

GENERALIZED HYERS - ULAM STABILITY OF AC FUNCTIONAL EQUATION IN FUZZY NORMED SPACE

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ABSTRACT. In this paper, the author has investigate the generalized Hyers-Ulam stability of additive cubic functional equation of the form

$$f(2x + x) + f(2x - x) = 4f(x + x) - 4f(x - x) - 6f(x)$$

in fuzzy normed spaces using Hyers method.

1.

INTRODUCTION AND PRELIMINARIES

The stability problem of functional equations originated from a question of S.M. Ulam [40] concerning the stability of group homomorphisms. D.H. Hyers [15] gave a first affirmative partial answer to the question of Ulam for Banach spaces.

Hyers' theorem was generalized by T. Aoki [2] for additive mappings and by Th.M. Rassias [35] for linear mappings by considering an unbounded Cauchy difference. The paper of Th.M. Rassias [35] has provided a lot of influence in the development of what we call generalized Hyers-Ulam stability of functional equations. A generalization of the Th.M. Rassias theorem was obtained by P. Gavruta [14] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias approach. In 1982, J.M. Rassias [32] followed the innovative approach of the Th.M. Rassias theorem [35] in which he replaced the factor $\|jxj\|^p + \|jxj\|^q$ by $\|jxj\|^p \|jxj\|^q$ for $p, q \in \mathbb{R}$ with $p + q = 1$:

In 2008, a special case of Gavruta's theorem for the unbounded Cauchy difference was obtained by Ravi et al., [37] by considering the summation of both the sum and the product of two p norms in the spirit of Rassias approach. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning

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this problem (see [1, 12, 16, 20]).

A.K. Katsaras [22] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have de-fined fuzzy norms on a vector space from various points of view [13, 24, 41]. In particular, T. Bag and S.K. Samanta [8], following S.C. Cheng and J.N. Morde-son [10], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [23]. They established a decom-position theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [9].

We use the definition of fuzzy normed spaces given in [8] and [27, 28, 29, 30].

Definition 1.1. Let X be a real linear space. A function $N : X \times \mathbb{R}^+ \rightarrow [0, 1]$ (the so-called fuzzy subset) is said to be a fuzzy norm on X if for all $x, y \in X$ and all

$s, t \in \mathbb{R}^+$,

(F 1) $N(x; c) = 0$ for $c \leq 0$;

(F 2) $x = 0$ if and only if $N(x; c) = 1$ for all $c > 0$;

(F 4) $N(x + y; s + t) \geq \min\{N(x; s), N(y; t)\}$;

(F 3) $N(cx; t) = N(x; \frac{t}{|c|})$ if $c \neq 0$;
 $\min\{N(x; s), N(y; t)\}$

(F 5) $N(x; \cdot)$ is a non-decreasing function on \mathbb{R}^+ and $\lim_{t \rightarrow \infty} N(x; t) = 1$;

(F 6) for $x \neq 0$; $N(x; \cdot)$ is (upper semi) continuous on \mathbb{R}^+ .

The pair $(X; N)$ is called a fuzzy normed linear space. One may regard $N(x; t)$ as the truth-value of the statement the norm of x is less than or equal to the real number t .

Example 1.2. Let $(X; \|\cdot\|)$ be a normed linear space. Then

$$N(x; t) = \begin{cases} \frac{t}{t + \|x\|} & ; t > 0; \\ 0 & ; x \neq 0; \\ 1 & ; x = 0 \end{cases}$$

is a fuzzy norm on X .

Definition 1.3. Let $(X; N)$ be a fuzzy normed linear space. Let x_n be a sequence in X . Then x_n is said to be convergent if there exists $x \in X$ such that $\lim_{n \rightarrow \infty} N(x_n$

$x; t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence x_n and we denote it by $N \lim_{n \rightarrow \infty} x_n = x$.

Definition 1.4. A sequence x_n in X is called Cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n; t) > 1 - \epsilon$.

Definition 1.5. Every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.

The stability of various functional equations in fuzzy normed spaces was investigated in [3, 4, 6, 17, 26, 27, 28, 29, 30, 38].

In this paper, the author has investigate the generalized Hyers-Ulam stability of additive cubic functional equation of the form

$$f(2x + x) - f(2x - x) = 4f(x + x) - 4f(x - x) + 6f(x) \tag{1.1}$$

in fuzzy normed spaces using Hyers method.

2. FUZZY STABILITY RESULTS: DIRECT METHOD

Throughout this section, assume that X ; $(Z; N^0)$ and $(Y; N^0)$ are linear space, fuzzy normed space and fuzzy Banach space, respectively. Now use the following notation for a given mapping $f : X \rightarrow Y$

$$Df(x; x) = f(2x + x) - f(2x - x) - 4f(x + x) + 4f(x - x) + 6f(x)$$

for all $x; y \in X$:

Now, we investigate the generalized Ulam-Hyers stability of AQCQ functional equation (1.1).

Theorem 2.1. Let $d \in [0, 1]$; $1/g$ be fixed and let $\phi : X^2 \rightarrow Z$ be a mapping such that for

$$N^0(\phi(2x; 2x); r) \geq N^0(d\phi(x; x); r) \tag{2.1}$$

for all $x \in X$ and all $r > 0$, $d > 0$, and

$$\lim_{k \rightarrow \infty} N^0(\phi(2^k x; 2^k y); r) = 1; \quad 2^k r = 1 \tag{2.2}$$

for all $x; y \in X$ and all $r > 0$. Suppose that a function $f : X \rightarrow Y$ satisfies the inequality

$$N(Df(x; y); r) \geq N^0(\phi(x; x); r) \tag{2.3}$$

for all $r > 0$ and all $x; y \in X$. Then the limit

$$A(x) = \lim_{k \rightarrow \infty} \frac{f(2^k x)}{2^k} \tag{2.4}$$

exists for all $x \in X$ and the mapping $A : X \rightarrow Y$ is a unique additive mapping such that

$$N(f(2x) - 8f(x) - A(x); r) \geq \min\{N^0(\phi(x; x); \frac{(2-d)r}{8}); N^0(\phi(x; 2x); \frac{(2-d)r}{4})\} \tag{2.5}$$

for all $x \in X$ and all $r > 0$.

Proof. First assume $\alpha = 1$. Replacing $(x; y)$ by $(x; x)$ in (2.3), we get

$$N(f(3x) - 4f(2x) + 5f(x); r) \geq N^0((x; x); r) \tag{2.6}$$

for all $x \in X$ and all $r > 0$. Replacing $(x; y)$ by $(x; 2x)$ in (2.3), we obtain

$$N(f(4x) - 4f(3x) + 6f(2x) - 4f(x); r) \geq N^0((x; 2x); r) \tag{2.7}$$

for all $x \in X$ and all $r > 0$. Now, from (2.6) and (2.7), we have

$$N\left(\frac{f(4x) - 10f(2x) + 16f(x); r}{4(f(3x) - 4f(2x) + 5f(x))}; r\right); N\left(\frac{f(4x) - 4f(3x) + 6f(2x) - 4f(x); r}{4f(x)}; r\right) \geq N^0((x; x); 8); N^0((x; 2x); 2) \tag{2.8}$$

for all $x \in X$ and all $r > 0$. Let $a : X \rightarrow Y$ be a mapping defined by $a(x) = \frac{f(2x) - 8f(x)}{r}$. Then we conclude that

$$N(a(2x) - 2a(x); r) \geq \min\left\{N^0((x; x); 8); N^0((x; 2x); 2)\right\} \tag{2.9}$$

for all $x \in X$ and all $r > 0$. Thus, we have

$$N\left(\frac{a(2x) - 2a(x)}{2}; \frac{r}{2}\right) \geq \min\left\{N^0((x; x); 8); N^0((x; 2x); 4)\right\} \tag{2.10}$$

for all $x \in X$ and all $r > 0$. Replace x by $2^k y$ in (2.10), we get

$$N\left(\frac{a(2^{k+1}x) - f(2^k x)}{2}; \frac{r}{2}\right) \geq \min\left\{N^0((2^k y; 2^k y); 8); N^0((2^k y; 2 \cdot 2^k y); 4)\right\} \tag{2.11}$$

for all $x \in X$ and all $r > 0$. Using (2.1), (F 3) in (2.11), we arrive

$$N\left(\frac{a(2^{k+1}x) - a(2^k x)}{2}; \frac{r}{2}\right) \geq N^0((2^k y; 2^k y); 8); N^0((2^k y; 2 \cdot 2^k y); 4) \tag{2.12}$$

for all $x \in X$ and all $r > 0$. Replacing r by $d^k r$ in (2.12), we get

$$N\left(\frac{a(2^{k+1}x) - a(2^k x)}{d^k}; \frac{r}{d^k}\right) \geq N^0((2^k y; 2^k y); 8); N^0((2^k y; 2 \cdot 2^k y); 4) \tag{2.13}$$

for all $x \in X$ and all $r > 0$. It is easy to see that

$$\frac{a(2^k x)}{d^k} \leq \frac{a(2^{i+1} x)}{d^{i+1}} \leq \frac{a(2^i x)}{d^i} \tag{2.14}$$

for all $x \in X$. From equations (2.13) and (2.14), we have

$$N_{\frac{a(2^k x) - a(x)}{2^{k-1} d r}} \geq \min_{i=0}^{k-1} N_{\frac{a(2^{i+1} x) - a(2^i x)}{2^{i-1} d r}} \quad (2.15)$$

for all $x \in X$ and all $r > 0$. Replacing x by $2^m x$ in (2.15) and using (2.1), (F 3), we obtain

$$N_{\frac{a(2^{k+m} x) - a(2^m x)}{2^{k-1} d r}} \geq \min_{i=0}^{m+k-1} N_{\frac{a(2^{i+1} x) - a(2^i x)}{2^{i-1} d r}} \quad (2.16)$$

for all $x \in X$ and all $r > 0$ and all $m; k \geq 0$. Replacing r by $d^m r$ in (2.16), we get

$$N_{\frac{a(2^{k+m} x) - a(2^m x)}{2^{k-1} d^{m+1} r}} \geq \min_{i=0}^{m+k-1} N_{\frac{a(2^{i+1} x) - a(2^i x)}{2^{i-1} d^m r}} \quad (2.17)$$

for all $x \in X$ and all $r > 0$ and all $m; k \geq 0$. Using (F 3) in (2.17), we obtain

$$N_{\frac{a(2^{k+m} x) - a(2^m x)}{2^{k-1} d^{m+1} r}} \geq \min_{i=0}^{m+k-1} N_{\frac{a(2^{i+1} x) - a(2^i x)}{2^{i-1} d^m r}} \quad (2.18)$$

for all $x \in X$ and all $r > 0$ and all $m; k \geq 0$. Since $0 < d < 2$ and $\lim_{i \rightarrow \infty} \frac{d^i}{2^i} = 0$,

the Cauchy criterion for convergence and (F 5) implies that $f_{\frac{a(2^k x)}{2^k} g}$ is a Cauchy sequence in $(Y; N)$. Since $(Y; N)$ is a fuzzy Banach space, this sequence converges to some point g . So one can define the mapping $A: X \rightarrow Y$ by $Ax = YAx$.

$$N_{\frac{a(2^k x)}{2^k} - A(x)} \geq \min_{i=0}^{k-1} N_{\frac{a(2^{i+1} x) - a(2^i x)}{2^{i-1} d r}} \quad (2.19)$$

for all $x \in X$ and all $r > 0$. Letting $k \rightarrow \infty$ in (2.19) and using (F 6), we arrive

$$N_{A(x) - A(x); r} \geq \min_{i=0}^{\infty} N_{\frac{(2-d)r}{2^i}}; N_{(x; 2x); \frac{(2-d)r}{2^i}}$$

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for all $x, y \in X$ and all $r > 0$. To prove A satisfies the (1.1), replacing $(x; x)$ by $(2x; 2x)$ in (2.3), respectively, we obtain

$$N_{\min \{N(A(2x+y), A(2x-y), 4A(x+y) + 4A(x-y) + 6A(y)); r\}}(2x; 2x) \geq \frac{1}{1 + 2^k D_f(2^k x; 2^k x) + r N^0(2^k x; 2^k x) + 2^k r} \quad (2.20)$$

for all $r > 0$ and all $x, y \in X$. Now,

$$N_{\min \{N(A(2x+y), A(2x-y), 4A(x+y) + 4A(x-y) + 6A(y)); r\}}(x+y; x+y) \geq \frac{1}{1 + 2^k D_f(x+y; x+y) + r N^0(x+y; x+y) + 2^k r} + \frac{1}{1 + 2^k D_f(x-y; x-y) + r N^0(x-y; x-y) + 2^k r} + \frac{1}{1 + 2^k D_f(y; y) + r N^0(y; y) + 2^k r} + \frac{1}{1 + 2^k D_f(2x+y; 2x+y) + r N^0(2x+y; 2x+y) + 2^k r} + \frac{1}{1 + 2^k D_f(2x-y; 2x-y) + r N^0(2x-y; 2x-y) + 2^k r} + \frac{1}{1 + 2^k D_f(4x+4y; 4x+4y) + r N^0(4x+4y; 4x+4y) + 2^k r} + \frac{1}{1 + 2^k D_f(4x-4y; 4x-4y) + r N^0(4x-4y; 4x-4y) + 2^k r} + \frac{1}{1 + 2^k D_f(6y; 6y) + r N^0(6y; 6y) + 2^k r} \quad (2.21)$$

for all $x, y \in X$ and all $r > 0$. Using (2.20) and (F 5) in (2.21), we arrive

$$N_{\min \{N(A(2x+y), A(2x-y), 4A(x+y) + 4A(x-y) + 6A(y)); r\}}(x+y; x+y) \geq \frac{1}{1 + 2^k D_f(x+y; x+y) + r N^0(x+y; x+y) + 2^k r} + \frac{1}{1 + 2^k D_f(x-y; x-y) + r N^0(x-y; x-y) + 2^k r} + \frac{1}{1 + 2^k D_f(y; y) + r N^0(y; y) + 2^k r} + \frac{1}{1 + 2^k D_f(2x+y; 2x+y) + r N^0(2x+y; 2x+y) + 2^k r} + \frac{1}{1 + 2^k D_f(2x-y; 2x-y) + r N^0(2x-y; 2x-y) + 2^k r} + \frac{1}{1 + 2^k D_f(4x+4y; 4x+4y) + r N^0(4x+4y; 4x+4y) + 2^k r} + \frac{1}{1 + 2^k D_f(4x-4y; 4x-4y) + r N^0(4x-4y; 4x-4y) + 2^k r} + \frac{1}{1 + 2^k D_f(6y; 6y) + r N^0(6y; 6y) + 2^k r} \quad (2.22)$$

for all $x, y \in X$ and all $r > 0$. Letting $k = 1$ in (2.22) and using (2.2), we see that

$N_{\min \{N(A(2x+y), A(2x-y), 4A(x+y) + 4A(x-y) + 6A(y)); r\}}(x+y; x+y) = 1$ (2.23) for all $x, y \in X$ and all $r > 0$. Using (F 2) in the above inequality gives

$$A(2x+y) - A(2x-y) = 4A(x+y) - 4A(x-y) - 6A(y)$$

for all $x, y \in X$. Hence A satisfies the cubic functional equation (1.1). In order to prove A(x) is unique, let $A^0(x)$ be another additive functional equation satisfying (1.1) and (2.5). Hence,

$$N_{\min \{N(A(x), A^0(x)); r\}}(x; x) \geq \frac{1}{1 + 2^k D_f(x; x) + r N^0(x; x) + 2^k r} + \frac{1}{1 + 2^k D_f(2x; 2x) + r N^0(2x; 2x) + 2^k r} + \frac{1}{1 + 2^k D_f(4x; 4x) + r N^0(4x; 4x) + 2^k r} + \frac{1}{1 + 2^k D_f(8x; 8x) + r N^0(8x; 8x) + 2^k r} + \frac{1}{1 + 2^k D_f(16x; 16x) + r N^0(16x; 16x) + 2^k r} + \frac{1}{1 + 2^k D_f(32x; 32x) + r N^0(32x; 32x) + 2^k r} + \frac{1}{1 + 2^k D_f(64x; 64x) + r N^0(64x; 64x) + 2^k r} + \frac{1}{1 + 2^k D_f(128x; 128x) + r N^0(128x; 128x) + 2^k r} + \frac{1}{1 + 2^k D_f(256x; 256x) + r N^0(256x; 256x) + 2^k r} + \frac{1}{1 + 2^k D_f(512x; 512x) + r N^0(512x; 512x) + 2^k r} + \frac{1}{1 + 2^k D_f(1024x; 1024x) + r N^0(1024x; 1024x) + 2^k r} + \frac{1}{1 + 2^k D_f(2048x; 2048x) + r N^0(2048x; 2048x) + 2^k r} + \frac{1}{1 + 2^k D_f(4096x; 4096x) + r N^0(4096x; 4096x) + 2^k r} + \frac{1}{1 + 2^k D_f(8192x; 8192x) + r N^0(8192x; 8192x) + 2^k r} + \frac{1}{1 + 2^k D_f(16384x; 16384x) + r N^0(16384x; 16384x) + 2^k r} + \frac{1}{1 + 2^k D_f(32768x; 32768x) + r N^0(32768x; 32768x) + 2^k r} + \frac{1}{1 + 2^k D_f(65536x; 65536x) + r N^0(65536x; 65536x) + 2^k r} + \frac{1}{1 + 2^k D_f(131072x; 131072x) + r N^0(131072x; 131072x) + 2^k r} + \frac{1}{1 + 2^k D_f(262144x; 262144x) + r N^0(262144x; 262144x) + 2^k r} + \frac{1}{1 + 2^k D_f(524288x; 524288x) + r N^0(524288x; 524288x) + 2^k r} + \frac{1}{1 + 2^k D_f(1048576x; 1048576x) + r N^0(1048576x; 1048576x) + 2^k r} + \frac{1}{1 + 2^k D_f(2097152x; 2097152x) + r N^0(2097152x; 2097152x) + 2^k r} + \frac{1}{1 + 2^k D_f(4194304x; 4194304x) + r N^0(4194304x; 4194304x) + 2^k r} + \frac{1}{1 + 2^k D_f(8388608x; 8388608x) + r N^0(8388608x; 8388608x) + 2^k r} + \frac{1}{1 + 2^k D_f(16777216x; 16777216x) + r N^0(16777216x; 16777216x) + 2^k r} + \frac{1}{1 + 2^k D_f(33554432x; 33554432x) + r N^0(33554432x; 33554432x) + 2^k r} + \frac{1}{1 + 2^k D_f(67108864x; 67108864x) + r N^0(67108864x; 67108864x) + 2^k r} + \frac{1}{1 + 2^k D_f(134217728x; 134217728x) + r N^0(134217728x; 134217728x) + 2^k r} + \frac{1}{1 + 2^k D_f(268435456x; 268435456x) + r N^0(268435456x; 268435456x) + 2^k r} + \frac{1}{1 + 2^k D_f(536870912x; 536870912x) + r N^0(536870912x; 536870912x) + 2^k r} + \frac{1}{1 + 2^k D_f(1073741824x; 1073741824x) + r N^0(1073741824x; 1073741824x) + 2^k r} + \frac{1}{1 + 2^k D_f(2147483648x; 2147483648x) + r N^0(2147483648x; 2147483648x) + 2^k r} + \frac{1}{1 + 2^k D_f(4294967296x; 4294967296x) + r N^0(4294967296x; 4294967296x) + 2^k r} + \frac{1}{1 + 2^k D_f(8589934592x; 8589934592x) + r N^0(8589934592x; 8589934592x) + 2^k r} + \frac{1}{1 + 2^k D_f(17179869184x; 17179869184x) + r N^0(17179869184x; 17179869184x) + 2^k r} + \frac{1}{1 + 2^k D_f(34359738368x; 34359738368x) + r N^0(34359738368x; 34359738368x) + 2^k r} + \frac{1}{1 + 2^k D_f(68719476736x; 68719476736x) + r N^0(68719476736x; 68719476736x) + 2^k r} + \frac{1}{1 + 2^k D_f(137438953472x; 137438953472x) + r N^0(137438953472x; 137438953472x) + 2^k r} + \frac{1}{1 + 2^k D_f(274877906944x; 274877906944x) + r N^0(274877906944x; 274877906944x) + 2^k r} + \frac{1}{1 + 2^k D_f(549755813888x; 549755813888x) + r N^0(549755813888x; 549755813888x) + 2^k r} + \frac{1}{1 + 2^k D_f(1099511627776x; 1099511627776x) + r N^0(1099511627776x; 1099511627776x) + 2^k r} + \frac{1}{1 + 2^k D_f(2199023255552x; 2199023255552x) + r N^0(2199023255552x; 2199023255552x) + 2^k r} + \frac{1}{1 + 2^k D_f(4398046511104x; 4398046511104x) + r N^0(4398046511104x; 4398046511104x) + 2^k r} + \frac{1}{1 + 2^k D_f(8796093022208x; 8796093022208x) + r N^0(8796093022208x; 8796093022208x) + 2^k r} + \frac{1}{1 + 2^k D_f(17592186044416x; 17592186044416x) + r N^0(17592186044416x; 17592186044416x) + 2^k r} + \frac{1}{1 + 2^k D_f(35184372088832x; 35184372088832x) + r N^0(35184372088832x; 35184372088832x) + 2^k r} + \frac{1}{1 + 2^k D_f(70368744177664x; 70368744177664x) + r N^0(70368744177664x; 70368744177664x) + 2^k r} + \frac{1}{1 + 2^k D_f(140737488355328x; 140737488355328x) + r N^0(140737488355328x; 140737488355328x) + 2^k r} + \frac{1}{1 + 2^k D_f(281474976710656x; 281474976710656x) + r N^0(281474976710656x; 281474976710656x) + 2^k r} + \frac{1}{1 + 2^k D_f(562949953421312x; 562949953421312x) + r N^0(562949953421312x; 562949953421312x) + 2^k r} + \frac{1}{1 + 2^k D_f(1125899906842624x; 1125899906842624x) + r N^0(1125899906842624x; 1125899906842624x) + 2^k r} + \frac{1}{1 + 2^k D_f(2251799813685248x; 2251799813685248x) + r N^0(2251799813685248x; 2251799813685248x) + 2^k r} + \frac{1}{1 + 2^k D_f(4503599627370496x; 4503599627370496x) + r N^0(4503599627370496x; 4503599627370496x) + 2^k r} + \frac{1}{1 + 2^k D_f(9007199254740992x; 9007199254740992x) + r N^0(9007199254740992x; 9007199254740992x) + 2^k r} + \frac{1}{1 + 2^k D_f(18014398509481984x; 18014398509481984x) + r N^0(18014398509481984x; 18014398509481984x) + 2^k r} + \frac{1}{1 + 2^k D_f(36028797018963968x; 36028797018963968x) + r N^0(36028797018963968x; 36028797018963968x) + 2^k r} + \frac{1}{1 + 2^k D_f(72057594037927936x; 72057594037927936x) + r N^0(72057594037927936x; 72057594037927936x) + 2^k r} + \frac{1}{1 + 2^k D_f(144115188075855872x; 144115188075855872x) + r N^0(144115188075855872x; 144115188075855872x) + 2^k r} + \frac{1}{1 + 2^k D_f(288230376151711744x; 288230376151711744x) + r N^0(288230376151711744x; 288230376151711744x) + 2^k r} + \frac{1}{1 + 2^k D_f(576460752303423488x; 576460752303423488x) + r N^0(576460752303423488x; 576460752303423488x) + 2^k r} + \frac{1}{1 + 2^k D_f(1152921504606846976x; 1152921504606846976x) + r N^0(1152921504606846976x; 1152921504606846976x) + 2^k r} + \frac{1}{1 + 2^k D_f(2305843009213693952x; 2305843009213693952x) + r N^0(2305843009213693952x; 2305843009213693952x) + 2^k r} + \frac{1}{1 + 2^k D_f(4611686018427387904x; 4611686018427387904x) + r N^0(4611686018427387904x; 4611686018427387904x) + 2^k r} + \frac{1}{1 + 2^k D_f(9223372036854775808x; 9223372036854775808x) + r N^0(9223372036854775808x; 9223372036854775808x) + 2^k r} + \frac{1}{1 + 2^k D_f(18446744073709551616x; 18446744073709551616x) + r N^0(18446744073709551616x; 18446744073709551616x) + 2^k r} + \frac{1}{1 + 2^k D_f(36893488147419103232x; 36893488147419103232x) + r N^0(36893488147419103232x; 36893488147419103232x) + 2^k r} + \frac{1}{1 + 2^k D_f(73786976294838206464x; 73786976294838206464x) + r N^0(73786976294838206464x; 73786976294838206464x) + 2^k r} + \frac{1}{1 + 2^k D_f(147573952589676412928x; 147573952589676412928x) + r N^0(147573952589676412928x; 147573952589676412928x) + 2^k r} + \frac{1}{1 + 2^k D_f(295147905179352825856x; 295147905179352825856x) + r N^0(295147905179352825856x; 295147905179352825856x) + 2^k r} + \frac{1}{1 + 2^k D_f(590295810358705651712x; 590295810358705651712x) + r N^0(590295810358705651712x; 590295810358705651712x) + 2^k r} + \frac{1}{1 + 2^k D_f(1180591620717411303424x; 1180591620717411303424x) + r N^0(1180591620717411303424x; 1180591620717411303424x) + 2^k r} + \frac{1}{1 + 2^k D_f(2361183241434822606848x; 2361183241434822606848x) + r N^0(2361183241434822606848x; 2361183241434822606848x) + 2^k r} + \frac{1}{1 + 2^k D_f(4722366482869645213696x; 4722366482869645213696x) + r N^0(4722366482869645213696x; 4722366482869645213696x) + 2^k r} + \frac{1}{1 + 2^k D_f(9444732965739290427392x; 9444732965739290427392x) + r N^0(9444732965739290427392x; 9444732965739290427392x) + 2^k r} + \frac{1}{1 + 2^k D_f(18889465931478580854784x; 18889465931478580854784x) + r N^0(18889465931478580854784x; 18889465931478580854784x) + 2^k r} + \frac{1}{1 + 2^k D_f(37778931862957161709568x; 37778931862957161709568x) + r N^0(37778931862957161709568x; 37778931862957161709568x) + 2^k r} + \frac{1}{1 + 2^k D_f(75557863725914323419136x; 75557863725914323419136x) + r N^0(75557863725914323419136x; 75557863725914323419136x) + 2^k r} + \frac{1}{1 + 2^k D_f(151115727451828646838272x; 151115727451828646838272x) + r N^0(151115727451828646838272x; 151115727451828646838272x) + 2^k r} + \frac{1}{1 + 2^k D_f(302231454903657293676544x; 302231454903657293676544x) + r N^0(302231454903657293676544x; 302231454903657293676544x) + 2^k r} + \frac{1}{1 + 2^k D_f(604462909807314587353088x; 604462909807314587353088x) + r N^0(604462909807314587353088x; 604462909807314587353088x) + 2^k r} + \frac{1}{1 + 2^k D_f(1208925819614629174706176x; 1208925819614629174706176x) + r N^0(1208925819614629174706176x; 1208925819614629174706176x) + 2^k r} + \frac{1}{1 + 2^k D_f(2417851639229258349412352x; 2417851639229258349412352x) + r N^0(2417851639229258349412352x; 2417851639229258349412352x) + 2^k r} + \frac{1}{1 + 2^k D_f(4835703278458516698824704x; 4835703278458516698824704x) + r N^0(4835703278458516698824704x; 4835703278458516698824704x) + 2^k r} + \frac{1}{1 + 2^k D_f(9671406556917033397649408x; 9671406556917033397649408x) + r N^0(9671406556917033397649408x; 9671406556917033397649408x) + 2^k r} + \frac{1}{1 + 2^k D_f(19342813113834066795298816x; 19342813113834066795298816x) + r N^0(19342813113834066795298816x; 19342813113834066795298816x) + 2^k r} + \frac{1}{1 + 2^k D_f(38685626227668133590597632x; 38685626227668133590597632x) + r N^0(38685626227668133590597632x; 38685626227668133590597632x) + 2^k r} + \frac{1}{1 + 2^k D_f(77371252455336267181195264x; 77371252455336267181195264x) + r N^0(77371252455336267181195264x; 77371252455336267181195264x) + 2^k r} + \frac{1}{1 + 2^k D_f(154742504910672534362390528x; 154742504910672534362390528x) + r N^0(154742504910672534362390528x; 154742504910672534362390528x) + 2^k r} + \frac{1}{1 + 2^k D_f(309485009821345068724781056x; 309485009821345068724781056x) + r N^0(309485009821345068724781056x; 309485009821345068724781056x) + 2^k r} + \frac{1}{1 + 2^k D_f(618970019642690137449562112x; 618970019642690137449562112x) + r N^0(618970019642690137449562112x; 618970019642690137449562112x) + 2^k r} + \frac{1}{1 + 2^k D_f(1237940039285380274899124224x; 1237940039285380274899124224x) + r N^0(1237940039285380274899124224x; 1237940039285380274899124224x) + 2^k r} + \frac{1}{1 + 2^k D_f(2475880078570760549798248448x; 2475880078570760549798248448x) + r N^0(2475880078570760549798248448x; 2475880078570760549798248448x) + 2^k r} + \frac{1}{1 + 2^k D_f(4951760157141521099596496896x; 4951760157141521099596496896x) + r N^0(4951760157141521099596496896x; 4951760157141521099596496896x) + 2^k r} + \frac{1}{1 + 2^k D_f(9903520314283042199192993792x; 9903520314283042199192993792x) + r N^0(9903520314283042199192993792x; 9903520314283042199192993792x) + 2^k r} + \frac{1}{1 + 2^k D_f(19807040628566084398385987584x; 19807040628566084398385987584x) + r N^0(19807040628566084398385987584x; 19807040628566084398385987584x) + 2^k r} + \frac{1}{1 + 2^k D_f(39614081257132168796771975168x; 39614081257132168796771975168x) + r N^0(39614081257132168796771975168x; 39614081257132168796771975168x) + 2^k r} + \frac{1}{1 + 2^k D_f(79228162514264337593543950336x; 79228162514264337593543950336x) + r N^0(79228162514264337593543950336x; 79228162514264337593543950336x) + 2^k r} + \frac{1}{1 + 2^k D_f(158456325028528675187087900672x; 158456325028528675187087900672x) + r N^0(158456325028528675187087900672x; 158456325028528675187087900672x) + 2^k r} + \frac{1}{1 + 2^k D_f(316912650057057350374175801344x; 316912650057057350374175801344x) + r N^0(316912650057057350374175801344x; 316912650057057350374175801344x) + 2^k r} + \frac{1}{1 + 2^k D_f(633825300114114700748351602688x; 633825300114114700748351602688x) + r N^0(633825300114114700748351602688x; 633825300114114700748351602688x) + 2^k r} + \frac{1}{1 + 2^k D_f(1267650600228229401496703205376x; 1267650600228229401496703205376x) + r N^0(1267650600228229401496703205376x; 1267650600228229401496703205376x) + 2^k r} + \frac{1}{1 + 2^k D_f(2535301200456458802993406410752x; 2535301200456458802993406410752x) + r N^0(2535301200456458802993406410752x; 2535301200456458802993406410752x) + 2^k r} + \frac{1}{1 + 2^k D_f(5070602400912917605986812821504x; 5070602400912917605986812821504x) + r N^0(5070602400912917605986812821504x; 5070602400912917605986812821504x) + 2^k r} + \frac{1}{1 + 2^k D_f(10141204801825835211973625643008x;$$

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for all $x \in X$ and all $r > 0$. Since

$$\lim_{k \rightarrow \infty} \frac{2^k(2-d)r}{8d^k} = \frac{1}{4} \text{ and } \lim_{k \rightarrow \infty} \frac{2^k(2-d)r}{4d^k} = \frac{1}{2} ;$$

we obtain

$$\lim_{k \rightarrow \infty} N^0(x; x); \frac{2^k(2-d)r}{8d^k} = \frac{1}{4} \text{ and } \lim_{k \rightarrow \infty} N^0(x; 2x); \frac{2^k(2-d)r}{4d^k} = \frac{1}{2}$$

for all $x \in X$ and all $r > 0$. Thus

$$N(A(x)) = A^0(x); r = 1$$

for all $x \in X$ and all $r > 0$, hence $A(x) = A^0(x)$. Therefore $A(x)$ is unique.

For $\phi = 1$, we can prove the result by a similar method. This completes the proof of the theorem.

From Theorem 2.1, we obtain the following corollaries concerning the Hyers-Ulam-Rassias and JMRassias stabilities for the functional equation (1.1).

Corollary 2.2. Suppose that a function $f : X \rightarrow Y$ satisfies the inequality

$$N^0(f(jx+jy) - jf(x) - jf(y); r) \geq N^0(\phi(jx) + \phi(jy); r) ; \quad (2.24)$$

for all $x, y \in X$ and all $r > 0$, where ϕ, ψ are constants with $\phi > 0$. Then there exists a unique additive mapping $A : X \rightarrow Y$ such that

$$N(f(2x) - 8f(x) - A(x); r) \geq \dots$$

$$> \quad (2.25)$$

for all $x \in X$ and all $r > 0$.

Theorem 2.3. Let f, g be fixed and let $d : X^2 \rightarrow Z$ be a mapping such that for

$$N^0(d(2y, 2y); r) \geq N^0(d(y, y); r) \quad (2.26)$$



for all $x \in X$ and all $r > 0, d > 0$, and

$$\lim_{k \rightarrow \infty} N^{(k)}_{X;ky} \leq \frac{d}{2^k} \quad (2.27)$$

for all $x, y \in X$ and all $r > 0$. Suppose that a function $f : X \rightarrow Y$ satisfies the inequality

$$N(Df(x, y); r) \leq N^0(x, x); r \quad (2.28)$$

for all $r > 0$ and all $x, y \in X$. Then the limit

$$C(x) = N \lim_{k \rightarrow \infty} \frac{a(2^{-k}x)}{2^k} \quad (2.29)$$

exists for all $x \in X$ and the mapping $C : X \rightarrow Y$ is a unique cubic mapping such that

$$N(f(2x) - 2f(x) - C(x); r) \leq \min \left\{ N^0(x, x); \frac{(2^3 - d)r}{8}; N^0(x, 2x); \frac{(2^3 - d)r}{4} \right\} \quad (2.30)$$

for all $x \in X$ and all $r > 0$.

Proof. It is easy to see from (2.8) that

$$N \left(\frac{f(4x) - 2f(2x) - 8f(x)}{r}; r \right) \leq \min \left\{ N \left(\frac{f(3x) - 4f(2x) + 5f(x)}{2r}; 2r \right); N \left(\frac{f(4x) - 4f(3x) + 6f(2x) - 4f(x)}{2r}; 2r \right); N^0(x, x); \frac{r}{8}; N^0(x, 2x); \frac{r}{2} \right\} \quad (2.31)$$

for all $x \in X$ and all $r > 0$. Let $h : X \rightarrow Y$ be a mapping defined by $h(x) =$

$f(2x) - 2f(x)$. Then we conclude that

$$N(h(2x) - 8h(x); r) \leq \min \left\{ N^0(x, x); \frac{r}{8}; N^0(x, 2x); \frac{r}{2} \right\} \quad (2.32)$$

for all $x \in X$ and all $r > 0$. The rest of the proof is similar to that of Theorem 2.1.

The following corollary is an immediate consequence of Theorem 2.3 concerning the Ulam-Hyers stability of the functional equation(1.1).

Corollary 2.4. Suppose that a function $f : X \rightarrow Y$ satisfies the inequality

$$\begin{aligned} & N^0(f_{ij}x_{ij}^s + jx_{ij}^s g; r) \leq N^0(x, x); r; \\ & N(Df(x, y); r) \leq N^0(f_{ij}x_{ij}^s + jx_{ij}^s g; r); \end{aligned} \quad \begin{aligned} & s = 3; \\ & s = \frac{32}{2}; \end{aligned} \quad (2.33)$$

$$(2.37)$$

for all $x \in X$ and all $r > 0$ and

$$N(f(2x) - 2f(x), C_1(x); r) \leq \min \left\{ N^0(x; x); \frac{(2^3 - d)r}{8}; N^0(x; 2x); \frac{(2^3 - d)r}{4} \right\} \tag{2.38}$$

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for all $x \in X$ and all $r > 0$. Now from (2.37) and (2.38), one can see that

$$\min N \left(\frac{1}{f(2x)} - \frac{1}{6f(x)}, \frac{1}{6A_1(x)} - \frac{1}{6C_1(x)}; 2r \right) \leq \min \left\{ N \left(\frac{1}{f(2x)} - \frac{1}{6f(x)}, \frac{1}{6A_1(x)} - \frac{1}{6C_1(x)}; r \right); N \left(\frac{1}{f(2x)} - \frac{1}{6f(x)}, \frac{1}{6A_1(x)} - \frac{1}{6C_1(x)}; 2r \right) \right\}$$

$$\min N(f(2x) - 8f(x), A_1(x); r); N(f(2x) - 2f(x), C_1(x); r)g$$

$$\min \left\{ N^0(x; x); \frac{(2 - d)r}{8}; N^0(x; 2x); \frac{(2 - d)r}{4}; N^0(x; x); \frac{(2^3 - d)r}{8} \right\}$$

for all $x \in X$ and all $r > 0$. Thus we obtain (2.36) by defining $A(x) = \frac{1}{6} A_1(x)$ and $C(x) = \frac{1}{6} C_1(x)$ for all $x \in X$ and all $r > 0$.

The following corollary is an immediate consequence of Theorem 2.5 concerning the Ulam-Hyers stability of the functional equation(1.1).

Corollary 2.6. Suppose that a function $f : X \rightarrow Y$ satisfies the inequality

$$N(Df(x; y); r) \leq N^0 \left(\sum_{s=1}^3 |x_s + x^s|; r \right); \quad s = 1, 3 \tag{2.39}$$

for all $x; y \in X$ and all $r > 0$, where $\alpha; s$ are constants with $\alpha > 0$. Then there exists a unique additive mapping $A : X \rightarrow Y$ and a unique Cubic mapping $C : X \rightarrow Y$ such that

$$N \left(\frac{f(x) - A(x) - C(x)}{n}; r \right) \geq \min \left\{ N^0 \left(\frac{1}{2^j r}; \frac{1}{8^4} \right); N^0 \left(\frac{1}{2^j r}; \frac{1}{8^4} \right) \right\}$$

REFERENCES

- [1] **J. Aczel and J. Dhombres**, Functional Equations in Several Variables, Cambridge Univ, Press, 1989.
- [2] **T. Aoki**, On the stability of the linear transformation in Banach spaces, J. Math. Soc. Japan, 2 (1950), 64-66.
- [3] **M. Arunkumar, S. Karthikeyan, S. Ramamoorthi**, Generalized Ulam-Hyers Stability of N-Dimensional Cubic Functional Equation in FNS and RNS: Various Methods, Middle-East Journal of Scientific Research, 24 (S2), 386-404, 2016.
- [4] **M. Arunkumar**, Three Dimensional Quartic Functional Equation In Fuzzy Normed Spaces, Far East Journal of Applied Mathematics, Vol 41, No. 2, (2010), 83-94.
- [5] **M. Arunkumar**, Stability of a functional equation in dq normed space, International Journal of pure and Applied Mathematics Vol 57, No.2, 2009, 241-250.
- [6] **M. Arunkumar, John M. Rassias, S. Karthikeyan**, Stability Of A Leibniz Type Additive And Quadratic Functional Equation In Intuitionistic Fuzzy Normed Spaces, Advances in Theoretical and Applied Mathematics, Vol 11, No. 2, 145-169, 2016.
- [7] **C. Baak and M. S. Moslenian**, On the stability of a orthogonally cubic functional equations, Kyungpook Math. J., 47 (2007), 69-76.
- [8] **T. Bag, S.K. Samanta**, Finite dimensional fuzzy normed linear spaces, J. Fuzzy Math. 11 (3) (2003) 687-705.
- [9] **T. Bag, S.K. Samanta**, Fuzzy bounded linear operators, Fuzzy Sets and Systems 151 (2005) 513-547.
- [10] **S.C. Cheng, J.N. Mordeson**, Fuzzy linear operator and fuzzy normed linear spaces, Bull. Cal-cutta Math. Soc. 86 (1994) 429-436.
- [11] **H. Y. Chu and D. S. Kang**, On the stability of an n dimensional cubic functional equation, J. Math. Anal. Appl., 325 No. 1 (2007), 595-607.
- [12] **S. Czerwik**, Functional Equations and Inequalities in Several Variables, World Scientific, River Edge, NJ, 2002.
- [13] **C. Felbin**, Finite dimensional fuzzy normed linear space, Fuzzy Sets and Systems 48 (1992) 239-248.
- [14] **P. Gavruta**, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, J. Math. Anal. Appl., 184 (1994), 431-436.
- [15] **D.H. Hyers**, On the stability of the linear functional equation, Proc.Nat. Acad.Sci.,U.S.A.,27 (1941) 222-224.
- [16] **D.H. Hyers, G. Isac, Th.M. Rassias**, Stability of functional equations in several variables, Birkhauser, Basel, 1998.
- [17] **Sun-Young Jang, Jung Rye Lee, Choonkil Park and Dong Yun Shin**, Fuzzy Stability of Jensen-Type Quadratic Functional Equations, Abstract and Applied Analysis, Volume 2009, Article ID 535678, 17 pages, doi:10.1155/2009/535678.
- [18] **K. W. Jun and H. M. Kim** On the Hyers-Ulam-Rassias stability of a general cubic functional equation, Math. Ineq. Appl., Vol. 6(2) (2003), 289-302.
- [19] **Y. S. Jung and I. S. Chang**, The stability of a cubic type functional equation with the fixed point alternative, J. Math. Anal. Appl., 306 (2005), 752-760.
- [20] **S.M. Jung**, Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis, Hadronic Press, Palm Harbor, 2001.
- [21] **Y. S. Jung and I. S. Chang**, The stability of a cubic type functional equation with the fixed point alternative, J. Math. Anal. Appl., 306 (2005), 752-760.
- [22] **A.K. Katsaras**, Fuzzy topological vector spaces II, Fuzzy Sets and Systems 12 (1984), 143-154.

- [23] **I. Kramosil, J. Michalek**, Fuzzy metric and statistical metric spaces, *Kybernetika* 11 (1975) 326-334.
- [24] **S.V. Krishna, K.K.M. Sarma**, Separation of fuzzy normed linear spaces, *Fuzzy Sets and Systems* 63 (1994) 207-217.
- [25] **B. Margolis, J. B. Diaz**, A fixed point theorem of the alternative for contractions on a generalized complete metric space, *Bull. Amer. Math. Soc.* Vol.126, no.74 (1968), 305-309.
- [26] **D. Mihet**, The fixed point method for fuzzy stability of the Jensen functional equation, *Fuzzy Sets and Systems* 160, (2009), 1663 -1667.
- [27] **A.K. Mirmostafae, M.S. Moslehian**, Fuzzy versions of Hyers-Ulam-Rassias theorem, *Fuzzy Sets and Systems*, Vol. 159, no. 6, (2008), 720729.
- [28] **A.K. Mirmostafae, M. Mirzavaziri, M.S. Moslehian**, Fuzzy stability of the Jensen functional equation, *Fuzzy Sets and Systems*, Vol. 159, no. 6, (2008), 730-738.
- [29] **A. K. Mirmostafae and M. S. Moslehian**, Fuzzy approximately cubic mappings, *Information Sciences*, Vol. 178, no. 19, (2008), 3791-3798.
- [30] **A. K. Mirmostafae and M. S. Moslehian**, Fuzzy almost quadratic functions, *Results in Mathematics*, Vol. 52, no. 1-2, (2008), 161-177.
- [31] **K. H. Park and Y.S. Jung** Stability for a cubic functional equation, *Bull. Korean Math. Soc.*, 41(2) (2004), 347-357.
- [32] **J.M. Rassias**, On approximately of approximately linear mappings by linear mappings, *J. Funct. Anal. USA*, 46, (1982) 126-130.
- [33] **J.M. Rassias**, On approximately of approximately linear mappings by linear mappings, *Bull. Sc. Math*, 108, (1984) 445-446.
- [34] **J. M. Rassias**, Solution of the Ulam stability problem for the cubic mapping, *Glasnik Matematika*, Vol. 36(56) (2001), 63-72.
- [35] **Th.M. Rassias**, On the stability of the linear mapping in Banach spaces, *Proc.Amer.Math. Soc.*, 72 (1978), 297-300.
- [36] **K.Ravi and M.Arunkumar**, On The Generalized Hyers-Ulam- Rassias Stability of a Cubic Functional Equation, *International Review of Pure and Applied Mathematics*, Vol.2, No.1, 2006, 13-23.
- [37] **K. Ravi, M. Arunkumar and J.M. Rassias**, On the Ulam stability for the orthogonally general Euler-Lagrange type functional equation, *International Journal of Mathematical Sciences*, Autumn 2008 Vol.3, No. 08, 36-47.
- [38] **K.Ravi, M. Arunkumar and P. Narasimman**, Fuzzy stability of a Additive functional equation, *International Journal of Mathematical Sciences*, Autumn 2011 Vol.9, No. S11, 88-105.
- [39] **P. K. Sahoo**, On a functional equation characterizing polynomials of degree three, *Bul. Inst. Math. Acad. Sin*, 32 (1), 2004.
- [40] **S.M. Ulam**, *Problems in Modern Mathematics*, Science Editions, Wiley, NewYork, 1964 (Chapter VI, Some Questions in Analysis: 1, Stability).
- [41] **J.Z. Xiao, X.H. Zhu**, Fuzzy normed spaces of operators and its completeness, *Fuzzy Sets and Systems* 133 (2003) 389-399.