RELATION BETWEEN ELECTRIC AND MAGNETIC FIELDS PRODUCED DUE TO LIGHTNING DISCHARGE

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Abstract: Maxwell's equations are a set of partial differential equations which relates the electric and magnetic fields that are generated by charges and currents. Maxwell's equations represent and relate the fundamentals of electricity and magnetism produced due to lightning discharge. The major consequence of these equations is that they visualize how varying electric and magnetic fields propagate at the speed of light. In Maxwell's electrodynamics, formulated as it is in terms of charge and current densities, a point charge must be regarded as the limit of an extended charge, when the size goes to zero, the retardation by a F-factor i. e. $(1 - r.\frac{v}{c})^{-1}$ of the charge.

Keywords: Maxwell's equations; Electric Field; Magnetic Field; F- factor.

1. Introduction

Maxwell's equations are a set of partial differential equations that describe how electric and magnetic fields are produced due to lightning discharge. The major consequence of these equations is that they visualize how varying electric and magnetic fields produced due to lightning discharge propagate at the speed of light. These equations are named after the physicists and mathematician James Clerk Maxwell. He first purposed the electromagnetic phenomenon of light and published an early form of these equations. Maxwell's equations represent one of the most elegant and concise ways to state the fundamentals of electricity and magnetism. Volume charge density (ρ) and Volume current density (**J**) generate the electric and magnetic field. Let **E(r, t)** and **B**(**r**, t) are the electric and magnetic field intensity, $\mu_\circ$ and $\varepsilon_\circ$ are the permeability and permittivity. The general form of Maxwell's equations are:

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{r},t) = \frac{\rho(\boldsymbol{r}',t_r)}{\varepsilon_\circ} \quad (1)$$

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{r},t) = 0 \quad (2)$$

$$\nabla \times \boldsymbol{E}(\boldsymbol{r},t) = -\frac{\partial \boldsymbol{B}(\boldsymbol{r},t)}{\partial t} \quad (3)$$

$$\nabla \times \boldsymbol{B}(\boldsymbol{r},t) = \mu_\circ \boldsymbol{J}(\boldsymbol{r}',t_r) + \frac{1}{c^2}\frac{\partial \boldsymbol{E}(\boldsymbol{r},t)}{\partial t} \quad (4)$$

Where c is the speed of light which relates $c^2 = \frac{1}{\mu_\circ \varepsilon_\circ}$

So the electric and magnetic fields are $\boldsymbol{E}(\boldsymbol{r},t) = -\nabla V(\boldsymbol{r},t)$ *and* $\boldsymbol{B}(\boldsymbol{r},t) = \nabla \times \boldsymbol{A}(\boldsymbol{r},t)$.

Hence $\nabla \times \boldsymbol{E}(\boldsymbol{r},t) + \frac{\partial \boldsymbol{A}(\boldsymbol{r},t)}{\partial t} = 0$ and $\nabla^2 V(\boldsymbol{r},t) + \frac{\partial \nabla \cdot \boldsymbol{A}(\boldsymbol{r},t)}{\partial t} = -\frac{\rho(\boldsymbol{r}',t_r)}{\varepsilon_\circ}$ (5)

Substituting equations of $\boldsymbol{E}(\boldsymbol{r},t)$ and $\boldsymbol{B}(\boldsymbol{r},t)$ into Faraday's law and as the term whose curl gives zero can be written as the gradient of scalar. Here t_r is the retarded time in the above equation (5) which is the combined form of the Maxwell's equations.

Again from the Maxwell's fourth relation,

$$\nabla^2 \boldsymbol{A}(\boldsymbol{r},t) - \mu_\circ\varepsilon_\circ \frac{\partial^2 \boldsymbol{A}(\boldsymbol{r},t)}{\partial t^2} - \nabla\left(\nabla\cdot\boldsymbol{A}(\boldsymbol{r},t) + \mu_\circ\varepsilon_\circ \frac{\partial V(\boldsymbol{r},t)}{\partial t}\right) = -\mu_\circ \boldsymbol{J}(\boldsymbol{r}',t_r) \quad (6)$$

Hence these two equation (5) and equation (6) combine contain all the information of Maxwell's equation.

For the non-static cases, $V(\boldsymbol{r},t) = \frac{1}{4\pi\varepsilon_\circ}\int \frac{\rho(\boldsymbol{r}',t_r)}{r} d\tau$ *and* $\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_\circ}{4\pi}\int \frac{\boldsymbol{J}(\boldsymbol{r}',t_r)}{r} d\tau$

Where $\rho(\boldsymbol{r}',t)$ is the charge density that prevailed at point **r'** at the retarded time where

$$t_r = t - \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}$$

We know from the Lorentz condition,

$$\nabla\cdot\boldsymbol{A}(\boldsymbol{r},t) + \mu_\circ\varepsilon_\circ \frac{\partial V(\boldsymbol{r},t)}{\partial t} = 0$$

For any scalar function in the operation of addition is possible. Hence there is no effect by addition or subtraction on the electric and magnetic field (E and B). Griffiths, (1999), described such changes in scalar potentials are called gauge transformation [1]. Hence the gradient of scalar potential gives,

$$\nabla V(\boldsymbol{r},t) = \frac{1}{4\pi\varepsilon_\circ}\int\left[-\frac{\dot{\rho}(\mathbf{r}',t_r)}{\mathrm{c}}\frac{\hat{\boldsymbol{r}}}{\mathrm{r}} - \rho(\boldsymbol{r}',t_r)\frac{\hat{\boldsymbol{r}}}{r^2}\right] d\tau$$

By substituting $\nabla\rho(\boldsymbol{r}',t_r) = \dot{\rho}(\mathbf{r}',t_r)\ \nabla t_r = -\frac{1}{\mathrm{c}}\dot{\rho}(\mathbf{r}',t_r)\nabla \mathrm{r}, \nabla\mathrm{r} = \hat{\boldsymbol{r}}$ and $\nabla\frac{1}{\mathrm{r}} = -\frac{\hat{\mathrm{r}}}{\mathrm{r}^2}$, $\hat{\boldsymbol{r}} = \frac{\mathrm{r}}{\mathrm{r}}$

On taking the divergence of the gradient of scalar potential we know,

$$\nabla^2 V(\boldsymbol{r},t) = \frac{1}{4\pi\varepsilon_\circ}\int\left[\frac{1}{\mathrm{c}^2}\frac{\ddot{\rho}(\mathbf{r}',t_r)}{\mathrm{r}} - 4\pi\rho(\boldsymbol{r}',t_r)\delta^3(\hat{\boldsymbol{r}})\right] d\tau$$

$$\nabla^2 V(\boldsymbol{r},t) = \frac{1}{c^2}\frac{\partial^2 V(\boldsymbol{r},t)}{\partial t^2} - \frac{1}{\varepsilon_\circ}\rho(\boldsymbol{r}',t_r)$$

By substituting $\nabla\dot{\rho}(\mathbf{r}',t_r) = -\frac{1}{\mathrm{c}}\ddot{\rho}(\mathbf{r}',t_r)\nabla\mathrm{r}$, $\nabla\cdot\frac{\hat{\mathbf{r}}}{\mathrm{r}} = \frac{1}{\mathrm{r}^2}$ $\nabla\cdot\frac{\mathrm{r}}{\mathrm{r}^2} = 4\pi\delta^3(\hat{\boldsymbol{r}})$

Where, $\delta^3(\hat{\boldsymbol{r}})$ is a three dimensional Dirac delta function. The time derivative of scalar potential A is $\frac{\partial \boldsymbol{A}(\boldsymbol{r},t)}{\partial t} = \frac{\mu_\circ}{4\pi}\int\frac{\dot{\mathrm{J}}(\boldsymbol{r}',t_r)}{r} d\tau$

Hence, $\boldsymbol{E}(\boldsymbol{r},t) = \frac{1}{4\pi\varepsilon_\circ}\int\left[\frac{\rho(\mathbf{r}',\mathrm{t})}{\mathrm{r}^2}\hat{\boldsymbol{r}} + \frac{\dot{\rho}(\mathbf{r}',\mathrm{t})}{\mathrm{cr}}\hat{\boldsymbol{r}} - \frac{\dot{\mathrm{J}}(\boldsymbol{r}',t_r)}{rc^2}\right] d\tau$ (7)

This is the time dependent generalization of coulombs law. Hence, equation (7) is known as the Jefimenko electric field equation derived from Maxwell's equation [2 - 4].

2. Theory and Discussion

In Maxwell's electrodynamics, formulated as it is in terms of charge and current densities, a point charge must be regarded as the limit of an extended charge, when the size goes to zero, the retardation by a factor $(1 - \mathrm{r}.\frac{\mathbf{V}}{\mathbf{C}})^{-1}$ of the charge [5 - 10].

$$\int \rho(\mathbf{r}', t_r)\, d\tau = \frac{q}{\left(1 - \boldsymbol{\mathscr{r}}.\frac{\mathbf{v}}{c}\right)}$$

Where v is the velocity of the charge at the retarded time.

$$\text{Then, } V(\mathbf{r}, t) = \frac{1}{4\pi \in_0}.\int \frac{\rho(\mathbf{r}', t_r)}{\mathscr{r}} d\tau = \frac{1}{4\pi \in_0} \frac{q}{\mathscr{r}.\left(1 - \frac{\boldsymbol{\mathscr{r}}.\mathbf{v}}{c}\right)}$$

$$V(\mathbf{r}, t) = \frac{1}{4\pi \in_0} \frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}}.\mathbf{v})}$$

where $\boldsymbol{\mathscr{r}}$ is the vector from the retarded position to the field point r.

$$\text{Again, } A(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\rho(\mathbf{r}', t_r)\, \mathbf{v}(t_r)}{\mathscr{r}} d\tau = \frac{\mu_0}{4\pi} \frac{\mathbf{v}}{\mathscr{r}} \int \rho(\mathbf{r}', t_r)\, d\tau$$

$$\frac{\mu_0}{4\pi} \frac{qc\mathbf{v}}{(\mathscr{r}c - \boldsymbol{\mathscr{r}}.\mathbf{v})} = \frac{\mathbf{v}}{c^2} V(\mathbf{r}, t)$$

$$\text{Gradient of } V = \nabla V = \nabla \left\{ \frac{1}{4\pi \in_0} \frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}}.\mathbf{v})} \right\}$$

$$= \frac{qc}{4\pi \in_0} . \frac{-1}{(\mathscr{r}c - \boldsymbol{\mathscr{r}}.\mathbf{v})^2} . \nabla(\mathscr{r}c - \boldsymbol{\mathscr{r}}.\mathbf{v})$$

Since $r = c\,(t - t_r)$ and $t_r = t - \frac{\mathscr{r}}{c}$, $\nabla r = -c.\, \nabla t_r$

$$\nabla(\boldsymbol{\mathscr{r}}.\mathbf{v}) = (\boldsymbol{\mathscr{r}}.\nabla)\mathbf{v} + (\mathbf{v}.\nabla)\boldsymbol{\mathscr{r}} + \boldsymbol{\mathscr{r}} \times \nabla \times \mathbf{v} + \mathbf{v} \times (\nabla \times \boldsymbol{\mathscr{r}})$$

Evaluating these terms (one by one)

$$(\boldsymbol{\mathscr{r}}.\nabla)\mathbf{v} = \left(\mathscr{r}_x \frac{\partial}{\partial x} + \mathscr{r}_y \frac{\partial}{\partial y} + \mathscr{r}_z \frac{\partial}{\partial z} \right) \mathbf{v}(t_r)$$

$$= \mathscr{r}_x . \frac{\partial \mathbf{v}}{\partial t_r} . \frac{\partial t_r}{\partial x} + \mathscr{r}_y . \frac{\partial \mathbf{v}}{\partial t_r} . \frac{\partial t_r}{\partial y} + \mathscr{r}_z . \frac{\partial \mathbf{v}}{\partial t_r} . \frac{\partial t_r}{\partial z}$$

$$= \frac{\partial \mathbf{v}}{\partial t_r} \left(\mathscr{r}_x \frac{\partial t_r}{\partial x} + \mathscr{r}_y \frac{\partial t_r}{\partial y} + \mathscr{r}_z \frac{\partial t_r}{\partial z} \right)$$

$$= \frac{\partial \mathbf{v}}{\partial t_r} \left(\mathscr{r}_x \frac{\partial}{\partial x} + \mathscr{r}_y \frac{\partial}{\partial y} + \mathscr{r}_z \frac{\partial}{\partial z} \right) t_r = \mathbf{a}(\boldsymbol{\mathscr{r}}.\nabla t_r)$$

where $\mathbf{a} \equiv \mathbf{v}$ is the acceleration of the particle at the retarded time.

$$(\mathbf{v}.\nabla)\mathscr{r} = (\mathbf{v}.\nabla)\mathbf{r} - (\mathbf{v}.\nabla)\mathbf{w} \qquad [\because \mathscr{r} = \mathbf{r} - \mathbf{w}]$$

and
$$(\mathbf{v}.\nabla)\mathbf{r} = \left(v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}\right)(X\mathbf{x} + Y\mathbf{y} + Z\mathbf{z})$$

$$= (V_x\mathbf{x} + V_y\mathbf{y} + V_z\mathbf{z}) = \vec{v}$$

While
$$(\mathbf{v}.\nabla)\mathbf{w} = \left(v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}\right)\mathbf{w}.(t_r)$$

$$= v_x.\frac{\partial \mathbf{w}}{\partial t_r}.\frac{\partial t_r}{\partial x} + v_y\frac{\partial \mathbf{w}}{\partial t_r}\frac{\partial t_r}{\partial y} + v_z\frac{\partial \mathbf{w}}{\partial t_r}\frac{\partial t_r}{\partial z}$$

$$= \frac{\partial \mathbf{w}}{\partial t_r}\left(v_x\frac{\partial t_r}{\partial x} + v_y\frac{\partial t_r}{\partial y} + v_z\frac{\partial t_r}{\partial z}\right) = \mathbf{v}(\mathbf{v}.\nabla t_r)$$

Hence,
$$(\mathbf{v}.\nabla)\mathscr{r} = \mathbf{v} - \mathbf{v}(\mathbf{v}.\nabla t_r)$$

For third term,
$$\nabla\times\mathbf{v} = -\mathbf{a}\times\nabla t_r$$

Similarly,
$$\nabla\times\mathbf{w} = -\mathbf{v}\times\nabla t_r$$

Again,
$$\nabla\times\mathscr{r} = \nabla\times(\mathbf{r} - \mathbf{w}) = \nabla\times\mathbf{r} - \nabla\times\mathbf{w}$$

$$= 0 - \nabla\times\mathbf{w} = -\mathbf{v}\times\nabla t_r$$

Substituting these all values then we get,

$$\nabla(\mathscr{r}.\mathbf{v}) = \vec{a}\ (\mathscr{r}.\nabla t_r) + \mathbf{v} - \mathbf{v}(\mathbf{v}.\nabla t_r) - \mathscr{r}\times\mathbf{a}\times\nabla t_r + \mathbf{v}\times\mathbf{v}\times\nabla t_r$$

$$= \mathbf{v} + (\mathscr{r}.\mathbf{a} - \mathbf{v}^2)\nabla t_r$$

Hence,
$$\nabla\vec{v} = \frac{qc}{4\pi\epsilon_0}.\frac{1}{(\mathscr{r}c - \mathscr{r}.\mathbf{v})^2}[\mathbf{v} + (c^2 - v^2 + \mathscr{r}.\mathbf{a})\nabla t_r]$$

Now, for ∇t_r, $\nabla r = -c\nabla t_r$

So,
$$-c\nabla t_r = \nabla r = \nabla\sqrt{\mathscr{r}.\mathscr{r}} = \frac{1}{2}\frac{1}{\sqrt{\mathscr{r}.\mathscr{r}}}\nabla(\mathscr{r}.\mathscr{r})$$

$$= \frac{1}{\mathscr{r}}[(\mathscr{r}.\nabla)\mathscr{r} + \mathscr{r}\times(\nabla\times\mathscr{r})]$$

But $(\mathscr{r}.\nabla)r = \mathscr{r} - \vec{v}(\mathscr{r}.\nabla t_r)$

Same as above,
$$\nabla\times\mathscr{r} = \mathbf{v}\times\nabla t_r$$

Thus, $-c\nabla t_r = \frac{1}{\mathscr{r}}\left[\boldsymbol{\mathscr{r}} - \mathbf{v}(\boldsymbol{\mathscr{r}}.\nabla t_r) + \boldsymbol{\mathscr{r}} \times (\mathbf{v} \times \nabla t_r)\right] = \frac{1}{\mathscr{r}}\left[\boldsymbol{\mathscr{r}} - (\boldsymbol{\mathscr{r}}.\mathbf{v})\nabla t_r\right]$

$\therefore$ $$\nabla t_r = \frac{-\boldsymbol{\mathscr{r}}}{\mathscr{r}c - \boldsymbol{\mathscr{r}}.\mathbf{v}}$$

Hence, the gradient of V = ∇V

$$= \frac{1}{4\pi \in_0}\frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}}.\mathbf{v})^3}[(\mathscr{r}c - \boldsymbol{\mathscr{r}}.\mathbf{v})\mathbf{v} - (c^2 - v^2) + (\boldsymbol{\mathscr{r}}.\mathbf{a})\boldsymbol{\mathscr{r}}]$$

Similarly, for $\frac{\partial \mathbf{A}}{\partial t} = \frac{1}{4\pi \in_0}\frac{qc}{(\mathscr{r}c - \hat{\mathscr{r}}.\vec{v})^3}[(\mathscr{r}c - \boldsymbol{\mathscr{r}}.\mathbf{v})\left(-\mathbf{v} + \frac{\mathscr{r}\mathbf{a}}{c}\right) + \frac{\mathscr{r}}{c}\{c^2 - v^2 + (\boldsymbol{\mathscr{r}}.\mathbf{a})\}\mathbf{v}]$

Then $\vec{u} \equiv c\boldsymbol{\mathscr{r}} - \mathbf{v}$

Then, E (**r**, t) = $\frac{q}{4\pi \in_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}} - \mathbf{u})^3}[(c^2 - v^2)\mathbf{u} + \boldsymbol{\mathscr{r}} \times (\mathbf{u} \times \mathbf{a})]$

Meanwhile, $\nabla \times$ **A**(**r**, t) = $\frac{1}{c^2}\nabla \times (v\mathbf{v}) = \frac{1}{c^2}[v(\nabla \times \mathbf{v}) - \mathbf{v} \times \nabla v]$

or, $\nabla \times$ **A** (**r**, t) = $\frac{1}{c}\frac{q}{4\pi \in_0}\frac{1}{(\mathbf{u}.\boldsymbol{\mathscr{r}})^3}\boldsymbol{\mathscr{r}} \times [(c^2 - v^2)\mathbf{v} + (\boldsymbol{\mathscr{r}}.\mathbf{a})\mathbf{v} + (\boldsymbol{\mathscr{r}}.\mathbf{u})\mathbf{a}]$

Hence, **B** (**r**, t) = $\frac{1}{c}\boldsymbol{\mathscr{r}} \times \mathbf{E}(\mathbf{r}, t)$

3. Conclusion

There is a relation between electric and magnetic fields produced due to lightning discharge. In electromagnetism, the movement of electric field gives magnetic field and the movement of magnetic field produces electric field. In this paper, the cross-product of simply vector **r** and the electric field vector **E** produce the magnetic field. i. e.

B (**r**, t) = $\frac{1}{c}\boldsymbol{\mathscr{r}} \times \mathbf{E}(\mathbf{r}, t)$

4. References

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