A STUDY ON HYPERGRAPH EXTENSION OF TURÁN'S THEOREM, DIRECTED HYPERGRAPHS AND ITS APPLICATIONS

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Abstract- In this paper, we establish some theorems on Hypergraph extensions of Turan's theorem, theorems on Hypergraph and its applications.

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1. INTRODUCTION

A family $\mathcal{F}$ of r −uniform hypergraphs (r −graphs for short), and an r −graph G, we say that G is $\mathcal{F}$ −free if G contains no member of $\mathcal{F}$ as a subhypergraph. The extremal number $ex(n,\mathcal{F})$ is the maximum number of edges in an $\mathcal{F}$ −free n −vertex r −graph (in case $\mathcal{F}$ is a single r −graph $\mathcal{F}$, we write $ex(n,\mathcal{F})$ instead of $ex(n,\{\mathcal{F}\})$. The Turán density of $\mathcal{F}$ is defined as

$$\pi(\mathcal{F}) := \lim_{n\to\infty} \frac{ex(n,\mathcal{F})}{\binom{n}{r}}.$$

When $\mathcal{F}$ is an r −graph, $\pi(\mathcal{F}) \neq 0$, and $r > 2$, determining $\pi(\mathcal{F})$ is a hard problem, even for very simple r −graphs $\mathcal{F}$. A result of Erdős and Simonovits implies that if H is an r −graph containing two vertices x, y such that $x \cup S \in H$ iff $y \cup S \in H$, and no edge contains both x and y, then $\pi(H) = \pi(H - y)$. Consequently, when studying $\pi(H)$, we may restrict to the case when H contains no two vertices x and y as above. In this case we say that H is irreducible. When $r = 3$, the value of $\pi(\mathcal{F})$ is known for very few irreducible r −graphs $\mathcal{F}$. This lack of knowledge of the behavior of π prevents us from understanding general phenomenon of the extremal theory of hypergraphs. It is therefore of interest to increase the list of irreducible hypergraphs with known Turán density.

Extending this method, the Mubayi and Rödl [15] determined π for about ten more irreducible 3 −graphs, but in each case the value was 3/4. They also conjectured that $\pi(\mathcal{F}) = 4/9$, where $\mathcal{F} = \{123,124,125,345\}$, and gave the lower bound. This conjecture was recently proved by Füredi-Pikhurko-Simonovits [9] and exact results and further extensions were obtained by the same authors in [10]. Another recent result, due to Keevash and Sudakov [12], determines $\pi(C_3^{(2r)})$, where $C_3^{(2r)}$ is the $(2r)$ −graph obtained by letting P_1, P_2, P_3 be pairwise disjoint sets of size r, and taking as edges the three sets $P_i \cup P_j$ with $i \neq j$. This result settled a conjecture of Frankl [6]. In spite of this recent activity, until the current work, there were only finitely many irreducible 3 −graphs whose Turán density was known.

Definition 1.1 Fix $l, r \geq 2$. Let $\mathcal{K}_l^{(r)}$ be the family of r −graphs with at most $\binom{l}{2}$ edges, that contain a set S, called the core, of l vertices, with an edge containing every pair of vertices in S. Let $H_l^{(r)} \in \mathcal{K}_l^{(r)}$ be the r −graph with vertex set $A\dot{\cup}(\dot{\cup}_{S\in\binom{A}{2}} B_S)$, where $|A| = l, |B_S| = r - 2$ for every S, and edge set $\{S \cup B_S : S \in \binom{A}{2}\}$.

2 INFINITELY MANY DENSITIES

Denote by $H(k)$ the r −graph obtained from H by replacing each vertex of H by k copies of itself. Call the k copies of vertex v clones of v. The supersaturation result of Erdős and Simonovits implies that if $k > 0$ is any fixed integer, then $\pi(H(k)) = \pi(H)$. We need a slightly stronger statement that follows immediately from their argument.

Lemma 2.1 Fix $k, t \geq 1, r \geq 2$, and let $F = \{H_1, \dots, H_t\}$ be a (finite) family of r −graphs. Suppose that H is an r −graph satisfying $H \subset H_i(k)$ for every $i \in [t]$. Then $\pi(H) \leq \pi(\mathcal{F})$.

Proof. In what follows, we write $a < \ < b$ to denote that b is much larger than a; for the sake of clarity, we prefer this notation to giving the explicit relationship. Choose $\varepsilon > 0$. Then there exists an $m > \ > 1/\varepsilon$ such that every r −graph on m vertices with more than $(\pi(\mathcal{F}) + \varepsilon/2)\binom{m}{r}$ edges contains a copy of some $H_i \in \mathcal{F}$. Choose $n > \ > m$.

Suppose that G is an r −graph on n vertices with $|G| > (\pi(\mathcal{F}) + \varepsilon)\binom{n}{r}$. Then an averaging argument [2] implies that at least $\gamma\binom{n}{m}$ of the m −sets of vertices in G induce an r −graph with more than $(\pi(\mathcal{F}) + \varepsilon/2)\binom{m}{r}$ edges, where $0 < \gamma = \gamma(\varepsilon)$. Each of these m −sets contains a copy of some member of $\mathcal{F}$. Therefore there is an i for which at least $(\gamma/t)\binom{n}{m}$ of the m −sets contain H_i. Consequently the number of copies of H_i in G is at least

$$\frac{(\gamma/t)\binom{n}{m}}{\binom{n-hi}{m-hi}} = \frac{\gamma}{t}\frac{(n)_{h_i}}{(m)_{h_i}},$$

Where $h_i = |V(H_i)|$. Now, since $n >> m$, a result of Erdős [3] implies that G contains a copy of $H_i(k)$. Consequently, $H \subset H_i(k) \subset G$, and therefore $\pi(H) \leq \pi(\mathcal{F})$.

Theorem 2.2 Let $\mathrm{n, l, r} \geq 2$. Then $\mathrm{ex(n, K_{l+1}^{(r)}) = tr(n, l)}$, and the unique r −graph on n vertices containing no copy of a member of $\mathrm{K_{l+1}^{(r)}}$ for which equality holds is $\mathrm{T_r(n, l)}$.

Proof. 1 We proceed by induction on l, with $l < r$ being trivial. When $r = 2$, the result is Turán's theorem. We therefore assume that $l \geq r > 2$. Let G be an n −vertex $K_{l+1}^{(r)}$ −free r −graph. If $n \leq l$, the result is again trivial, so from now on we assume that $n \geq l + 1 \geq r + 1 > 3$.

Pick a vertex $x \in V(G)$ of maximum degree Δ. Let $N = N(x)$ be the set of vertices y for which $\text{codeg}_G(x, y) > 0$. Consider the r −graph $G[N]$ induced by N, and suppose that it contains a copy H of a member of $K_l^{(r)}$. Let $S \subset V(H)$ be the core of H. Form H' from H by adding the vertex x and one edge containing each pair x, v with $v \in S$. These edges exist by the definition of N. Altogether we have added at most l edges, giving $|H'| \leq |H| + l \leq \binom{l}{2} + l = \binom{l+1}{2}$. Therefore $H' \in K_{l+1}^{(r)}$ which is a contradiction. Consequently, $G[N]$ is $K_l^{(r)}$ −free.

Next consider the $(r-1)$ −graph $L(x)$. If $L(x)$ contains a copy H of a member of $K_l^{(r-1)}$ then by enlarging every edge of H to contain x, we obtain a copy of an $H' \in K_{l+1}^{(r)}$, since $|H'| = |H| < \binom{l+1}{2}$. Therefore $L(x)$ is $K_l^{r-1)}$ −free. Set $k = n - |N|$. By the induction hypothesis, $|G[N]| \cdot\leq t_r(n-k, l-1)$ and $\Delta = |L(x)| \leq t_{r-1}(n-k, l-1)$.

Since all vertices outside N have degree at most Δ, we conclude that

$$|G| \leq |G[N]| + k \cdot \Delta\, t_r(n-k, l-1) + k \cdot t_{r-1}(n-k, l-1) \leq t_r(n, l),$$

where the last inequality followed by

$$t_r(n-k, l-1) + k \cdot t_{r-1}(n-k, l-1) \leq t_r(n, l).$$

If equality holds above, then no edge of G contains two vertices in $V(G) - N$, since this would result in over-counting edges in the first inequality. Also, by the discussion after (3.1), we may assume that $k = \lfloor n/l \rfloor$ or $\lceil n/l \rceil$. Further, by induction we conclude that $G[N]$ is a copy of $T_r(n-k, l-1)$ and the link of each vertex outside N is a copy of $T_{r-1}(n-k, l-1)$. Let us first assume that $l > r$, and fix $z \notin N$. We have already argued that $L(z)$ (which is isomorphic to $T_{r-1}(n-k, l-1)$) has vertex set N. Next we argue that its vertex partition $V_1 \cup \cdots \cup V_{l-1}$ respects that of $G[N]$.

Suppose to the contrary that $G[N]$ has $(l-)$ −partition $W_1 \cup \cdots \cup W_{l-1}$, and $\{v_1, v_2\} \in W_1$, where $v_i \in V_i$. Note that since v_1 and v_2 lie in different parts of $L(z)$, there is an edge of G containing them both. Now pick a vertex $w_j \in W_j$ for each $j > 1$, and consider $S = \{w_2, \ldots, w_{l-1}, v_1, v_2\}$. In order for $G[N]$ to contain at least one edge, we need $n - k \geq l - 1 \geq r$. This follows since $n - k \geq n - \lceil n/l \rceil \geq (l+1) - 2 = l - 1 \geq r$. Therefore every two vertices in different parts of $G[N]$ lie in an edge of $G[N]$. Consequently, for $j \neq j'$, we have $codeg_{\,G[N]}(w_j, w_{j'}) > 0$, and $\text{codeg}_{\,G[N]}(w_j, v_i) > 0$ for $i = 1,2$. Since v_1 and v_2 also lie in an edge of G (that also contains z), this produces a copy of a member of $\mathcal{K}_l^{(r)}$ with core S. Together with z, we obtain a copy of a member of $\mathcal{K}_{l+1}^{(r)}$, with core $S \cup z$, a contradiction. Therefore each $L(z)$ respects the $(l-1)$ −partition of $G[N]$, and G is $T_r(n, l)$ as required.

If $l = r$, then $G[N]$ has no edges, so we cannot use the argument above. In this case we must show that for any two $z, z' \notin N$, the $(r-1)$ −partitions of $L(z)$ and $L(z')$ are the same.

Proof. 2 For this proof, we need the recurrence

$$t_r(n-1, l) + t_{r-1}(n - \lceil n/l \rceil, l-1) = t_r(n, l).$$

This follows by removing one vertex from $T_r(n, l)$ and counting edges among the remaining $n-1$ vertices, together with edges containing the removed vertex.

Again we proceed by induction on l. Let G be an n −vertex $\mathcal{K}_{l+1}^{(r)}$ −free r −graph with $|G| \geq t_r(n, l)$. As in the first proof, we may assume that $n \geq l + 1 \geq r + 1 > 3$. We know that $t_r(n, l) > t_r(n, l-1)$, so by induction we may assume that $H \subset G$ for some $H \in \mathcal{K}_l^{(r)}$. Let $S = \{w_1, \ldots, w_l\}$ be the core of H. For each $v \in V(G)$, let $s(v)$ be the number of i for which $\text{codeg}(v, w_i) > 0$. If $s(v) = l$ for some v, then $S \cup v$ is the core of a copy of some member of $\mathcal{K}_{l+1}^{(r)}$. We may therefore assume that $s(v) < l$ for each v. Recall that for a vertex x, $|N(x)|$ is the number of y for which $\text{codeg}(x, y) > 0$. By double counting,

$$\textstyle\sum_{i=1}^{l} |N(w_i)| = \sum_{v \in V(G)} s(v) \leq n(l-1).$$

Consequently, there is an i, for which $|N(w_i)| \leq \lfloor n(l-1)/l \rfloor = n - \lceil n/l \rceil$. As in Proof 1, we know that $L(w_i)$ is $\mathcal{K}_l^{(r-1)}$ −free. Therefore by induction

$$|G| \leq |L(w_i)| + |G[V(G) - w_i]| \leq t_{r-1}(n - \lceil n/l \rceil, l-1) + t_r(n-1, l) = t_r(n, l).$$

Although this proof can be extended to give the case of equality, the arguments are not as clean as in Proof 1.

Proof. 3 This proof only gives the bound on the number of edges when $l|n$, however for this purpose it is ideally suited. Given an n −vertex r −graph G, define the polynomial

$$f(G, x_1, \ldots, x_n) = \sum_{E \in G} \prod_{i \in E} x_i.$$

The Lagrange function of G is

$$\lambda(G) = \max\left\{ f(G, x_1, \ldots, x_n) : x_i \geq 0 \ and \ \sum_{i=1}^{n} x_i = 1 \right\}.$$

Now let G be an n −vertex $\mathcal{K}_{l+1}^{(r)}$ −free r −graph, and let $x_i, i \in [n]$ be chosen for which $f(G, x_1, \ldots, x_n) = \lambda(G)$. Define the support of G by $supp(G) = \{i : x_i > 0\}$. It follows from a lemma of Frankl and Rödl [7] (proved earlier for $r = 2$ by

Motzkin and Straus [14]) that if $\{i,j\} \subset supp(G)$, then $\text{codeg}_G(i,j) > 0$. Since G is $\mathcal{K}_{l+1}^{(r)}$ −free, we conclude that $|supp(G)| \le l$. An easy optimization now implies that $\lambda(G) \le \binom{l}{r}(1/l)^r$. On the other hand, setting each $x_i = 1/n$ gives the lower bound $\lambda(G) \ge |G|/n^r$. Putting this together yields $|G| \le \binom{l}{r}(1/l)^r$ as needed.

Theorem 2.3 Let $l \ge r \ge 2$. Then

$$\pi(H_{l+1}^{(r)}) = \frac{(l)_r}{l^r},$$

where $(l)_r = l(l-1)\cdots(l-r+1)$.

Proof. We first show that $H_{l+1}^{(r)} \subset H(\binom{l+1}{r}+1)$ for every $H \in \mathcal{K}_{l+1}^{(r)}$.

Pick $H \in \mathcal{K}_{l+1}^{(r)}$, and let $H' = H(\binom{l+1}{r}+1)$. For each vertex $v \in V(H)$, suppose that the clones of v are $v = v^1, v^2, \ldots, v\binom{l+1}{2}+1$. In particular, identify the first clone of v with v.

Let $S = \{w_1, \ldots, w_{l+1}\} \subset V(H)$ be the core of H. For every $1 \le i < j \le l+1$, let $E_{ij} \in H$ with $E_{ij} \supset \{w_i, w_j\}$. Replace each vertex z of $E_{ij} - \{w_i, w_j\}$ by z^q where $q > 1$, to obtain an edge $E'_{ij} \in H'$. Continue this procedure for every i, j, making sure that whenever we encounter a new edge it intersects the previously encountered edges only in H. Since the number of clones is $\binom{l+1}{r}+1$, this procedure can be carried out successfully and results in a copy of $H_{l+1}^{(r)}$ with core S. Therefore $H_{l+1}^{(r)} \subset H' = H\binom{l+1}{r}+1)$. Consequently, Lemma 3.2.1 implies that $\pi(H_{l+1}^{(r)}) \le \pi(\mathcal{K}_{l+1}^{(r)})$.

As $H_{l+1}^{(r)}$ contains a core of size $l+1$, we conclude that $H_{l+1}^{(r)} \not\subset T_r(n,l)$. Therefore

$$\lim_{n\to\infty} \frac{t_r(n,l)}{\binom{n}{r}} \le \pi(H_{l+1}^{(r)}) \le \pi(\mathcal{K}_{l+1}^{(r)}) \le \lim_{n\to\infty} \frac{t_r(n,l)}{\binom{n}{r}},$$

where the last inequality follows from Theorem 3.2.2. Since

$$t_r(n,l) = [(l)_r/(l^r)]\binom{n}{r} + o(n^r).$$

3 DIRECTED HYPERGRAPHS

A hypergraphis a pair $H = (V,E)$, where $V = \{v_1, v_2, \ldots, v_n\}$ is the set of vertices (or nodes) and $E = \{E_1, E_2, \ldots, E_m\}$, with $E_i \subseteq V$ for $i = 1,2,\ldots,m$, is the set of hyperedges. Clearly, when $|E_i| = 2$, $i = 1,2,\ldots,m$, the hypergraph is a standard graph.

While the size of a standard graph is uniquely defined by n and m, the size of a hypergraph depends also on the cardinality of its hyperedges; we define the size of H as the sum of the cardinalities of its hyperedges:

$$size(H) = \sum_{E_i \in E} |E_i|.$$

It is worth noting that there is a one-to-one correspondence between hypergraphs and Boolean matrices. Indeed, any $n \times m$ matrix $A = [a_{ij}]$ such that $a_{ij} \in \{0,1\}$ may be considered as the incidence matrix of a hypergraph H where each row i is associated with a vertex v_i and each column j with a hyperedge E_j.

A directed hyperedge or hyperarcis an ordered pair, $E = (X,Y)$, of (possibly empty) disjoint subsets of vertices; X is the tail of E while Y is its head. In the following, the tail and the head of hyperarc E will be denoted by $T(E)$ and $H(E)$, respectively.

A directed hypergraph is a hypergraph with directed hyperedges. In the following, directed hypergraphs will simply be called hypergraphs. An example of hypergraph is illustrated. Note that hyperarc E_5 has an empty head.

As for directed graphs, the incidence matrix of a hypergraph H is a $n \times m$ matrix $[a_{ij}]$ defined as follows:

$$a_{ij} = \begin{cases} -1, & if \quad v_i \in T(E_j), \\ 1, & if \quad v_i \in H(E_j), \\ 0, & otherwise. \end{cases}$$

Clearly, there is a one-to-one correspondence between hypergraphs and $(-1,0,1)$ matrices.

A Backward hyperarc, or simply B −arc, is a hyperarc $E = (T(E), H(E))$ with $|H(E)| = 1$ A Forward hyperarc, or simply F −arc, is a hyperarc $E = (T(E), H(E))$ with $|T(E)| = 1$

A B −graph (or B −hypergraph) is a hypergraph whose hyperarcs are B −arcs. A F −graph (or F −hypergraph) is a hypergraph whose hyperarcs are F −arcs. A BF −graph (or BF −hypergraph) is a hypergraph whose hyperarcs are either B −arcs or F −arcs.

Given a hypergraph $H = (V,E)$, we define its symmetric image the hypergraph $\tilde{H} = (V, \tilde{E})$, where $\tilde{E} = \{(X,Y) : (Y,X) \in E\}$. Note that the symmetric image of a B −graph is a F −graph, and viceversa.

Note that it is always possible to transform a general hypergraph into a BF −graph, by adding a dummy node to each hyperarc which is neither a B −arc nor a F −arc, and thus replacing the hyperarc by one backward and one forward hyperarc.

Let $FS(v) = \{E \in E : v \in T(E)\}$ and $BS(v) = \{E \in E : v \in H(E)\}$ denote the Forward Star and the Backward Star of node v, respectively.

CONCLUSION

In this paper, we have established some theorems on Hypergraph extensions of Turan's theorem, theorems on Hypergraph and its applications.

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