CONVEXITY FOR THE PROPORTION OF ONE MEAN WITH RESPECT TO ANOTHER PROPORTION OF MEAN

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Abstract: Convexity/Concavity nature proportion of contrast of means are for the most part discussed. In any case, in this paper relative investigation of Convexity/Concavity between proportion of distinction of means are found and these outcomes are deciphered in Vander Monde determinants.

1. Introduction

The outstanding means in writing, for example, Arithmetic mean, Geometric mean, Harmonic mean and Contra Harmonic mean are displayed by Pappus of Alexandria. In Pythagorean School based on extent and furthermore a portion of alternate means like Heron mean and Centriodal Mean are characterized as takes after [1, 2] and some intriguing outcomes on above said implies are talked about in [3-8].

For $a,b>0$ be two real numbers, then

$$A(a,b)=\frac{a+b}{2}$$

$$G(a,b)=\sqrt{ab}$$

$$H(a,b)=\frac{2ab}{a+b}$$

$$H_e(a,b)=\frac{a+b+\sqrt{ab}}{3}$$

$$C(a,b)=\frac{a^2+b^2}{a+b}$$

$$C_d(a,b)=\frac{2\left(a^2+b^2+ab\right)}{3(a+b)}$$

are respectively called Arithmetic mean, Geometric mean, Harmonic mean, Heron mean, Contra Harmonic Mean and Centriodal Mean

J.Rooin and M Hassni , introduced the homogeneous functions, the function

$f(x)=\frac{a^x-b^x}{c^x-d^x}$ and $g(x)=\frac{a^x-b^x}{c^x-d^x}$. For $a>b\geq c>d>0$ where $x\in(-\infty,\infty)$ is

(i) Convex, if $ad-bc>0$

(ii) Concave, if $ad-bc<0$ and

(iii) Equality holds, if $ad-bc=0$

In [16], authors studied the convexity(concavity) of the following proportion of difference of means.

(1.2) $$M_{CAGH}(a,b)=\frac{C(a,b)-A(a,b)}{G(a,b)-H(a,b)}$$

(1.3) $$M_{AH_eGH}(a,b)=\frac{A(a,b)-H_e(a,b)}{G(a,b)-H(a,b)}$$

(1.4) $$M_{CH_eGH}(a,b)=\frac{C(a,b)-H_e(a,b)}{G(a,b)-H(a,b)}$$

(1.5) $$M_{C_dAGH}(a,b)=\frac{C_d(a,b)-A(a,b)}{G(a,b)-H(a,b)}$$

(1.6) $$M_{CAH_eG}(a,b)=\frac{C(a,b)-A(a,b)}{H_e(a,b)-G(a,b)}$$

(1.7) $$M_{CC_dGH}(a,b)=\frac{C(a,b)-C_d(a,b)}{G(a,b)-H(a,b)}$$

Some fruitful results related to Schur convexities were also found in [15-22]. In this paper, we study the convexity of proportion of difference of means. $M_{CAGH}(a,b)$, $M_{AH_eGH}(a,b)$, $M_{CH_eGH}(a,b)$ and $M_{C_dAGH}(a,b)$ and some applications of these proportion of difference of means.

In[21],[22] Taneja has established chain of inequality for the binary means as follows

(1.8) $$H(a,b)\le G(a,b)\le N_1(a,b)\le H_e(a,b)\le N_2(a,b)\le A(a,b)\le S(a,b)$$

are respectively called arithmetic mean, geometric mean, harmonic mean, Heron mean, root square mean, square-root. where For $a,b>0$ be two real numbers. Many proportion of difference of well known means are studied in [16]

(1.9) $$M_{SA}(a,b)=S(a,b)-A(a,b)$$
(1.10) $$M_{SN_2}(a,b)=S(a,b)-N_2(a,b)$$
(1.11) $$M_{SH_e}(a,b)=S(a,b)-H_e(a,b)$$
(1.12) $$M_{SN_1}(a,b)=S(a,b)-N_1(a,b)$$
(1.13) $$M_{SG}(a,b)=S(a,b)-G(a,b)$$
(1.14) $$M_{SH}(a,b)=S(a,b)-H(a,b)$$
(1.15) $$M_{AN_2}(a,b)=A(a,b)-N_2(a,b)$$
(1.16) $$M_{AG}(a,b)=A(a,b)-G(a,b)$$
(1.17) $$M_{AH}(a,b)=A(a,b)-H(a,b)$$
(1.18) $$M_{N_2N_1}(a,b)=A(a,b)-N_2(a,b)$$
(1.19) $$M_{N_2G}(a,b)=N_2(a,b)-G(a,b)$$

The above difference of means are nonnegative and convex in $\Re^2{}_+ = (0,\infty)\times(0,\infty)$. We set of this section with the difference of means defined in the equations [1.9] to [1.19] The difference between of two means are as follows,

(1.20) $$M_{SN_2}(a,b) - M_{SA}(a,b) = A - N_2$$

(1.21) $$M_{SH_e}(a,b) - M_{SN_2}(a,b) = N_2 - H_e$$

(1.22) $$M_{SH_e}(a,b) - M_{SA}(a,b) = A - H_e$$

(1.23) $$M_{SN_1}(a,b) - M_{SH_e}(a,b) = H_e - N_1$$

(1.24) $$M_{SN_1}(a,b) - M_{SN_2}(a,b) = N_2 - N_1$$

(1.25) $$M_{N_1H}(a,b) - M_{GH}(a,b) = N_1 - G$$

(1.26) $$M_{SH}(a,b) - M_{SG}(a,b) = G - H$$

The above difference of means are convex for all positive real value of 't'.

Now, we established the proportion of difference of above means as follows:

(1.27) $$\frac{M_{SH_e} - M_{SA}}{M_{SN_1} - M_{SH_e}} = \frac{A - H_e}{H_e - N_1}$$

(1.28) $$\frac{M_{SN_2} - M_{SA}}{M_{SH_e} - M_{SN_2}} = \frac{A - N_2}{N_2 - H_e}$$

(1.29) $$\frac{M_{N_1H} - M_{GH}}{M_{SH} - M_{SG}} = \frac{N_1 - G}{G - H}$$

Results on convexity of one function with respect to another function were in detail discussed by Bullen [1] and also some convexity results on various important means and their applications to mean inequalities were found in [9, 11, 12, 14].

Zhen-Gang Xiao et al. [15] and various other authors have obtained some interesting and valuable results on generalization of heron mean, centriodal mean, root square means using the generalized Vander Monde's determinants. These type of generalizations and applications have generated an impressive amount of work in this field.

Let φ be a continuous function on an interval I $\subseteq$ R, $x = (x_0, x_1, ...x_n)$ and $x_i \in I$, $x_i \neq x_j$ for $i \neq j$ (see[6]) Setting

$$V(x,\varphi) = \begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} & \varphi(x_0) \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & \varphi(x_1) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} & \varphi(x_n) \end{vmatrix}$$

Suppose that $\varphi(x) = x^{n+r} \ln^k x$ then we have

$$V(x;r,k)=\begin{vmatrix}1 & x_0 & x_0^2 & \dots & x_0^{n-1} & x_0^{n+r}\ln^k x_0\\ 1 & x_1 & x_1^2 & \dots & x_1^{n-1} & x_1^{n+r}\ln^k x_1\\ \dots & \dots & \dots & \dots & \dots & \dots\\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} & x_n^{n+r}\ln^k x_n\end{vmatrix}.$$

2. Definitions and Lemma

In this section, we presented some definitions and lemmas ,which are necessary to develop this paper.

Definition 2.1. A mean is defined as a function $M : R_+^2 \to R_+$ which has the property

$$a \wedge b \le M(a,b) \le a \vee b,\ \forall a,b > 0,$$

where $a \wedge b = \min(a,b)$ *and* $a \vee b = \max(a,b)$.

Lemma 2.1: For $\varphi(x)= x^2$ and $x = (x_0, x_1, ...x_n)$ Eq.(1.2) is the Vander Monde's determinant of order three in the form:

$$(2.1)\qquad v(x;r=0,k=0)=\begin{vmatrix}1 & x_0 & x_0^2\\ 1 & x_1 & x_1^2\\ 1 & x_2 & x_2^2\end{vmatrix}$$

which is equivalently

$$(2.2)\qquad v(x;r=0,k=0)=(x_1-x_0)(x_2-x_0)(x_2-x_1)$$

Lemma 2.2: For $\varphi(x)= x^{1/2}$ and $x = (x_0, x_1, ...x_n)$ Eq.(1.2) is the Vander Monde's determinant of order three in the form:

$$(2.3)\qquad v\left(x;r=-\tfrac{3}{2},k=0\right)=\begin{vmatrix}1 & x_0 & \sqrt{x_0}\\ 1 & x_1 & \sqrt{x_1}\\ 1 & x_2 & \sqrt{x_2}\end{vmatrix}$$

which is equivalently

$$(2.4)\qquad v\left(x;\ r=-\tfrac{3}{2},k=0\right)=\left(\sqrt{x_1}-\sqrt{x_0}\right)\left(\sqrt{x_2}-\sqrt{x_0}\right)\left(\sqrt{x_2}-\sqrt{x_1}\right)$$

Setting $a = x$ and $b = 1$ in arithmetic mean, geometric mean, harmonic mean, Heron mean, Contra Harmonic Mean and Centriodal Mean takes the following functions form;

$$A(x,1)=\frac{x+1}{2}$$

$$G(x,1)=\sqrt{x}$$

$$H(x,1)=\frac{2x}{x+1}$$

$$H_e(x,1) = \frac{x+1+\sqrt{x}}{3}$$

$$C(x,1) = \frac{x^2+1}{x+1}$$

$$C_d(x,1) = \frac{2(x^2+1+x)}{3(x+1)}$$

Lemma 2.3: Let $f(x)$ and $g(x)$ are two functions, then $f(x)$ is said to be convex with respect to $g(x)$ for $a \le b \le c$ if and only if

(2.5) $$\begin{vmatrix} 1 & f(a) & g(a) \\ 1 & f(b) & g(b) \\ 1 & f(c) & g(c) \end{vmatrix} \ge 0$$

which is equivalently

(2.6) $$\begin{vmatrix} 1 & f(a) & g(a) \\ 1 & f(b)-f(a) & g(b)-g(a) \\ 1 & f(c)-f(a) & g(c)-g(a) \end{vmatrix} \ge 0$$

or

(2.7) $$[f(b)-f(a)][g(c)-g(a)]-[f(c)-f(a)][g(b)-g(a)] \ge 0$$

3. MAIN RESULTS

In this area, the fundamental and sufficient conditions for convexity proportion of contrast mean with deference to another distinction of means are talked about and the conditions are communicated as far as Vander Monde's determinants.

Theorem 3.1. In the proportion of difference of means

$$M_{CAGH}(a,b) = \frac{C(a,b)-A(a,b)}{G(a,b)-H(a,b)}$$

the difference of mean $C(a,b)-A(a,b)$ is convex(concave) with respect to another difference of mean $G(a,b)-H(a,b)$ if and only if $v\left(a; r=-\frac{3}{2}, k=0\right) \ge (\le) 0$

Proof. Consider the difference of mean $C(x,1)-A(x,1)$ and $G(x,1)-H(x,1)$ in the form;

$$C(x,1)-A(x,1) = \frac{(x-1)^2}{2(x+1)} \quad \text{and} \quad G(x,1)-H(x,1) = \sqrt{x} - \frac{2x}{x+1}$$

Let

$$f(x) = \frac{(x-1)^2}{2(x+1)} \quad \text{and} \quad g(x) = \sqrt{x} - \frac{2x}{x+1}$$

then by Lemma 2.3 we have

$$\begin{vmatrix} 1 & f(a) & g(a) \\ 1 & f(b)-f(a) & g(b)-g(a) \\ 1 & f(c)-f(a) & g(c)-g(a) \end{vmatrix} \geq 0$$

or

$$[f(b)-f(a)][g(c)-g(a)]-[f(c)-f(a)][g(b)-g(a)] \quad \geq 0$$

on simplifying the determinant leads to

$$f(b)[g(c)-g(a)]+f(c)[g(b)-g(a)]+f(a)[g(b)-g(c)] \quad \geq 0$$

$$\sum f(b)[g(c)-g(a)] \geq 0$$

$$\sum \frac{(a-1)^2}{2(a+1)}\left[\sqrt{b}-\sqrt{c}-\frac{2(b-c)}{(b+1)(c+1)}\right]$$

(3.1) $$\sum \frac{(a-1)^2\left(\sqrt{b}-\sqrt{c}\right)}{2(a+1)(b+1)(c+1)}\left[(b+1)(c+1)-2\left(\sqrt{b}+\sqrt{c}\right)\right] \geq 0$$

From Lemma 2.2

(3.2) $$\sum (a-1)^2\left(\sqrt{b}-\sqrt{c}\right)\left[(b+1)(c+1)-2\left(\sqrt{b}+\sqrt{c}\right)\right]=v\left(a;r=-\tfrac{3}{2},k=0\right) \geq 0$$

similarly by considering

$$f(x)=\sqrt{x}-\frac{2x}{x+1} \text{ and } g(x)=\frac{(x-1)^2}{2(x+1)}$$

(3.2) $$\sum (a-1)^2\left(\sqrt{b}-\sqrt{c}\right)\left[(b+1)(c+1)-2\left(\sqrt{b}+\sqrt{c}\right)\right]=v\left(a;r=-\tfrac{3}{2},k=0\right) \leq 0$$

By combining equations 3.2 and 3.3 ,the proof of Theorem 3.1 is completes.

Theorem 3.2. In the proportion of difference of means

$$M_{AH_eGH}(a,b)=\frac{A(a,b)-H_e(a,b)}{G(a,b)-H(a,b)}$$

the difference of mean $A(a,b)-H_e(a,b)$ is convex(concave) with respect to another difference of mean $G(a,b)-H(a,b)$ if and only if $v\left(x;r=-\tfrac{3}{2},k=0\right) \geq (\leq) 0$

Proof. Consider the difference of mean $A(x,1)-H_e(x,1)$ and $G(x,1)-H(x,1)$ in the form;

$$A(x,1)-H_e(x,1)=\frac{x+1-2\sqrt{x}}{6} \text{ and } G(x,1)-H(x,1)=\sqrt{x}-\frac{2x}{x+1}$$

Let

$$f(x)=\frac{x+1-2\sqrt{x}}{6} \text{ and } g(x)=\sqrt{x}-\frac{2x}{x+1}$$

then by Lemma 2.3 we have

$$\begin{vmatrix} 1 & f(a) & g(a) \\ 1 & f(b)-f(a) & g(b)-g(a) \\ 1 & f(c)-f(a) & g(c)-g(a) \end{vmatrix} \geq 0$$

or

$$[f(b)-f(a)][g(c)-g(a)]-[f(c)-f(a)][g(b)-g(a)] \geq 0$$

on simplifying the determinant leads to

$$f(b)[g(c)-g(a)]+f(c)[g(b)-g(a)]+f(a)[g(b)-g(c)] \geq 0$$

$$\sum f(b)[g(c)-g(a)] \geq 0$$

$$\sum \frac{(\sqrt{a}-1)^2}{6}\left[\sqrt{b}-\sqrt{c}-\frac{2(b-c)}{(b+1)(c+1)}\right]$$

(3.4) $$\sum \frac{(\sqrt{a}-1)^2(\sqrt{b}-\sqrt{c})}{6(b+1)(c+1)}\left[(b+1)(c+1)-2(\sqrt{b}+\sqrt{c})\right] \geq 0$$

From Lemma 2.2

(3.5) $$\sum(\sqrt{a}-1)^2(\sqrt{b}-\sqrt{c})\left[(b+1)(c+1)-2(\sqrt{b}+\sqrt{c})\right]=v\left(a;r=-\tfrac{3}{2},k=0\right) \geq 0$$

similarly by considering

$$f(x)=\sqrt{x}-\frac{2x}{x+1} \text{ and } g(x)=\frac{x+1-2\sqrt{x}}{6}$$

(3.2) $$\sum(\sqrt{a}-1)^2(\sqrt{b}-\sqrt{c})\left[(b+1)(c+1)-2(\sqrt{b}+\sqrt{c})\right]=v\left(a;r=-\tfrac{3}{2},k=0\right) \leq 0$$

By combining equations 3.4 and 3.5,the proof of Theorem 3.2 is completes.

Theorem 3.3. In the proportion of difference of means

$$\frac{M_{SH_e} - M_{SA}}{M_{SN_1} - M_{SH_e}} = \frac{A - H_e}{H_e - N_1}$$

the difference of mean $A(a,b) - H_e(a,b)$ is convex(concave) with respect to another difference of mean $H_e(a,b) - N_1(a,b)$ if and only if $v\left(a; r = -3/2, k = 0\right) \geq (\leq) 0$

Proof. Consider the difference of mean $A(x,1) - H_e(x,1)$ and $H_e(x,1) - N_1(x,1)$ in the form;

$$A(x,1) - H_e(x,1) = \frac{x + 1 - 2\sqrt{x}}{6} \text{ and } H_e(x,1) - N_1(x,1) = \frac{\left(\sqrt{x} - 1\right)^2}{12}$$

Let

$$f(x) = \frac{x + 1 - 2\sqrt{x}}{6} \text{ and } g(x) = \frac{\left(\sqrt{x} - 1\right)^2}{12}$$

then by Lemma 2.3 we have

$$\begin{vmatrix} 1 & f(a) & g(a) \\ 1 & f(b) - f(a) & g(b) - g(a) \\ 1 & f(c) - f(a) & g(c) - g(a) \end{vmatrix} \geq 0$$

or

$$[f(b) - f(a)][g(c) - g(a)] - [f(c) - f(a)][g(b) - g(a)] \geq 0$$

on simplifying the determinant leads to

$$f(b)[g(c) - g(a)] + f(c)[g(b) - g(a)] + f(a)[g(b) - g(c)] \geq 0$$

$$\sum f(b)[g(c) - g(a)] \geq 0$$

$$\sum \frac{\left(\sqrt{a} - 1\right)^2}{6} \left[\frac{\left(\sqrt{b} - \sqrt{c}\right)\left(\sqrt{b} + \sqrt{c} - 2\right)}{12} \right]$$

(3.7) $$\sum \frac{\left(\sqrt{a} - 1\right)^2}{72} \left(\left(\sqrt{b} - \sqrt{c}\right)\left(\sqrt{b} + \sqrt{c} - 2\right)\right) \geq 0$$

From Lemma 2.2

(3.8) $$\sum \left(\sqrt{a} - 1\right)^2 \left(\left(\sqrt{b} - \sqrt{c}\right)\left(\sqrt{b} + \sqrt{c} - 2\right)\right) = v\left(x; r = -3/2, k = 0\right) \geq 0$$

similarly by considering

$$f(x)=\frac{\left(\sqrt{x}-1\right)^2}{12} \text{ and } g(x)=\frac{x+1-2\sqrt{x}}{6}$$

(3.9) $$\sum\left(\sqrt{a}-1\right)^2\left(\left(\sqrt{b}-\sqrt{c}\right)\left(\sqrt{b}+\sqrt{c}-2\right)\right)=v\left(x;r=-\tfrac{3}{2},k=0\right)\leq 0$$

By combining equations 3.8 and 3.9,the proof of Theorem 3.3 is completes.

Theorem 3.4. In the proportion of difference of means

$$\frac{M_{SN_2}-M_{SA}}{M_{SH_e}-M_{SN_2}}=\frac{A-N_2}{N_2-H_e}$$

the difference of mean $A(a,b)-N_2(a,b)$ is convex(concave) with respect to another difference of mean $N_2(a,b)-H_e(a,b)$ if and only if $v\left(a;r=-\tfrac{3}{2},k=0\right)\geq(\leq)0$

Proof. Consider the difference of mean $A(x,1)-N_2(x,1)$ and $N_2(x,1)-H_e(x,1)$ in the form;

$$A(x,1)-N_2(x,1)=\sqrt{\frac{x+1}{2}}\left(\frac{\sqrt{x+1}}{\sqrt{2}}-\frac{\sqrt{x}+1}{2}\right) \text{ and } N_2(x,1)-H_e(x,1)=\sqrt{\frac{x+1}{2}}\left(\frac{\sqrt{x}+1}{2}\right)-\frac{x+1+\sqrt{x}}{3}$$

Let

$$f(x)=\sqrt{\frac{x+1}{2}}\left(\frac{\sqrt{x+1}}{\sqrt{2}}-\frac{\sqrt{x}+1}{2}\right) \text{ and } g(x)=\sqrt{\frac{x+1}{2}}\left(\frac{\sqrt{x}+1}{2}\right)-\frac{x+1+\sqrt{x}}{3}$$

then by Lemma 2.3 we have

$$\begin{vmatrix}1 & f(a) & g(a)\\ 1 & f(b)-f(a) & g(b)-g(a)\\ 1 & f(c)-f(a) & g(c)-g(a)\end{vmatrix}\geq 0$$

or

$$[f(b)-f(a)][g(c)-g(a)]-[f(c)-f(a)][g(b)-g(a)]\quad\geq 0$$

on simplifying the determinant leads to

$$f(b)[g(c)-g(a)]+f(c)[g(b)-g(a)]+f(a)[g(b)-g(c)]\quad\geq 0$$

$$\sum f(b)[g(c)-g(a)]\geq 0$$

(3.10) $$\sum\sqrt{\frac{a+1}{2}}\left(\frac{\sqrt{a+1}}{\sqrt{2}}-\frac{\sqrt{a}+1}{2}\right)\left(\frac{\left(\sqrt{b}+1\right)\left(3\sqrt{b}+1-2\sqrt{2}\right)-\left(\sqrt{c}+1\right)\left(3\sqrt{c}+1-2\sqrt{2}\right)-2\sqrt{2}(b+c)}{6\sqrt{2}}\right)\geq 0$$

From Lemma 2.2

$$(3.11)\quad \sum\sqrt{\frac{a+1}{2}}\left(\frac{\sqrt{a+1}}{\sqrt{2}}-\frac{\sqrt{a}+1}{2}\right)\left(\frac{\left(\sqrt{b}+1\right)\left(3\sqrt{b}+1-2\sqrt{2}\right)-\left(\sqrt{c}+1\right)\left(3\sqrt{c}+1-2\sqrt{2}\right)-2\sqrt{2}(b+c)}{6\sqrt{2}}\right)=$$
$$v\left(a;r=-\tfrac{3}{2},k=0\right)\geq 0$$

similarly by considering

$$f(x)=\sqrt{\frac{x+1}{2}}\left(\frac{\sqrt{x}+1}{2}\right)-\frac{x+1+\sqrt{x}}{3} \text{ and } g(x)=\sqrt{\frac{x+1}{2}}\left(\frac{\sqrt{x+1}}{\sqrt{2}}-\frac{\sqrt{x}+1}{2}\right)$$

$$(3.12)\quad \sum\sqrt{\frac{a+1}{2}}\left(\frac{\sqrt{a+1}}{\sqrt{2}}-\frac{\sqrt{a}+1}{2}\right)\left(\frac{\left(\sqrt{b}+1\right)\left(3\sqrt{b}+1-2\sqrt{2}\right)-\left(\sqrt{c}+1\right)\left(3\sqrt{c}+1-2\sqrt{2}\right)-2\sqrt{2}(b+c)}{6\sqrt{2}}\right)=$$
$$v\left(a;r=-\tfrac{3}{2},k=0\right)\leq 0$$

By combining equations 3.11 and 3.12,the proof of Theorem 3.4 is completes.

Theorem 3.5. In the proportion of difference of means

$$\frac{M_{N_1H}-M_{GH}}{M_{SH}-M_{SG}}=\frac{N_1-G}{G-H}$$

the difference of mean $N_1(a,b)-G(a,b)$ is convex(concave) with respect to another difference of mean $G(a,b)-H(a,b)$ if and only if $v\left(a;r=-\tfrac{3}{2},k=0\right)\geq(\leq)0$

Proof. Consider the difference of mean $N_1(x,1)-G(x,1)$ and $G(x,1)-H(x,1)$ in the form;

$$N_1(x,1)-G(x,1)=\left(\frac{\sqrt{x}-1}{2}\right)^2 \text{ and } G(x,1)-H(x,1)=\sqrt{x}-\frac{2x}{x+1}$$

Let

$$f(x)=\left(\frac{\sqrt{x}-1}{2}\right)^2 \text{ and } g(x)=\sqrt{x}-\frac{2x}{x+1}$$

then by Lemma 2.3 we have

$$\begin{vmatrix}1 & f(a) & g(a)\\ 1 & f(b)-f(a) & g(b)-g(a)\\ 1 & f(c)-f(a) & g(c)-g(a)\end{vmatrix}\geq 0$$

or

$$[f(b)-f(a)][g(c)-g(a)]-[f(c)-f(a)][g(b)-g(a)] \geq 0$$

on simplifying the determinant leads to

$$f(b)[g(c)-g(a)]+f(c)[g(b)-g(a)]+f(a)[g(b)-g(c)] \geq 0$$

$$\sum f(b)[g(c)-g(a)] \geq 0$$

(3.15) $$\sum\left(\frac{\sqrt{a}-1}{2}\right)^2\left(\sqrt{b}-\sqrt{c}\right)(b+1)(c+1)-2\left(\sqrt{b}+\sqrt{c}\right) \geq 0$$

From Lemma 2.2

(3.16) $$\sum\left(\frac{\sqrt{a}-1}{2}\right)^2\left(\sqrt{b}-\sqrt{c}\right)(b+1)(c+1)-2\left(\sqrt{b}+\sqrt{c}\right)=v\left(a;r=-\tfrac{3}{2},k=0\right)\geq 0$$

similarly by considering

$$f(x)=\sqrt{x}-\frac{2x}{x+1} \text{ and } g(x)=\left(\frac{\sqrt{x}-1}{2}\right)^2$$

(3.17) $$\sum\left(\frac{\sqrt{a}-1}{2}\right)^2\left(\sqrt{b}-\sqrt{c}\right)(b+1)(c+1)-2\left(\sqrt{b}+\sqrt{c}\right)=v\left(a;r=-\tfrac{3}{2},k=0\right)=v\left(a;r=-\tfrac{3}{2},k=0\right)\leq 0$$

By combining equations 3.16 and 3.17,the proof of Theorem 3.5 is completes.

4. Conclusion

Therefore, for various mixes of standard means communicated in practical shape, it has been conceivable to express concavity or convexity conditions as far as Vander monde determinants. It would hold any importance with think about more capacities at once and the conditions communicated regarding Vander monde determinants, might be diminished to inclining structure or triangular shape. This will help in promptly utilizing them as conditions at whatever point required in advance applications. Pertinence of these conditions in computational work must be investigated and there is a degree for additionally think about.

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