GENERALSED CLOSED SETS GENERATED BY MULTIFUNCTIONS

Dr. Nandini Bandyopadhyay,
Assistant Professor, Department of Mathematics,
Panskura Banamali College, Panskura R. S. ,
Purba Medinipur,West Bengal, India.

Prof . Paritosh Bhattacharyya,
Professor, Department of Mathematics, University of Kalyani,
Kalyani, Nadia, West Bengal, India.

Abstract. Semi-precontinuous multifunction generates semi-preclosed sets under certain condtitions.This property of the semi precontinuous multifunction has been exhibited in this paper.

Key Words: spo, spc, α- set.

AMS subject classification. 54C99

1. Introduction.

In 1932, Kuratowski [5] and Bouligand [3] independently introduced two kinds of semi-continuity of a multifunction which are termed as upper semi-continuous and lower semi-continuous multifunctions. In 1986, Andrijević [1] introduced the notion of semi-preopen sets. In 1992, Przemski [8] defined semi-precontinuous function with the aid of semi-preopen sets. Bandyopadhay [2] defined semi-precontinuous multifunction as an extension of single valued function to multivalued function and carried further investigation in [2]. In course of the invesitgaton it has been found that semi -precontinuous multifunction can generate semi-preclosed sets.These observations have been incorporated in this paper.

2. PRELIMNARIES

Throughout the paper (X, τ) or simply X always denotes nontrivial topological spaces. The closure and interior of the subset A is denoted by Cl(A)(resp.Clx(A)) and Int(A)(resp.Intx(A) respectively.The family of all open sets containing a is denoted by Σ (a) The following definitions and results have been frequently utilised in this paper.

Definition 2.1. [1**]** In (X, τ), $A \subset X$ is called a semi-preopen set (briefly s.p.o. set) (resp. α- set) iff $A \subset$ Cl (Int (Cl (A))) (resp. $A\subset$ Int (Cl(Int (A)))).The family of all s.p.o. sets (resp.α- sets) is denoted by SPO(X)(resp. τ^{α}). For each $x \in X$, the family of all s.p.o. sets containing x is denoted by SPO(X, x).

Definition 2.2. [1**]** The complement of a s.p.o. set is called semi-preclosed. Equivalently a set F is semi-preclosed iff Int (Cl (Int (A))) $\subset$ F. The family of all semi-preclosed sets is denoted by SPF (X).

Definition 2.3. [8] A single valued function $f : X \rightarrow Y$ is said to be semi- precontinuous if the inverse image of every open set in Y is semi-preopen in X.

Definition 2.4 .[1] The semi-preclosure of $A \subset X$ is denoted by spcl (A) and is defined by spcl (A) = ∩ {B : B is semi-preclosed and $B \supset A$}.

Definition 2.5 .[4] A space X is called sp-T_2 if for every pair of dstinct points x,y of X there exist two disjoint s.p.o. sets U and V such that $x \in U$ and $y \in V$.

For a multifunction $F : X \rightarrow Y$, $F^+[B]$ and $F^-[B]$, respectively denote the upper and lower inverses of the set $B \subset Y$, where $F^+[B] = \{x \in X : F(x) \subset B\}$and $F^-[B] = \{x \in X : F(x) \cap B \neq \phi\}$.

Definition 2.6. [2] A multifunction $F : X \rightarrow Y$ is termed upper semi-precontinuous (resp. lower semi-precontinuous), briefly uspc (resp. lspc), iff for each closed (resp. open) set

$$A \subset Y, F^-[A] \in SPF(X) \text{ (resp. } F^-[A] \in SPO(X)).$$

Definition 2.7. [2] A multifunction $F : X \rightarrow Y$ is semi-precontinuous (briefly spc) iff F is both uspc and lspc.

Definition 2.8. [9] **.** A space (X, τ) will be said to have the property P if the closure is preserved under finite intersection.

Definition 2.9. [6] A subset A of X is said to be generalised closed set (briefly g-closed set) iff $Cl(A) \subset O$ whenever $A \subset O \in \tau$.

3.Results and discussions.

An interesting property enjoyed by this multifunction is that it can, under certain conditions, generate semi-preclosed sets. To prove the next theorem the following lemma is required.

Lemma 3.1. If A and B are two disjoint compact subsets of a Hausdorff space X, then there exist open sets U, V such that $A \subset U$, $B \subset V$ and $U \cap V = \phi$.

Proof. Let $a \in A$. Then the Hausdorffness of X indicates there exist $G_a \in \Sigma(a)$, $H_x \in \Sigma(x)$ such that $G_a \cap H_x = \phi$ …(1). Clearly $\{H_x : x \in B\}$ is an open cover of B. Since B is compact there exists a finite subfamily $\{H_{X1}, H_{X2}, \ldots, H_{Xn}\}$such that
$B \subset \cup \{H_{Xi} : i=1,2,\ldots,n\}$. Let $G_a^1, G_a^2, \ldots, G_a^n$ be the corresponding sets containing a and satisfying (1).
Let $U(a) = \cap \{G_a^i : i=1,2,\ldots n\}$and $V = \cup\{H_{Xi} : i=1,2,\ldots,n\}$. Clearly $U(a) \in \Sigma(a)$ and $V \in \Sigma$

(B).Then the family $\{U(a) : a \in A\}$ is an open cover of A. Since A is compact there exists a finite subfamily $\{U(a_1), U(a_2),\dots,U(a_t)\}$ such that $A \subset \underset{i}{\cup}\{U(a_k) : k=1,2,\dots t\} = U$ (say).Then $U \in \Sigma(a)$.Now $U \cap V = \cup(\{U(a_k):k=1,2,\dots,t\}) \cap V = \cup(\{(U(a_k) \cap V): k=1,2,\dots,t\})$…(1). Again $U(a_k) \cap V = (\cap\{G^i_{ak} : k=1,2,\dots,n\}) \cap (\cup Hx) = \cup(\cap\{G^i_a ; i=1,2,\dots,n\} \cap H_X) = \cup(G_a^1 \cap G_a^2 \cap \dots \cap G_a^n \cap H_X) = \phi$. So from above $U \cap V = \phi$. Thus there exist two open sets U, V such that

$$A \subset U, B \subset V \text{ and } U \cap V = \phi.$$

Theorem 3.1. If $F_i : (X, \tau) \to (Y, \sigma)$ are punctually compact uspc multifunction from X with property P to the Hausdorff space Y, then the set $A = \{x : F_1(x) \cap F_2(x) \neq \phi\} \in SPF(X)$.

Proof. Let $x \notin A$. Then $F_1(x) \cap F_2(x) = \phi$.Since F_i $(i = 1, 2)$ are punctually compact, $F_1(x)$, $F_2(x)$ are each compact. By (1), these compact sets are disjoint. The Hausdorffness of Y, by Lemma 3.1, gives the existence of $V_1 \in \Sigma(F_1(x))$, $V_2 \in \Sigma(F_2(x))$ in Y such that $V_1 \cap V_2 = \phi$. Again the uspc of F_1 and F_2 assures the existence of $U_1, U_2 \in SPO(X)$ such that $F_1(z) \subset V_1$ $\forall z \in U_1$and $F_2(z) \subset V_2$ $\forall z \in U_2$. Thus $F_1(z) \cap F_2(z) \subset V_1 \cap V_2 = \phi \Rightarrow F_1(z) \cap F_2(z) = \phi$.Let $U = U_1 \cap U_2$.Since X enjoys the property P, $U \in SPO(X)$.Now each $z \in U$ is a member of both U_1 and U_2 for which $F_1(z) \cap F_2(z) = \phi \Rightarrow z \notin A$. So, $U \cap A = \phi$.Therefore, $x \notin spcl(A) \Rightarrow spcl(A) \subset A \Rightarrow A \in SPF(X)$.

Definition 3.1. A Multifunction $F: X \to Y$ is upper α-continuous (briefly uαc) at $x \in X$ if for every $V \in \Sigma(F(x))$ there is a $U \in \tau^\alpha$ such that $F(y) \subset V$ $\forall y \in U$.F is uαc if it is so at each $x \in X$.

Theorem 3.2. Let $F_i : X \to Y$, $i = 1, 2$ be a multifunction where Y is a normal space with the following properties: (a)F_i is punctually closed, i = 1, 2,; (b)F_1 is uαc; (c)F_2 is uspc.

Then the set $A = \{x : F_1(x) \cap F_2(x) \neq \phi\} \in SPF(X)$.

Proof. Let $x \notin A$. Then $F_1(x) \cap F_2(x) = \phi$….(1) .Since F's $(i = 1, 2)$ are punctually closed $F_1(x)$, $F_2(x)$ are each closed. By (1), these closed sets are disjoint. Hence by the normality of Y there exists $V_i \in \Sigma(F_i(x))$ $i = 1, 2$ with $V_1 \cap V_2 = \phi$ …(2).Again the uαc of F_1 at x gives a $U_1 \in \tau^\alpha$ containing x such that $F_1(z) \subset V_1 \forall z \in U_1 \Rightarrow F_1[U_1] \subset V_1$. Since F_2 is uspc at x and $F_2(x) \subset V_2$ there exists a $U_2 \in SPO(X, x)$ such that $F_2(U_2) \subset V_2$. Now let $U = U_1 \cap U_2$.Since every open set is an α-set it follows that $U \in SPO(X, x)$.Again if $y \in U$, then above $F_1(y) \subset V_1$ and $F_2(y) \subset V_2$ whence $F_1(y) \cap F_2(y) \subset V_1 \cap V_2 = \phi \Rightarrow F_1(y)$

$\cap\ F_2(y) = \phi \Rightarrow F_1[U] \cap F_2[U] = \phi \Rightarrow y \in X - A \Rightarrow U \subset X - A$. Thus $x \in U \subset X - A \Rightarrow X - A$ is sp-nbd of each of its points $\Rightarrow X - A \in SPO(X) \Rightarrow A \in SPF(X)$.

Theorem 3.3. If $F : X \to Y$ is a punctually compact uspc multifunction into a Hausdorff space Y, then $G_F \in SPF(X \times Y)$.

Proof. Let $(x, y) \in X \times Y - G_F$. Clearly $y \notin F(x)$.The Hausdorffness of Y ensures the existence of $G_Z \in \Sigma(z)$ and $H_Z \in \Sigma(y)$ in Y such that $G_Z \cap H_Z = \phi$. …(1)

The family $C = \{G_Z : Z \in F(x)\}$ is an open cover of F (x). Since F is punctually compact, F(x) is compact. Therefore there exists a finite subfamily of C such that

$\{G_z, G_z, \ldots, G_z\}$. $F(x) \subset \cup \{G_z : k = 1, 2, \ldots, n\}$.Let $U = \cup \{G : k = 1, 2, \ldots, n\}$.

and $V = \cap \{G : k = 1, 2, \ldots, n\}$.where H (k = 1, 2, …, n) corresponds to G

satisfying (1). Then $V \in \Sigma(y)$ in Y and $U \cap V = \phi$ …(2).Obviously, $U \in \Sigma(F(x))$, $V \in \Sigma(y)$ in Y. Now the uspc-ness of F gives a $W \in SPO(X, x)$ such that $F[W] \subset U$, which, in its turn, implies by (2) that $F[W] \cap V = \phi$.Again $(x, y) \in W \times V \in SPO(X \times Y)$ whence one observes that $(x, y) \in W \times V \subset X \times Y - G_F \Rightarrow X \times Y - G_F$ contains a sp-nbd of each of its points $\Rightarrow X \times Y - G_F \in SPO(X \times Y) \Rightarrow G_F \in SPF(X \times Y)$.

Lemma 3.2. If (X, τ) is a normal space and $F \cap A = \phi$ where F is closed and A is g-closed then there exist disjoint open sets O_1 and O_2 such that $F \subset O_1$ and $F \subset O_2$.

It is known that a multifunction $F : X \to Y$ is said to be punctually compact (resp.closed) iff for each $x \in X$, F (x) is compact (resp.closed).

Definition 3.2. A multifunction $F : X \to Y$ is said to be punctually g-closed if F (x) is g-closed for every $x \in X$.

Theorem 3.4. Let $F_i : X \to Y$, i = 1, 2 be a multifunction into the normal space Y with the following properties;

(a) F_1 is punctually closed,

(b) F_2 is punctually g-closed,

(c) F_i is uspc (i = 1, 2).

Then the set $A = \{(x_1, x_2) : F_1(x_1) \cap F_2(x_2) \neq \phi\} \in SPF(X \times X)$.

Proof. Let $(x_1, x_2) \notin A$. Then$F_1(x_1) \cap F_2(x_2) = \phi$ …(1)Since F_1 is punctually closed, $F_1(x_1)$ is closed in Y. Also punctual g-closedness of F_2 indicates that $F_2(x_2)$ is g-closed in Y. By (1), the sets $F_1(x_1)$ and $F_2(x_2)$

are disjoint.Hence there exist open sets $V_1 \in \Sigma (F_1 (x_1))$, $V_2 \in \Sigma (F_2 (x_2))$ such that $V_1 \cap V_2 = \phi$.Now uspc-ness of F_i (i = 1, 2) ensures the existence of $U_i \in SPO (X, x_i)$ (i = 1, 2) such that $F_1 (x) \subset V_1 \;\forall x \in U_1$ and $F_2 (x) \subset V_2 \;\forall x \in U_2$.Set $U = U_1 \times U_2$.Then $U \in SPO (X \times X)$ and $(x_1, x_2) \in U$. Let $(y_1, y_2) \in U$ be any arbitrary point. Then one observes that $F_1 (y_1) \subset V_1$ and $F_2 (y_2) \subset V_2 \Rightarrow F_1 (y_1) \cap F_2 (y_2) \subset V_1 \cap V_2 = \phi \Rightarrow F_1 (y_1) \cap F_2 (y_2) = \phi \Rightarrow (y_1, y_2) \in X \times X - A \Rightarrow U \subset X \times X - A$ whence $(x_1, x_2) \in U \subset X \times X - A$ i.e. $X \times X - A$ contains a sp-nbd of each of its points. Hence $X \times X - A \in SPO (X \times X)$. In otherwords $A \in SPF (X \times X)$.

Theorem 3.5. If $F_i : (X, \tau) \to (Y, \sigma)$ i = 1, 2 are punctually compact uspc multifunctions from a space X with the property P to the Hausdorff space Y such that$F_1 (x) \cap F_2 (x) \neq \phi$ for all $x \in X$ then the multifunction $F : (X, \tau) \to (Y, \sigma)$ defined by $F (x) = F_1 (x) \cap F_2 (x)$ is uspc.

Proof. Let $x_0 \in X$ and suppose $V \in \Sigma (F (x_0))$ in Y. Now $F (x_0) \subset V \Rightarrow F_1 (x_0) \cap F_2 (x_0) \subset V$. Set $A = F_1 (x_0) - V$ and $B = F_2 (x_0) - V$. Then $A = F_1 (x_0) \cap V$.Since F_1 is punctually compact $F_1 (x_0)$ is compact and $V \in F(\sigma)$. Therefore, A is compact.Pursuing the same reasoning we can show that B is compact. Moreover by construction $A \cap B = \phi$.Hence there exist$V_1 \in \Sigma (A)$ and $V_2 \in \Sigma (B)$ such that $V_1 \cap V_2 = \phi$. Now $V_1 \cup V \in \Sigma (F_1 (x_0))$.The uspc of F_1 at x_0 ensures the existence of $U_1 \in SPO (X, x_0)$ such that $F_1 [U_1] \subset V_1 \cup V \Rightarrow F_1 (x_0) \subset V_1 \cup V$.Pursuing the same argument we obtain$F_2 (x_0) \subset F_2 [U_2] \subset V_2 \cup V$where$U_2 \in SPO (X. x_0)$.Therefore $F_1 (x_0) \cap F_2 (x_0) \subset (V_1 \cup V) \cap (V_2 \cup V) \Rightarrow F (x_0) \subset (V_1 \cap V_2) \cup (V \cap V_2) \cup (V_1 \cap V) \cup (V \cap V)$.Let $U = U_1 \cap U_2$.Since X enjoys the property P, $U \in SPO (X)$. Now each $Z \in U$ is a member of both U_1 and U_2 for which $F (z) \subset (V_1 \cap V_2) \cup (V \cap V_2) \cup (V_1 \cap V) \cup (V \cap V)$. Suppose $y \in F (z)$. Hence the foregoing induces $y \in (V \cap V_2) \cup (V_1 \cap V) \cup V$.This in any case $y \in V$ and for any $y \in F (z)$, $y \in V$. Hence $F (z) \subset V \;\; \forall z \in U \Rightarrow F$ is uspc.

Theorem 3.6. Let $F : (X, \tau) \to (Y, \sigma)$ be a punctually closed uspc multifunction into a normal space Y such that $F (x_1) \cap F (x_2) = \phi$ where $x_1 \neq x_2$. Then X is a sp-T_2 space.

Proof. Let $x_1 \neq x_2$.Then $F(x_1) \cap F(x_2) = \phi$.Since F is punctually closed $F(x_1), F(x_2) \in F (\sigma)$.The normality of Y guarantees the existence of sets $V_i \in \Sigma (F (x_i))$ in Y, i = 1, 2 with $V_1 \cap V_2 = \phi$. Since F is uspc for $x_1, x_2 \in X$ satisfying the relations $F (x_1) \subset V_1$, $F (x_2) \subset V_2$ there exist $U_i \in SPO (X, x_i)$, i = 1, 2 such that $F[U_1] \subset V_1$, $F[U_2] \subset V_2$whence $F[U_1] \cap F[U_2] \subset V_1 \cap V_2 = \phi \Rightarrow F[U_1 \cap U_2] = \phi \Rightarrow U_1 \cap U_2 = \phi$.Thus $x_1 \neq x_2$ implies there exist $U_1 \in$ SPO (X, x_1), $U_2 \in$ SPO (X, x_2) such that $U_1 \cap U_2 = \phi$.So, X is sp-T_2.

References

1. D. Andrijević, Semi-preopen Sets, Mat. Vensik 38 (1986), 24 – 32.
2. N.Bandyopadhyay, Semi-precontinuous multifunction, Research review international journal of multidisciplinary, (2019), 157-159.
3. Bouligand , Ens. Math. (1932), p-14.
4. P. K.Ghosh, Ph.D. Thesis, University of Kalyani, Kalyani, West Bengal, 2005.
5. Kuratowski , Fund. Math. 18 (1932), p-148. (1983), 425 – 432.
6. N. Levine, Generalised closed sets in topology, rend,Circ,Mat,Palermo(2),19(1970),89-96.
7. O.Njåstad ,On some classes of nearly open sets, Pacific. J Math. 15 (3) (1965), 961 –970.
8. Prezemeski M., On some forms of cliquishness on topological spaces, Serdica Bulgarieae mathematicae publications 18 (1992), 99 – 117.
9. R. Paul and P.Bhattacharyya, On pre-Urysohn spaces, Bull Malaysian Math. Soc. Second Series 22 (1999), 23 – 34.