SOLVING INTERVAL ASSIGNMENT PROBLEM USING DYNAMIC PROGRAMMING METHOD

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Abstract:
In this paper, an idea of dynamic programming method is used for solving the interval assignment problem. First, the concept of separation method is proposed without using the midpoint of the intervals. Further, it is progressed to find the optimal solution of upper bound and lower bound by the help of dynamic programming approach. Finally, the optimal solution of interval assignment problem is determined.

Keywords: Separation Method, Upper Bound, Lower Bound, Dynamic Programming Approach.

1. INTRODUCTION

Dynamic programming is a useful mathematical technique for making a sequence of interrelated decisions. It provides a systematic procedure for determining the optimal combination of decisions. Dynamic Programming is the only technique used to solve the problems in area such as inventory, chemical engineering design, and control theory. It has the advantage of effectively decomposing highly complex problems with a large number of variables into a series of sub problems which are solved recursively. Unlike linear programming, for a dynamic programming problem, there is no standard form that allows only one algorithm to be used in solving problems. The method of dynamic programming was first developed in [2], through the works of Richard Bellman. In Dynamic Programming Problem, there does not exist any standard mathematical formulation and particular equations must be developed to fit for each individual situation [Goel & Mittal [4]]. In other words, a Dynamic programming is a decision making problem in n-variable, the problem being subdivided into n- sub problems where each sub problem is a decision making problem in one variable only [Swaroup [10]]. Powell [6], developed a stochastic model of the dynamic vehicle allocation problem. Crainic.T, et al., [3] proposed a dynamic stochastic models for the allocation of empty containers. Assignment problems deal with the allocation of items to locations, one to each, in such a way that some optimum return is obtained. In this paper, we use the dynamic programming method for finding an optimal solution to the interval assignment problem. Sarangam Majumdar [9] solved interval linear assignment problems using a new method named interval Hungarian method. Ramesh and Ganesan [7] have proposed a new computational technique to solve assignment problems with generalized interval Hungarian method. Ramesh Kumar et al., [8] have introduced a matrix ones interval linear assignment method and compared with the existing method. Amutha et al. [1] has introduced a method of solved extension of the interval in assignment problem. A new method for finding an optimal solution of fully interval integer transportation problems have proposed by Pandian et al., [5].

The Paper is organized as follows; The Section 2 discussed about the interval assignment problem and mathematical formulation of interval assignment problem followed by the definition with the characteristics of dynamic programming problem related to it. The recursive relationship of dynamic programming method to solve the assignment problem is given in Section 3. Also, A concept of Separation method and computational procedure is developed for solving the interval assignment problem. Further, A numerical example is given to show the optimality of interval assignment problem using dynamic programming approach. Finally, the conclusion of the paper is given in Section 4.

2. INTERVAL ASSIGNMENT PROBLEM

Assume there are n works to be completed and n persons are offered for doing the works and also, each person can do each work at a time, yet with a variable grade of efficiency. Let $\underline{C}_{ij}$ be the interval cost if the i^{th} person is allocated the j^{th} work , the problem is to discover a minimum interval cost through optimal interval assignment. The Mathematical problem with interval cost cab be represented as follows:

Table 1: Interval assignment with interval cost

Resource	Activity A_1	A_2	………	A_n	Available
R_1	$\underline{C}_{11}$	$\underline{C}_{12}$	………	$\underline{C}_{1n}$	1
R_2	$\underline{C}_{21}$	$\underline{C}_{22}$	………	$\underline{C}_{2n}$	1
⋮	⋮	⋮	………	⋮	⋮
⋮	⋮	⋮	………	⋮	⋮
R_n	$\underline{C}_{n1}$	$\underline{C}_{n2}$	………	$\underline{C}_{nn}$	1
Required	1	1	………	1	

2.1 MATHEMATICAL FORMULATION OF AN INTERVAL ASSIGNMENT PROBLEM

The interval assignment problem represented in the table 1 can be written as:

$$(P) \text{ Minimize } \underline{Z} = \sum_{i=1}^{m}\sum_{j=1}^{n} \underline{c}_{ij}\, x_{ij}, i = 1,2,\dots,n$$

subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = 1, \text{and } \sum_{i=1}^{m} x_{ij} = 1\,, x_{ij} = 0 \text{ or } 1$$

Let x_{ij} denote the assignment of i^{th} person to j^{th} work, such that

$$x_{ij} = \begin{cases} 1, \text{ if resource i is assigned to activity j} \\ 0, \text{ otherwise} \end{cases}$$

and $\underline{c}_{ij}$ is an interval cost to i^{th} person for doing j^{th} work, $\sum_{i=1}^{m}\sum_{j=1}^{n} \underline{c}_{ij}$ is the total interval cost for carrying out all the works.

2.2 DEFINITIONS

2.2.1 Arithmetic Operations in interval:

Let D= {[a, b], a ≤ b and a and b are in R}, denote the closed set of bounded intervals on the real line R.

Let A= [a, b] and B= [c, d] be in D. Then,

(i) $A \oplus B=[a + c, b + d]$
(ii) $A - B = [a - d, c - b]$ provide if $B \neq [0, 0]$,
(iii) $A \otimes B= [p, q]$ where p= min{ac, ad, bc, bd} and q= max{ac, ad, bc, bd}

(iv) $A \div B$ =[min{$a \div d$, $c \div b$, $a \div b$, $c \div d$}, max{$a \div d$, $c \div b$, $a \div b$, $c \div d$}] Provided $B \neq [0, 0]$,

(v) $A \leq B$ if $a \leq c$ and $b \leq d$

(vi) $A \geq B$ if $a \geq c$ and $b \geq d$ and

(vii) $A = B$ if $a = c$ and $b = d$

2.3 CHARACTERISTICS OF DYNAMIC PROGRAMMING METHOD

The basic features which characterize the dynamic programming problem are as follows:

(i) The problem can be subdivided into stages with a policy decision required at each stage. An stage is a device to sequence the decisions.

(ii) Every stage consists of a number of states associated with it. The states are the different possible conditions in which the system may find itself at that stage of the problem.

(iii) Decision at each stage converts the current stage into state associated with the next stage. The state of the system at a stage is described by a set of variables, called state variables.

(iv) To identify the optimum policy for each state of the system, a recursive equation is formulated with n stage remaining, given the optimal policy for each state with (n-1) stages left.

3. RECURSIVE RELATIONSHIP OF DYNAMIC PROGRAMMING METHOD TO SOLVE ASSIGNMENT PROBLEM:

Recursive equations are used to structure a multistage decision problem as a sequential process. A multistage problem is solved by breaking into a number of single stage problems through recursion. Each recursive equation represents a stage at which a decision is required and the series of equations are successively solved each equation depending on the output values of the previous equations. This recursive approach can be done in a backward manner or in a forward manner. The recursive dynamic programming equation is also called the functional equation or optimization equation.

Suppose that the desired objective is to minimize the n-stage objective function 'f', and then n Machines can be taken as n stages. The state at n stage will be taken as$(a_1, \dots, a_n)$, when $a_1 = A_1$ or 0, $a_2 = A_2$ or 0, $a_3 = A_3$ or 0 etc. Hence $A_1, \dots, A_n$ indicate that the jobs are available for assignment to the machine while $a_n = 0$ indicates that the n jobs are not available for the assignment.

Let $f_n(a_1, \dots, a_n)$ be the minimum value of the total cost for the n stage problem with the state$(a_1, \dots, a_n)$. Also, let $c_n(x_n)$ be the cost at the stage nth stage.

$$f_1(a_1, \dots, a_n) = Min\, c_1(x_1)$$

Thus the recurrence relation is

$$f_n(a_1, \dots, a_n) = Min\, [\, c_n(x_n) + f_{n-1}(a_1', \dots a_n')]$$

In an Assignment problem one job is to be assigned to one machine only, so that any stage n, x_n will have only one of the values A_1 or A_2 or …or A_n definitely. In this Paper, the recurrence relations are formulated using the backward approach so, the relations are solved forwardly.(i.e) beginning the problem with the first decision.

3.1 SEPARATION METHOD FOR SOLVING INTERVAL ASSIGNMENT PROBLEM

Consider the interval Assignment problem (P) as

$$\text{Minimize } \underline{Z} = \sum_{i=1}^{m} \sum_{j=1}^{n} \underline{c}_{ij}\, x_{ij}, i = 1,2, \dots, n$$

subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = 1, \text{and } \sum_{i=1}^{m} x_{ij} = 1\, , x_{ij} = 0 \text{ or } 1$$

where $\underline{c}_{ij} = [c_{ij}^1, c_{ij}^2]$ and if $[(x_1^*, \dots, x_n^*), (y_1^*, \dots, y_n^*)]$ are the optimal solution of the lower bound(LB) and the upper bound (UB) respectively then, the intervals $[(x_1^*, \dots, x_n^*), (y_1^*, \dots, y_n^*)]$ will be an optimal solution of the problem (P), provided $(x_1^*, \dots, x_n^*) < (y_1^*, \dots, y_n^*)$.

3.2 COMPUTATIONAL PROCEDURE

The procedure for implementing the dynamic programming approach to solve an assignment problem is as follows:
Step 1: Read the coefficients of the variables, lower and upper bounds of the interval assignment problem and separate it into two sub problems.
Step 2: Treat the sub problems obtained in step 1 as a multistage decision problem (Dynamic Programming Approach). Each sub problem is solved by breaking into a number of single stage problems. The number of single stage problem is equal to the number of variables.
Step 3: Set $(x_{11}, x_{21}, \dots, x_{n1}), (x_{12}, x_{22}, \dots, x_{n2}), \dots, (x_{1n}, x_{2n}, \dots, x_{nn})$ be the lower bound of the decision variables and determine the optimal value of variable $x_1^*, \dots, x_n^*$ for first, second, and n stages respectively, using recursive equation which is detailed in section 3. Proceed to step 4.
Step 4: Set $(y_{11}, y_{21}, \dots, y_{n1}), (y_{12}, y_{22}, \dots, y_{n2}), \dots, (y_{1n}, y_{2n}, \dots, y_{nn})$ be the upper bound of the decision variables, determine the optimal value of variable $y_1^*, \dots, y_n^*$ for first, second, and n stages respectively, and using recursive equation which is detailed in section 3.
Step 5: The optimal allocation schedule will be $[(x_1^*, \dots, x_n^*), (y_1^*, \dots, y_n^*)]$, provided by the condition $(x_1^*, \dots, x_n^*) < (y_1^*, \dots, y_n^*)$.Thus the optimal solution for the original interval assignment problem is determined by calculating the sum of those upper bound optimal decision values and lower bound decision values.

3.3 NUMERICAL EXAMPLE :

1. A Company wishes to assign 3 jobs to 3 machines in such a way that each job is assigned to some machine and no machine works on more than one job. The cost of assigning job i to machine j is given by the matrix below in interval data:

Table 2: Cost Matrix

	X	Y	Z
A	[7,9]	[6,8]	[5,7]
B	[4,6]	[6,8]	[7,9]
C	[5,7]	[7,9]	[6,8]

Find the minimum cost of making assignment.

The three machines can be taken as 3 stages. The state at any stage will be taken as a_1, a_2, a_3 when a_1 = A or = 0, a_2 = B or 0, a_3 = C or 0.

Hence A, B, C indicate that the jobs are available for assignment to the machines. While $a_3 = 0$ indicates that the third jobs are not available for the assignment. Let $(x_{11}, x_{21}, x_{n1}), (x_{12}, x_{22}, x_{n2}), (x_{13}, x_{23}, x_{33})$, be the lower bound of the decision variables at first, second, third stages respectively. At first, the lower bound of the given interval assignment problem is considered below as

Table 3: Cost Matrix of the lower bound

	X	Y	Z
A	7	6	5
B	4	6	7
C	5	7	6

Let $f_n(a_1, \dots, a_n)$, $n = 1,2,3$,be the minimum value of total cost for the problem with the state (a_1,a_2,a_3). Also, Let $c_n(x_n), n = 1,2,3$, be the cost calculated in the succeeding stages.

First stage:

By using lower bound of the interval cost ,we find the following relations:-

$$f_1(a_1, a_2, a_3) = Min\ c_1(x_1)$$
(i.e) x_1^* = A or B or C

Second stage:

$$f_2(a_1, a_2, a_3) = Min\ [\ c_2(x_2) + \ f_1(a_1', a_2', a_3')]$$
(i.e) x_2^* = A or B or C

Where a_1', a_2', a_3' is the resulting state due to decision variable, x_1^*

Third stage:

Thus the recurrence relation is

$$f_3(a_1, a_2, a_3) = Min\ [\, c_3(x_3) + \ f_2(a_1', a_2', a_3')]$$
$$\text{(i.e) } x_3^* = \text{A or B or C}$$

Where a_1', a_2', a_3' is the resulting state due to decision x_2^*

FIRST STAGE PROBLEM:

Table 4: Computations for first stage problem

State	$c_1(x_1)$			$f_1(a_1, a_2, a_3)$	x_1^*
	$x_{11} = A$	$x_{21} = B$	$x_{31} = C$		
AOO	7	-	-	A	7
OBO	-	4	-	B	4
OOC	-	-	5	C	5
ABO	7	4	-	B	4
AOC	7	-	5	C	5
OBC	-	4	5	B	4
ABC	7	4	5	B	4

SECOND STAGE PROBLEM:

Table 5: Computations for second stage problem

State	$c_2(x_2) + \ f_1(a_1', a_2', a_3')$			$f_2(a_1, a_2, a_3)$	x_2^*
	$x_{12} = A$	$x_{22} = B$	$x_{32} = C$		
ABO	6+7=13	6+7=13	-	A , B	13
AOC	6+4=10	-	7+7=14	A	10
OBC	-	6+4=10	7+4=11	B	10
ABC	6+5=11	6+5=11	7+5=12	A , B	11

THIRD STAGE PROBLEM:

Table 6 : Computations for third stage problem

State	$c_3(x_3) + \ f_2(a_1', a_2', a_3')$			$f_3(a_1, a_2, a_3)$	x_3^*
	$x_{13} = A$	$x_{23} = B$	$x_{33} = C$		
ABC	5+13=18	7+10=17	6+10=16	C	16

Let $(y_{11}, y_{21}, y_{31}), (y_{12}, y_{22}, y_{32}), (y_{13}, y_{23}, y_{33})$, be the upper bound of the decision variables at first, second, third stages respectively.

Next , the upper bound of the given interval assignment problem is considered below as

Table 7: Cost Matrix of the upper bound

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} X & Y & Z \end{array} \\ \begin{bmatrix} 9 & 8 & 7 \\ 6 & 8 & 9 \\ 7 & 9 & 8 \end{bmatrix} \end{array}$$

Let $f_n(a_1, \ldots, a_n)\, , n = 1,2,3$,be the minimum value of total cost for the problem with the state (a_1,a_2,a_3). Also, Let $c_n(y_n), n = 1,2,3$, be the cost calculated in the succeeding stages.

First stage:
By using upper bound of the interval cost, the following relations are calculated:-

$$f_1(a_1, a_2, a_3) = Min\ c_1(y_1)$$

$$\text{(i.e) } y_1^* = \text{A or B or C}$$

Second stage:

$$f_2(a_1, a_2, a_3) = Min\ [\, c_2(y_2) + \ f_1(a_1', a_2', a_3')]$$

$$\text{(i.e) } y_2^* = \text{A or B or C}$$

Where a_1', a_2', a_3' is the resulting state due to decision y_1^*

Third stage:
Thus the recurrence relation is

$$f_3(a_1, a_2, a_3) = Min\,[\,c_3(y_3) + f_2(a_1', a_2', a_3')]$$

$$\text{(i.e) } y_3^* = \text{A or B or C}$$

Where a_1', a_2', a_3' is the resulting state due to decision y_2^*

FIRST STAGE PROBLEM:

Table 8: Computations for first stage problem

State	$c_1(y_1)$			$f_1(a_1, a_2, a_3)$	y_1^*
	$y_{11} = A$	$y_{21} = B$	$y_{31} = C$		
AOO	9	-	-	A	9
OBO	-	6	-	B	6
OOC	-	-	7	C	7
ABO	9	6	-	B	6
AOC	9	-	7	C	7
OBC	-	6	7	B	6
ABC	9	6	7	B	6

SECOND STAGE PROBLEM:

Table 9: Computations for second stage problem

State	$c_2(y_2) + f_1(a_1', a_2', a_3')$			$f_2(a_1, a_2, a_3)$	y_2^*
	$y_{12} = A$	$y_{22} = B$	$y_{32} = C$		
ABO	8+9=17	8+9=17	-	A,B	17
AOC	8+6=14	-	9+9=18	A	14
OBC	-	8+6=14	9+6=15	B	14
ABC	8+7=15	8+7=15	9+7=16	A,B	15

THIRD STAGE PROBLEM:

Table 10: Computations for third stage problem

State	$c_3(y_3) + f_2(a_1', a_2', a_3')$			$f_3(a_1, a_2, a_3)$	y_3^*
	$y_{13} = A$	$y_{23} = B$	$y_{33} = C$		
ABC	7+17=24	9+14=23	8+14=22	C	22

Thus, the optimum value of the Interval assignment problem is [19, 22] and the optimal schedule is [(A,Y),(B,X),(C,Z)].

4. CONCLUSION

In this paper, dynamic programming approach is elucidated to solve interval assignment problem with help of the separation method. Dynamic programming works on the principle of optimality, if a stage has reached an optimal solution, and then its variables at preceding stages must also be optimized. This method is used to save time, since each sub-problem is solved only once. Using Dynamic programming one can store preceding values to avoid multiple calculations.

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