to each client’s situation and balance sheet position. For example, if portfolio is heavily weighted in callable agencies, then one might consider selling callable agencies that are approaching the next call date and higher coupon agencies that are experiencing fast prepayments. These two types of bonds experience a large amount of negative convexity; thus, prices aren’t going to increase much above today’s levels.

The goal of the strategy is to sell a basket of negatively convex securities at breakeven or a gain, depending on the call structure, and reinvest the sales proceeds into corporates. In addition to improving the portfolio’s convexity, this strategy will help to diversify the portfolio and often leads to an increase in net income.

By executing a strategy similar to the one outlined above, the portfolio will perform well under different interest-rate environments. If interest rates fall further, positively convex securities should experience greater price appreciation. Furthermore, corporates will not experience an increase in cash flow if there aren't any embedded call options. In this problem, monthly cash flow increases due to an increase in mortgage refinancing. This reduction of cash flow is favorable as there is no need to invest this excess cash flow at lower interest rates.

If interest rates remain at today's levels, the ability to increase net income by adding higher-yielding to the portfolio might be the most important aspect. Many institutions are experiencing margin compression as loan demand has weakened and high-yielding assets are few and far between. If the institution is adequately capitalized, the yields can be relatively attractive.

If interest rates rise, we will have lower-yielding securities in the portfolio for a longer period of time. Furthermore, if rates rise because of economic improvement, spread tightening is experienced. This can help mitigate a decrease in market price.

It is important to remember that when buying a bond with a call option, the investor has effectively sold the option to the borrower. Therefore, any action the borrower takes as it relates to the call option will be to his benefit and the investor's disadvantage. When rates go higher and you want excess cash flow to invest at higher yields, the borrower won't exercise the call option and you will be stuck holding the security until maturity. When rates move lower and we want to lock out cash flow, our securities will be called and we will be forced to invest at a lower interest rate. If we take the call option out of the equation, and shift a portion of the portfolio to assets with positive convexity, we might be pleasantly surprised with the performance of our portfolio.

Benefits of Positive Convexity in Mathematical Finance

Mathematical finance, also known as quantitative finance, is a field of applied mathematics, concerned with mathematical modeling of financial markets. Generally, mathematical finance will derive and extend the mathematical or numerical models without necessarily establishing a link to financial theory, taking observed market prices as input. Mathematical consistency is required, not compatibility with economic theory. Thus, for example, while a financial economist might study the structural reasons why a company may have a certain share price, a financial mathematician may take the share price as a given, and attempt to use stochastic calculus to obtain the corresponding value of derivatives of the stock. The fundamental theorem of arbitrage-free pricing is one of the key theorems in mathematical finance, while the Black–Scholes equation and formula are amongst the key results. [1]

Mathematical finance also overlaps heavily with the fields of computational finance and financial engineering. The latter focuses on applications and modeling, often by help of stochastic asset models, while the former focuses, in addition to analysis, on building tools of implementation for the models. In general, there exist two separate branches of finance that require advanced quantitative techniques: derivatives pricing on the one hand, and risk- and portfolio management on the other. [2]

There are two separate branches of finance that require advanced quantitative techniques: derivatives pricing, and risk and portfolio management. One of the main differences is that they use different probabilities, namely the risk-neutral probability and the actual probability.

Bond Convexity

In finance, bond convexity is a measure of the non-linear relationship of bond prices to changes in interest rates, the second derivative of the price of the bond with respect to interest rates (duration is the first derivative). In general, the higher the duration, the more sensitive the bond price is to the change in interest rates. Bond convexity is one of the most basic and widely used forms of convexity in finance.

Convexity is a measure of the curvature or second derivative of how the price of a bond varies with interest rate, i.e. how the duration of a bond changes as the interest rate changes. Specifically, one assumes that the interest rate is constant across the life of the bond and that changes in interest rates occur evenly. Using these assumptions, duration can be formulated as the first derivative of the price function of the bond with respect to the interest rate in question. Then the convexity would be the second derivative of the price function with respect to the interest rate.

The price sensitivity to parallel changes in the term structure of interest rates is highest with a zero-coupon bond and lowest with an amortizing bond (where the payments are front-loaded). Although the amortizing bond and the zero-coupon bond have different sensitivities at the same maturity, if their final maturities differ so that they have identical bond durations then they will have identical sensitivities. That is, their prices will be affected equally by small, first-order, (and parallel) yield curve shifts. They will, however, start to change by different amounts with each *further* incremental parallel rate shift due to their differing payment dates and amounts.

For two bonds with same par value, same coupon and same maturity, convexity may differ depending on at what point on the price yield curve they are located.

Suppose both of them have at present the same price yield (p-y) combination; also we have to take into consideration the profile, rating, etc. of the issuers: let us suppose they are issued by different entities. Though both bonds have same p-y combination bond A may be located on a more elastic segment of the p-y curve compared to bond B. This means if yield increases further, price of bond A may fall drastically while price of bond B won't change, i.e. bond B holders are expecting a price rise any moment and are therefore reluctant to sell it off, while bond A holders are expecting further price-fall and ready to dispose of it.

This means bond B has better rating than bond A.

So the higher the rating or credibility of the issuer the less the convexity and the less the gain from risk-return game or strategies; less convexity means less price-volatility or risk; less risk means less return.

The goal of derivatives pricing is to determine the fair price of a given security in terms of more liquid securities whose price is determined by the law of supply and demand. The meaning of "fair" depends, of course, on whether one considers buying or selling the security. Examples of securities being priced are plain vanilla and exotic options, convertible bonds, etc.

Risk and portfolio management aims at modeling the statistically derived probability distribution of the market prices of all the securities at a given future investment horizon. The portfolio-selection work of Markowitz and Sharpe introduced mathematics to investment management. With time, the mathematics has become more sophisticated. [5]

Conclusion

Duration and convexity are two metrics used to help investors understand how the price of a bond will be affected by changes in interest rates. How a bond's price responds to changes in interest rates is measured by its duration, and can help investors understand the implications for a bond's price should interest rates change. The change in a bond's duration for a given change in yields can be measured by its convexity.

- If rates are expected in increase, consider bonds with shorter durations. These bonds will be less sensitive to a rise in yields and will fall in price less than bonds with higher durations.
- If rates are expected to decline, consider bonds with higher durations. As yields decline and bond prices move up, higher duration bonds stand to gain more than their lower duration counterparts.

Over the years, increasingly sophisticated mathematical models and derivative pricing strategies have been developed, but many a times their credibility has been damaged by the financial crisis.

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