Schur-complement of m-Symmetric Matrices in Minkowski Space *M*

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Abstract

In this paper, the notion of Generalized Schur Complement of m-symmetric matrices in Minkowski space are derived. Block Diagonalization of Schur complement in Minkowski space also derived. Finally Positive semi definite(and Definite) m-symmetric matrices are derived in Minkowski space

M .

Keywords and Phrases: Schur complement, Minkowski Inverse, Positive semidefinite (and Definite)Matrices.

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1 Introduction

Let C^n be the space of complex complex vector in n -tuples. We shall index the components of a C^n from 0 to $n-1$. That is $u = (u_0, u_1, u_2, ..., u_{n-1})$.

Let G be the Minkowski metric tensor defined by $Gu = (u_0, -u_1, -u_2, ..., -u_{n-1})$. Clearly the Minkowski metric matrix

$$G = \begin{bmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{bmatrix}, G = G^* \ \ and \ \ G^2 = I_n. \quad (1.1)$$

Minkowski inner product on C^n is defined by $(u, v) = < u, Gv >$ where $< . , . >$ denotes the conventional Hilbert space inner product. A space with Minkowski inner product is called a Minkowski space denoted as M.

For $A \in C^{n\times n}$, $x, y \in C^n$, by using(1.1),

$(Ax, y) = [Ax, Gy] = [x, A^* Gy] = [x, G(GA^* G)y] = [x, GA^{\sim} y] = (x, A^{\sim} y)$,

where $A^{\sim} = GA^* G$, $A^{\sim}$ is called the Minkowski adjoint of A in M (A^* is the usual hermitian adjoint of A). Suppose that the square matrix M written as 2×2 block matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad (1.2)$$

where A is a $s \times s$ matrix and D is a $t \times t$ matrix with $n = s + t$. Here A and D are square matrices, but B and C are not square unless $n = m$. Entries are generally assumed to be complex. If A is square and nonsingular, then M can be decomposed as

$$M = \begin{bmatrix} I_m & 0 \\ CA^{-1} & I_l \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} I_m & A^{-1}B \\ 0 & I_l \end{bmatrix}. \quad (1.3)$$

This decomposition is often called Aitken block-diagonalization formula in the literature, see Puntanen and Styan [12]. Moreover, if both M and A are nonsingular, then the Schur complement $S = D - CA^{-1}B$ is nonsingular too, and the inverse of M can be written in the following form

$$M^{-1} = \begin{bmatrix} I_m & -A^{-1}B \\ 0 & I_l \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} I_m & 0 \\ -CA^{-1} & I_l \end{bmatrix}$$

$$= \begin{bmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{bmatrix} \quad (1.4)$$

This well-known formula is called the Banachiewicz inversion formula for the inverse of a nonsingular matrix in the literature, see Puntanen and Styan [12], and can be found in most linear algebra books. The two formulas in (1.3) and (1.4)

and their consequences are widely used in manipulating partitioned matrices and their operations. When both A and M in (1.2) are singular, the two formulas in (1.3) and (1.4) can be extended to generalized inverses of matrices.

2 Preliminaries

Definition 2.1. *If A is nonsingular, the Schur complement of M with respect to A is defined as*

$M/A = D - CA^{-1}B.$

If D is nonsingular, the Schur complement of M with respect to D is defined as

$M/D = A - BD^{-1}C.$

Matrices of the above form are called the Schur complement of A in M and the Schur complement of D in M respectively.

Definition 2.2. *For $A \in C^{n\times n}$, A is said to be Hermitian positive semi-definite denoted as $A \geq 0$ if A is Hermitian and $[Ax, x] \geq 0$, for all $X \in C^n$.*

Definition 2.3. *For any $A \in C^{m\times n}$, A^m is the Minkowski inverse of A if $AA^mA = A$, $A^mAA^m = A^m$, AA^m and A^mA are m-symmetric.*

For $A \in C^{m\times n}$, $A^{\dagger}$ exists but the Minkowski inverse A^m need not exists in m. The existence of A^m has been established by Meenakshi [9], by modifying the definition of Minkowski adjoint as $A^{\sim} = GA^{*}G$, where G is the Minkowski metric matrix of order n.

Theorem 2.4. *(Theorem 1 [9]) For $A \in C^{m\times n}$, A^m exists if and only if $rk(AA^{\sim}) = rk(A^{\sim}A) = rk(A)$.*

Definition 2.5. *For $A \in C^{n\times n}$, A is said to be m -symmetric in Minkowski space M if $A = A^{\sim}$.*

Definition 2.6. *A matrix $A \in C^{n\times n}$ is said to be range symmetric in m if and only if $N(A) = N(A^{\sim})$.*

Theorem 2.7. *(Theorem 2.2.8([7])) For* $A \in C^{m\times n}$, *if* A^m *exists then* $(GA)^m$ *exists and* A^mG *is the Minkowski metric tensor of order* m *. Conversely if* $(GA)^m$ *exists, then* A^m *exists.*

3 Schur complement of m-symmetric matrices using Minkowski inverse

In this section I have obtained Schur-complement of m-symmetric matrices using Minkowski Inverse.

Theorem 3.1. *Let M be a block matrix with* $r(M) = r(A) + r(D)$. *If* $AA^mB = BDD^m$ *and* $CAA^m = DD^mC$, *then M has the Minkowski inverse* M^m *and*

$$M^m = \begin{bmatrix} A^m+A^mB(D-CA^mB)^mCA^m & A^mB(D-CA^mB)^m \\ -(D-CA^mB)^mCA^m & -(D-CA^mB)^m \end{bmatrix}$$

$$= \begin{bmatrix} (A-BD^mC)^m & -(A-BD^mC)^mBD^m \\ -D^mC(A-BD^mC)^m & D^m+D^mC(A-BD^mC)^mBD^m \end{bmatrix} \quad (3.1)$$

Proof : Using linear transformation

$$\begin{bmatrix} I & 0 \\ -CA^m & I \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} I & A^mB \\ 0 & -I \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & D-CA^mB \end{bmatrix}$$

From the rank additivity condition $r(M) = r(A) + r(D)$ and $r(D) = r(D - CA^mB)$. Also since $B = BDD^m$ and $C = DD^mC$.

If we demonstrate $A^m + A^mB(S_D)^mCA^m$ satisfies the Minkowski inverses, then we can say that $(S_A)^m = A^m + A^mB(S_D)^mCA^m$. In fact,

$(A - BD^mC)(A^mB(D - CA^mB)^mCA^m) = AA^m + AA^mB(D - CA^mB)^mCA^m - BD^mCA^m - BD^mCA^mB(D - CA^mB)^mCA^m$.

$= \quad AA^m + (D - CA^mB)^m(AA^mBCA^m - BD^mCA^mBCA^m) - BD^mCA^m$

$= \quad AA^m - BD^mCA^m + (B - BD^mCA^mB)(D - CA^mB)^mCA^m$

$= \quad AA^m - BD^mCA^m + BD^m(D - CA^mB)(D - CA^mB)^mCA^m$

$= \quad AA^m - BD^mCA^m + BD^mCA^m = AA^m$.

Therefore

$((A-BD^mC)(A^m + A^mB(D-CA^mB)^mCA^m))^\sim = (A-BD^mC)(A^m + A^mB(D-CA^mB)^mCA^m)$.

On the other hand,

$(A^m + A^mB(D - CA^mB)^mCA^m)(A - BD^mC) = A^mA + A^mB(D - CA^mB)^mCA^mA - A^mBD^m - A^mB(D - CA^mB)^mCA^mBD^mC$

$= \quad A^mA - A^mBD^mC + A^mB(D - CA^mB)^m(D - CA^mB)D^mC$

$= \quad A^mA - A^mBD^mC + A^mBD^mC = A^mA$.

Hence, $((A^m + A^mB(D - CA^mB)^m)(A - BD^mC))^\sim = (A^m + A^mB(D - CA^mB)^mCA^m)(A - BD^mC.)$

From the above two equalities, we have $(A - BD^mC)(A^m + A^mB(D - CA^mB)^mCA^m)(A - BD^mC) = AA^m(A - BD^mC)$

$= \quad A - BD^mC(A^m + A^mB(D - CA^mB)^mCA^m)(A - BD^mC)(A^m + A^mB(D - CA^mB)^mCA^m)$

$= \quad A^mA(A^m + A^mB(D - CA^mB)^m)CA^m$

$= \quad A^m + A^mB(D - CA^mB)^mCA^m$.

Hence (3.1) holds. Next to show that the matrix

$$\begin{bmatrix} A^m+A^mB(D-CA^mB)^mCA^m & A^mB(D-CA^mB)^m \\ -(D-CA^mB)^mCA^m & -(D-CA^mB)^m \end{bmatrix}$$

is the m-inverse of M. Since

$$\begin{bmatrix} A^m+A^mB(D-CA^mB)^mCA^m & A^mB(D-CA^mB)^m \\ -(D-CA^mB)^mCA^m & -(D-CA^mB)^m \end{bmatrix} M = \begin{bmatrix} AA^m & 0 \\ 0 & DD^m \end{bmatrix}$$

Further,

$$M \begin{bmatrix} A^m+A^mB(D-CA^mB)^mCA^m & A^mB(D-CA^mB)^m \\ -(D-CA^mB)^mCA^m & -(D-CA^mB)^m \end{bmatrix} M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} AA^m & 0 \\ 0 & D^mD \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = M$$

and

$$\begin{bmatrix} A^m+A^mB(D-CA^mB)^mCA^m & -A^mB(D-CA^mB)^m \\ -(D-CA^mB)^mCA^m & (D-CA^mB)^m \end{bmatrix} M \begin{bmatrix} A^m+A^mB(D-CA^mB)^mCA^m & -A^mB(D-CA^mB)^m \\ -(D-CA^mB)^mCA^m & (D-CA^mB)^m \end{bmatrix}$$

$$= \begin{bmatrix} A^mA & 0 \\ 0 & D^mD \end{bmatrix} \begin{bmatrix} A^m+A^mB(D-CA^mB)^mCA^m & -A^mB(D-CA^mB)^m \\ -(D-CA^mB)^mCA^m & (D-CA^mB)^m \end{bmatrix}$$

$$= \begin{bmatrix} A^m+A^mB(D-CA^mB)^mCA^m & -A^mB(D-CA^mB)^m \\ -(D-CA^mB)^mCA^m & (D-CA^mB)^m \end{bmatrix}$$

From the above, we can say that M has Minkowski inverse.

Theorem 3.2. *Let M be of the form*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

then M is m-symmetric in M iff A is m-symmetric and $B = -C^{\sim}G_1$, $C = -G_1B^{\sim}$ *and* $D = D^*$, *where G is the Minkowski metric tensor of order as that of A.*

Proof : Since A is m-symmetric in M, then $A = A^{\sim}$, where $A^{\sim} = GA^*G$.

A is m-symmetric iff AG is Hermitian iff GA is Hermitian.

A is m-symmetric iff $(Ax, x) = (x, A^{\sim}x)$ for every $X \in C^n$. AG is Hermitian implies that $(AG)^* = AG$

$G(AG)^*G = GAGG$

$(AG)^{\sim} = GA.$

By a theorem(Theorem 3.3 [8]), $B = AA^mB$ and $C = CA^mA$ and $D = CA^mB$. Taking Minkowski adjoint and by using $G^{\sim}_1 = G_1$ and by a theorem, we get $C = -G_1(A^mB)^{\sim}A$.

$= -G_1B^{\sim}(A^m)^{\sim}A$

$= -G_1B^{\sim}A^mA$

$= -G_1B^{\sim}.$

$= (-G_1(A^mB)^{\sim}A)^{\sim}$

$= -A^{\sim}A^mBG^{\sim}_1$

$= -AA^mBG_1$

$B = -BG_1$

$B^{\sim} = -(BG_1)^{\sim} = -G_1{}^{\sim}B^{\sim} = -G_1B^{\sim} = C.$

Also $D - CA^mB \geq 0$. $D = CA^mB$ $D^* = (CA^mB)^* = B^*(A^m)^*C^*$

$= (BA^mC)^* = D$. Thus by using the above $M^{\sim}$ can be expressed in the form

$$M^{\sim} = \begin{bmatrix} A^{\sim} & C^{\sim}G_1 \\ -G_1B^{\sim} & -D^{*} \end{bmatrix},$$

4 Partitioned Inverses of the Schur complement in Minkowski space

If $A^{\sim}$ is non-singular, then

$$(M^{\sim})^m = \begin{bmatrix} A^{\sim} & C^{\sim}G_1 \\ -G_1B^{\sim} & -D^{*} \end{bmatrix}^m$$

$$= \begin{bmatrix} I & (A^{\sim})^m C^{\sim}G_1 \\ 0 & I \end{bmatrix} \begin{bmatrix} (A^{\sim})^m & 0 \\ 0 & (M^{\sim}/A^{\sim})^m \end{bmatrix} \begin{bmatrix} I & 0 \\ (G_1B^{\sim})(A^{\sim})^m & I \end{bmatrix}$$

$$= \begin{bmatrix} (A^{\sim})^m & (A^{\sim})^m C^{\sim}G_1 (M^{\sim}/A^{\sim})^m \\ 0 & (M^{\sim}/A^{\sim})^m \end{bmatrix} \begin{bmatrix} I & 0 \\ (G_1B^{\sim})(A^{\sim})^m & I \end{bmatrix}$$

$$= \begin{bmatrix} (A^{\sim})^m + (A^{\sim})^m C^{\sim}G_1 (M^{\sim}/A^{\sim})^m (G_1B^{\sim})(A^{\sim})^m & (A^{\sim})^m C^{\sim}G_1 (M^{\sim}/A^{\sim})^m \\ (M^{\sim}/A^{\sim})^m G_1B^{\sim}(A^{\sim})^m & (M^{\sim}/A^{\sim})^m \end{bmatrix}$$

Suppose that D^{*} is non-singular, then

$$(M^{\sim})^m = \begin{bmatrix} A^{\sim} & C^{\sim}G_1 \\ -G_1B^{\sim} & -D^{*} \end{bmatrix}^m$$

$$= \begin{bmatrix} (M^{\sim}/D^{*})^m & (M^{\sim}/D^{*})^m C^{\sim}G_1 (A^{\sim})^m \\ (D^{*})^m G_1B^{\sim}(M^{\sim}/D^{*})^m & (D^{*})^m + (D^{*})^m G_1B^{\sim}(M^{\sim}/D^{*})^m C^{\sim}G_1 (D^{*})^m \end{bmatrix}$$

If $A^{\sim}$ and D^{*} are both Schur complements then $M^{\sim}/A^{\sim}$ and $M^{\sim}/D^{*}$ are all invertible. By comparing the above two expressions for $(M^{\sim})^m$, we get the (non-obvious) formula

$$(A^{\sim} - C^{\sim}(D^{\sim})^m B^{\sim})^m$$
$$= (A^{\sim})^m - (A^{\sim})^m G_1B^{\sim}(D^{*} - G_1B^{\sim}(A^{\sim})^m C^{\sim}G_1)^m (-G_1B^{\sim})(A^{\sim})^m.$$

Using this formula, we obtain another expression for the inverse of $M^{\sim}$ involving the Schur complements of $A^{\sim}$ and D^{*},

$$\begin{bmatrix} A^{\sim} & -C^{\sim}G_1 \\ -G_1B^{\sim} & D^{*} \end{bmatrix}^m$$

$$= \begin{bmatrix} (A^{\sim} - C^{\sim}(D^{\sim})^m B^{\sim})^m & (A^{\sim})^m C^{\sim}G_1 (D^{*} - G_1B^{\sim}(A^{\sim})^m C^{\sim}G_1)^m \\ -(D^{*} - G_1B^{\sim}(A^{\sim})^m C^{\sim}G_1)^m (-G_1B^{\sim})(A^{\sim})^m & (D^{*} - G_1B^{\sim}(A^{\sim})^m C^{\sim}G_1)^m \end{bmatrix}$$

If we set $D^{*} = I$ and $-C^{\sim}G_1$ to $C^{\sim}G_1$ we get

$$(A^{\sim} - C^{\sim}(D^{\sim})^m B^{\sim})^m = (A^{\sim})^m + (A^{\sim})^m C^{\sim}G_1 (I + G_1B^{\sim}(A^{\sim})^m C^{\sim}G_1)^m (-G_1B^{\sim})(A^{\sim})^m,$$

a formula known as the Matrix Inversion formula.

Theorem 4.1. *For any m-symmetric matrix $M^{\sim}$ of the form*

$$M^{\sim} = \begin{bmatrix} A^{\sim} & -C^{\sim}G_1 \\ -G_1C & -G_1B^{\sim} \end{bmatrix},$$

If $-G_1B^{\sim}$ is invertible then the following properties hold.

(i) If $M^{\sim} \succ 0$ if and only if $(-G_1B^{\sim}) \succ 0$ and $A^{\sim}+C^{\sim}G_1(G_1B^{\sim})^m(G_1C) \succ 0$ (ii)If $(-G_1B^{\sim}) \succ 0$, then $M^{\sim} \geq 0$ if and only if $A^{\sim} + C^{\sim}G_1(G_1B^{\sim})^m(G_1C) \geq$ 0.

Proof : We know that for any m-symmetric matrix T and any invertible matrix N , the matrix T is positive definite ($T \succ$ 0) if and only if $N\,T\,N^{\sim}$ is positive definite. That is $N\,T\,N^{\sim} \succ 0$. But a block diagonal matrix is positive definite if
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and only if each diagonal block is positive definite. Hence (i) satisfied. Similarly we can show that for any m-symmetric matrix T and any invertible matrix N , we have $T \geq 0$ if and only if $N\,T\,N^{\sim} \geq 0$.

Theorem 4.2. *For any m-symmetric matrix $M^{\sim}$ of the form*

$$M^{\sim} = \begin{bmatrix} A^{\sim} & -C^{\sim}G_1 \\ -G_1C & -G_1B^{\sim} \end{bmatrix},$$

If $A^{\sim}$ is invertible then the following properties hold.

(i) $M^{\sim} \succ 0$ if and only if $A^{\sim} \succ 0$ and $-G_1B^{\sim} - G_1C(A^{\sim})^mC^{\sim}G_1 \succ 0$ (ii)If $A^{\sim} \succ 0$, then $M^{\sim} \geq 0$ if and only if $-G_1B^{\sim} - G_1C(A^{\sim})^mC^{\sim}G_1 \geq 0$.

Proof : When $-G_1B^{\sim}$ is singular ((or) $A^{\sim}$ is singular), it is still possible to characterize when a symmetric matrix $M^{\sim}$, as above is positive semidefinite but this requires using a version of the Schur complement involving the Minkowski inverse of $-G_1B^{\sim}$, namely $A^{\sim}+C^{\sim}G_1(G_1B^{\sim})^mG_1C$ (or the Schur complement, $-G_1B^{\sim} + G_1C(A^{\sim})^mC^{\sim}G_1$ of $A^{\sim}$).

5 Singular Value Decomposition using Minkowski Inverses

Every square $n \times n$ matrix $M^{\sim}$ has a singular value decomposition (SVD). We can write $M^{\sim} = U\Sigma V^{\sim}$, where U and V are orthogonal matrices and Σ is a diagonal matrix of the form $\Sigma = diag(\sigma_1, \sigma_2, ..., \sigma_r, 0, ..., 0)$ and r is the rank of $M^{\sim}$. The $(\sigma_i)'s$ are called the singular values of $M^{\sim}$ and they are the positive square roots of the non zero eigenvalues of $M^{\sim}M$ and $M\,M^{\sim}$. Also U and V are not unique. Furthermore, the columns of V are eigenvectors of $M\,M^{\sim}$ and the columns of U are eigenvectors of $M^{\sim}M$. Hence $M^{\sim} = U\Sigma V^{\sim}$ is a singular value decomposition of $M^{\sim}$.

If $rk(M^{\sim}M) = rk(M^{\sim}) = rk(M\,M^{\sim})$, then $(M^{\sim})^m$ exists. It is easy to check that $M^{\sim}(M^{\sim})^mM^{\sim} = M^{\sim}$, $(M^{\sim})^mM^{\sim}(M^{\sim})^m = (M^{\sim})^m$ and both $M^{\sim}(M^{\sim})^m$ and $(M^{\sim})^mM^{\sim}$ are symmetric matrices.

$$\text{In fact } M^{\sim}(M^{\sim})^m = U\Sigma V^{\sim}V\Sigma^{\sim}U^{\sim} = U\Sigma\Sigma^{\sim}U^{\sim} = U\begin{bmatrix} I_r & 0 \\ 0 & 0_{n-r} \end{bmatrix}U^{\sim}$$
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$$\text{and } (M^{\sim})^mM^{\sim} = V\Sigma^{\sim}U^{\sim}U\Sigma V^{\sim} = U\Sigma^{\sim}\Sigma V^{\sim} = V\begin{bmatrix} I_r & 0 \\ 0 & 0_{n-r} \end{bmatrix}V^{\sim}.$$

We immediately get $(M^{\sim}(M^{\sim})^m)^2 = M^{\sim}(M^{\sim})^m$, $((M^{\sim})^m M^{\sim})^2 = (M^{\sim})^m M^{\sim}$.

) $M^{\sim}$) = ($M^{\sim}$)

So both $M^{\sim}(M^{\sim})^m$ and $(M^{\sim})^m M^{\sim}$ are orthogonal projections. (since they both are symmetric). We claim that $M^{\sim}(M^{\sim})^m$ is the orthogonal projection onto $ker(M^{\sim})^{\perp}$, the orthogonal complement of $ker(M^{\sim})$. Obviously, $range(M^{\sim})(M^{\sim})^m \subseteq range(M^{\sim})$ and for any $y = M^{\sim}x \in range(M^{\sim})$, as $M^{\sim}(M^{\sim})^m M^{\sim} = M^{\sim}$, we have $M^{\sim}(M^{\sim})^m y = M^{\sim}(M^{\sim})^m M^{\sim}x = M^{\sim}x$ =

y. Therefore $M^{\sim}(M^{\sim})^m y = y$. so the image of $M^{\sim}(M^{\sim})^m$ is indeed the range of $M^{\sim}$. It is also clear that $Ker(M^{\sim}) \subseteq ker((M^{\sim})^m M^{\sim})$ and since $M^{\sim}(M^{\sim})^m M^{\sim} = M^{\sim}$.

We also have $ker((M^{\sim})^m M^{\sim}) \subseteq ker(M^{\sim})$ and so $Ker((M^{\sim})^m M^{\sim}) = ker(M^{\sim})$. Since $(M^{\sim})^m M^{\sim}$ is Hermitian, range $((M^{\sim})^m M^{\sim}) = ker((M^{\sim})^m M^{\sim})^{\perp} = ker(M^{\sim})^{\perp}$ as claimed. It will also be useful to see that $range(M^{\sim}) = range(M^{\sim}(M^{\sim})^m)$

consists of all vector $y \in C^n$ such that $U^{\sim}y = \begin{bmatrix} Z \\ 0 \end{bmatrix}$ with $Z \in C^r$, Indeed if

$$y = M^{\sim}x, \text{ then } U^{\sim}y = U^{\sim}M^{\sim}x = U^{\sim}U\Sigma V^{\sim}x = \Sigma V^{\sim}x = \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0_{n-r} \end{bmatrix} V^{\sim}x = \begin{bmatrix} Z^{\sim} \\ 0 \end{bmatrix},$$

where Σ_r is the $r \times r$ diagonal matrix $diag(\sigma_1, \ldots, \sigma_r)$.

Conversely, if

$U^{\sim}y = \begin{bmatrix} Z \\ 0 \end{bmatrix}$ then $y = U \begin{bmatrix} Z \\ 0 \end{bmatrix}$

and

$$M^{\sim}(M^{\sim})^m y = U \begin{bmatrix} I_r & 0 \\ 0 & 0_{n-r} \end{bmatrix} U^{\sim}y$$

$$= U \begin{bmatrix} I_r & 0 \\ 0 & 0_{n-r} \end{bmatrix} U^{\sim}U \begin{bmatrix} Z \\ 0 \end{bmatrix}$$

$$= U \begin{bmatrix} I_r & 0 \\ 0 & 0_{n-r} \end{bmatrix} \begin{bmatrix} Z \\ 0 \end{bmatrix} = U \begin{bmatrix} Z \\ 0 \end{bmatrix} = y.$$

Therefore $M^{\sim}(M^{\sim})^m y = y$ which shows that y belongs to the range of $M^{\sim}$.

Similarly, we claim that $range((M^{\sim})^m M^{\sim}) = ker(M^{\sim})^{\perp}$ consists of all vector

$y \in C^n$ such that

$V^{\sim}y = \begin{bmatrix} Z \\ 0 \end{bmatrix}$, with $Z \in C^r$.

$$If\ y = (M^{\sim})^{\sim}M^{\sim}u, \text{ then } y = (M^{\sim})^{\sim}M^{\sim}u = V \begin{bmatrix} I_r & 0 \\ 0 & 0_{n-r} \end{bmatrix} V^{\sim}u = V \begin{bmatrix} Z \\ 0 \end{bmatrix}, \text{ for some } Z \in C^r.$$

Conversely, if

$V^{\sim}y = \begin{bmatrix} Z \\ 0 \end{bmatrix}$ then $y = V \begin{bmatrix} Z \\ 0 \end{bmatrix}$

and so,

$$(M^{\sim})^m M^{\sim} V \begin{bmatrix} Z \\ 0 \end{bmatrix} = V \begin{bmatrix} I_r & 0 \\ 0 & 0_{n-r} \end{bmatrix} V^{\sim}V \begin{bmatrix} Z \\ 0 \end{bmatrix} = V \begin{bmatrix} Z \\ 0 \end{bmatrix} = y,$$

which shows that $y \in range((M^{\sim})^m M^{\sim})$. If $M^{\sim}$ is a symmetric matrix, then in general there is no SVD, $U\Sigma V^{\sim}$ of $M^{\sim}$ with $U = V$. However, if $M^{\sim} \geq 0$, then the eigenvalues of $M^{\sim}$ are nonnegative and so the nonzero eigenvalues of $M^{\sim}$ are equal to the singular values of $M^{\sim}$ and singular value decompositions of $M^{\sim}$ of the form $M^{\sim} = U\Sigma V^{\sim}$. In this case U and V are unitary matrices, $M^{\sim}M$ and $M M^{\sim}$ are Hermitian orthogonal projections. If $M^{\sim}$ is a normal matrix which means that $M^{\sim}M = M M^{\sim}$, then there is an intimate relationship between singular value decompositions of $M^{\sim}$ and block diagonalization of $M^{\sim}$. If $M^{\sim}$ is a normal matrix, then it can be block diagonalized with respect to an orthogonal matrix U, as $M^{\sim} = U\Lambda U^{\sim}$, where Λ is the block diagonal matrix, where $\Lambda = diag(B_1, B_2, ..., B_n)$, consisting either of 2×2 blocks of the form

$$B_j = \begin{bmatrix} \lambda_j & -\mu_j \\ \mu_j & \lambda_j \end{bmatrix}$$

with $\mu_j \neq 0$ or one dimensional blocks $B_k = \lambda_k$. Assume that $B_1, B_2, ..., B_p$ are 2×2 blocks and that $\lambda_{2p+1}, ..., \lambda_n$ are the scalar entries. We know that the

numbers $\lambda_j \pm i\mu_j$, and the λ_{2p+k} are the eigenvalues of A.

Let $\rho_{2j-1} = \rho_{2j} = \sqrt{(\lambda_j)^2 + (\mu_j)^2}$ for $j = 1, 2, ..., p$. $\rho_{2p+j} = \lambda_j$ for $j = 1, 2, ..., n - 2p$ and assume that the blocks are ordered so that

$\rho_1 \geq \rho_2 \geq ... \geq \rho_n$. Then it is easy to see that

$U U^{\sim} = U^{\sim}U = U\Lambda U^{\sim}U\Lambda^{\sim}U^{\sim} = U\Lambda\Lambda^{\sim}U^{\sim}$, with $\Lambda\Lambda^{\sim} = diag(\rho_1^2, ..., \rho_n^2)$.

So, the singular values $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_n$ of A which are the nonnegative square roots of the eigenvalues of $AA^{\sim}$, are such that $\sigma_j = \rho_j$, $1 \leq j \leq n$. For any $A \in C_{n\times n}$ in m, If A is m-normal if and only if $A^{\sim}GA = AGA^{\sim}$ (By Theorem 2.1.3. [7]). We can define the diagonal matrices $\Sigma = diag(\sigma_1, ..., \sigma_r, 0, ..., 0)$ where $r = rk(A)$, $\sigma_1 \geq ... \geq \sigma_r \geq 0$, and B = $diag(\sigma_1^{-1}B_1, ..., \sigma_p^{-1}B_p, 1, ..., 1)$ so that B is an orthogonal matrix and $\Lambda = B\Sigma = (B_1, ..., B_p, \lambda_{2p+1}, ..., \lambda_r, 0, ..., 0)$. But then, we can write $A = U\Lambda U^{\sim} = UB\Sigma U^{\sim}$ and if we let $V = UB$, as U is orthogonal and B is also orthogonal, V is also
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orthogonal and $A = V\Sigma U^{\sim}$ is an SVD for A. Now, we get $A^m = U\Sigma^m V^{\sim} = U\Sigma^m B^{\sim}U^{\sim}$. However, since B is an orthogonal matrix, $(B)^{\sim} = (B)^m$ and a simple calculation shows that $\Sigma^m B^{\sim} = \Sigma^m B^{-1} = \Lambda^m$, which yields the formula

$A^m = U\Lambda^m U^{\sim}$. Also observe that if we write $\Lambda_r = (B_1, ..., B_p, \lambda_{2p+1}, ..., \lambda_r)$, then Λ_r is invertible and

$$\Lambda^m = \begin{bmatrix} (\Lambda_r)^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore, the Minkowski inverse of a normal matrix can be computed directly from any block diagonalization of A, as claimed. Next we will use the Minkowski inverses to generalize the result to the symmetric matrices

$$M^{\sim} = \begin{bmatrix} A^{\sim} & -C^{\sim}G_1 \\ -G_1C & -G_1B^{\sim} \end{bmatrix}$$

where $-G_1B^{\sim}$ $(orA^{\sim})$ is singular.

Theorem 5.1. *If P is an invertible symmetrix matrix, then the function $f(x) = \frac{1}{2}x^{\sim}Px + x^{\sim}b$ has a minimum value if and only if $P \geq 0$, in whihc case this optimal value is obtained for a unique value of x, namely $x^* = -P^m b$, and with $f(P^m b) = -\frac{1}{2}b^{\sim}P^m b$.*

Proof : Observe that $\frac{1}{2}(x + P^m b)^{\sim}P(x + P^m b) = \frac{1}{2}x^{\sim}Px + x^{\sim}b + \frac{1}{2}b^{\sim}P^m b$.

Thus, $f(x) = \frac{1}{2}$ [illegible]

If P has some negative eigenvalue, say $-\lambda$ (*with* $\lambda \geq 0$), if we pick any eigenvector u of P associated with λ, then for any $\alpha \in \Re$ with $\alpha \neq 0$, if we let $x = \alpha u - P^{-1}b$, then as $P u = -\lambda u$ we get

$$f(x) = \frac{1}{2}(x + P^{m}b)^{\sim}P(x + P^{m}b) - \frac{1}{2}b^{\sim}P^{m}b$$

$$= \frac{1}{2}\alpha u^{\sim}P\alpha u - \frac{1}{2}b^{\sim}P^{m}b$$

$$= -\frac{1}{2}\alpha^{2}\lambda\|u\|_2^{2} - \frac{1}{2}b^{\sim}P^{m}b,$$

and as α can be made as large as we want and $\lambda \geq 0$, we see that f has no minimum. Consequently, in order for f to have a minimum, we must have

P ≥ 0. In this case, as $(x + P^{m}b)^{\sim}P(x + P^{m}b) \geq 0$, it is clear that the minimum value of f is achieved when $x + P^{m}b = 0$. That is $x = -P^{m}b$.

Theorem 5.2. *If P is a symmetric matrix, then the function*

$f(x) = \frac{1}{2}x^{\sim}Px + x^{\sim}b$ *has a minimum value if and only if* $P \geq 0$ *and*
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$(I - PP^{m})b = 0$, *in which case this minimum value is* $P^{*} = -\frac{1}{2}b^{\sim}P^{m}b$. *Furthermore, if* $P = U^{\sim}\Sigma U$ *is an SV D of P, then the optimal value is achieved*

by all $x \in C^{n}$ *of the form* $x = -P^{m}b + U^{\sim}\begin{bmatrix}0\\z\end{bmatrix}$, *for any* $Z \in C^{n-r}$ *where r is the rank of* P.

Proof : The case where P is invertible is taken care of by Theorem 4.1. so, we may assume that P is singular. If P has rank $r \prec n$, then we can diagonalize P as

$$P = U^{\sim}\begin{bmatrix}\Sigma_r & 0\\ 0 & 0\end{bmatrix}U,$$

where U is an orthogonal matrix and where Σ_r is an $r \times r$ diagonal invertible matrix. Then, we have

$$f(x) = \frac{1}{2}x^{\sim}U^{\sim}\begin{bmatrix}\Sigma_r & 0\\ 0 & 0\end{bmatrix}Ux + x^{\sim}U^{\sim}Ub$$

$$= \frac{1}{2}(Ux)^{\sim}\begin{bmatrix}\Sigma_r & 0\\ 0 & 0\end{bmatrix}Ux + (Ux)^{\sim}Ub.$$

$$Ux = \begin{bmatrix}y\\z\end{bmatrix} \text{ and } Ub = \begin{bmatrix}c\\d\end{bmatrix} \text{with } y, c \in C^{r} \text{ and } z, d \in C^{n-r}.$$

$$\text{we get } f(x) = \frac{1}{2}(Ux)^{\sim}\begin{bmatrix}\Sigma_r & 0\\ 0 & 0\end{bmatrix}Ux + (Ux)^{\sim}Ub$$

$$= \frac{1}{2}(y^{\sim}, z^{\sim})\begin{bmatrix}\Sigma_r & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}y\\z\end{bmatrix} + \begin{bmatrix}y^{\sim}, z^{\sim}\end{bmatrix}\begin{bmatrix}c\\d\end{bmatrix}$$

$$= \frac{1}{2}y^{\sim}\Sigma_r y + y^{\sim}c + z^{\sim}d.$$

For $y = 0$, we get $f(x) = z^{\sim}d$, so if $d \neq 0$, the function f has no minimum.

Therefore, if f has a minimum, then $d = 0$. However, $d = 0$ means that c and we m 1 ~

know that b is in the range of P (here U is U) which is

equivalent to $(I - P P\ )b = 0$. If $d = 0$, then $f(x) = \frac{}{2}\ y^{\sim}\Sigma_r y + y^{\sim}c$ and as Σ_r is invertible, By Theorem 4.1, the function f has a minimum if and only if Σ_r ≥ 0, which is equivalent to $P \geq 0$. Therefore, we proved that if f has a minimum, then $(I - P P^{m})b = 0$ and $P \geq 0$.

Conversely, if $(I - P P^{m})b = 0$ *and* $P \geq 0$. To prove that f does have a minimum.

When the above conditions hold, the minimum is achieved if $y = -\Sigma_r^{-1}c, z = 0$ and $d = 0$. That is For x given by $U x^{*} =$ $-\Sigma_r^{-1}c$ *and* $U b =$ c ,

$* \ \Sigma_r^{-1}c$ $[\ 0\ ]\Sigma_r^{-1}c\ 0$ $c\ [\ 0\ ]$

from which we deduce that $x^{*} = -U^{\sim}[\ 0\ ] = -U^{\sim}[\ 0\ 0\ ][\ 0\ ] =$ 13

$-U^{\sim}\ [0\ \ \Sigma_r^{-1}\ c\ \ 0_n]\ U b = -P^{\sim} b$ and the minimum value of f is $f(x^{*}) = -\frac{1}{2} b^{\sim} P^{m} b$

for any $x \in C$ *of the form* $x = -P\ b + U^{\sim}\ [z]$ for any $z \in C$.

Our previous calculations shows that $f(x) = -\frac{1}{2}\, b^{\sim}P^{m}b$. When a symmetric

matrix $\begin{bmatrix} A^{\sim} & -C^{\sim}G_1 \\ -G_1C & -G_1B^{\sim} \end{bmatrix}$ is positive semidefinite. Thus we want to know when the function

$$f(x, y) = (x^{\sim}, y^{\sim}) \begin{bmatrix} A^{\sim} & -C^{\sim}G_1 \\ -G_1C & -G_1B^{\sim} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$f(x, y) = x^{\sim}A^{\sim}x - 2x^{\sim}C^{\sim}G_1y - y^{\sim}G_1B^{\sim}y$$

has a minimum with respect to both x and y. Holding y constant, Theorem 4.2 implies that $f(x, y)$ has a minimum if and only if $A \geq 0$ and $(I - AA^{\sim})By = 0$ and then, the minimum value is

$$f(x^{*}, y) = -y^{\sim}B^{\sim}A^{m}By + y^{\sim}cy = y^{\sim}(c - B^{\sim}A^{m}B)y.$$

Since we want $f(x, y)$ to be uniformly bounded from below for all x, y we must have $(I - AA^{m})B =$ 0. Now, $f(x^{*}, y)$ has a minimum if and only if

$A \geq 0$, $(I - AA^{m})B = 0$, $C - B^{\sim}A^{m}B \geq 0$. Therefore we established that $f(x, y)$ has a minimum over all x, y if and only if $A \geq 0$, $(I - AA^{m})B =$ 0, $C - B^{\sim}A^{m}B \geq 0$. A similar reasoning applies if we first minimize with respect to y and then with respect to x, but this time, the Schur complement

$A^{\sim} - C^{\sim}G_1(G_1B^{\sim})^{m}(C^{\sim}G_1)^{\sim}$, of $-G_1B^{\sim}$ is involved. Putting all these facts together we get our main result.

Theorem 5.3. *Given any symmetric matrix*

$$M^{\sim} = \begin{bmatrix} A^{\sim} & -C^{\sim}G_1 \\ -G_1C & -G_1B^{\sim} \end{bmatrix},$$

the

following conditions are equivalent. (*i*) $M^{\sim} \geq 0$ ($M^{\sim}$ *is positive semidefinite.*)

(ii) $A^{\sim} \geq 0, (I - A^{\sim}(A^{\sim})^{m})(-C^{\sim}G_1) = 0, \quad -G_1B^{\sim} - G_1C(A^{\sim})^{m}C^{\sim}G_1 \geq 0.$

(iii) $-G_1B^{\sim} \geq 0, \ (I - G_1B^{\sim}BG_1)(-G_1C)^{*} = 0$

Proof : If $M^{\sim} \geq 0$, then by Theorem 4.1 and 4.2 it is clear that the above conditions are equivalent (using the fact $A^{m}AA^{m} = A^{m}$ and $C^{m}CC^{m} = C^{m}$).

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Example 5.4.

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -i \\ 0 & -i & 2 \end{bmatrix}$$

be the partitioned matrix of the form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A and D are square and nonsingular matrices.Here $M = M^{\sim}$ then we say that the matrix M is m-symmetric.

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