NON HOMOGENEOUS CUBIC EQUATION WITH THREE UNKNOWNS $6(x^2+y^2)-11xy+3(x+y)+9=192z^3$

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Abstract

The non-homogeneous cubic equation with three unknowns represented by $6(x^2+y^2)-11xy+3(x+y)+9=192z^3$ is analyzed for finding its non-zero distinct integral solutions. Five different methods have been presented for determining the integral solutions of the non-homogeneous cubic equation under consideration. Employing the integral solutions of the above equation, a few interesting relations between special numbers are exhibited.

Keywords: Ternary, Cubic, Non-homogeneous, Integral solutions.

1. Introduction

Integral solutions for the cubic homogeneous or non-homogeneous Diophantine equations is an interesting concept, as it can be seen from [1,2,3]. In [4-23], a few special cases of ternary cubic Diophantine equations are analyzed. In this communication, we present the integral solutions of yet another cubic equation $6(x^2+y^2)-11xy+3(x+y)+9=192z^3$ along with a few interesting properties.

Notations

$t_{m,n}=n\left(1+\frac{(n-1)(m-2)}{2}\right)$

$P_n^m=\left(\frac{n(n+1)}{6}\right)[(m-2)n+(5-m)]$

$SO_n=n(2n^2-1)$

$s_n=6n(n-1)+1$

$Pr_n=n(n+1)$

$Ky_n=2^{2n}+2^{n+1}-1$

$Gno_n=2n-1$

$CP_n^6=n^3$

$PP_n=\frac{1}{2}n^2(n+1)$

$T_n=\frac{1}{2}n(n+1)$

$CS_n=n^2+(n-1)^2$

$CP_a^3=\frac{n(n^2+1)}{2}$

$HXP_a=\frac{1}{6}n(n+1)(4n-1)$

2. Method of analysis

Consider the non-homogeneous ternary cubic Diophantine equation

$$6(x^2+y^2)-11xy+3(x+y)+9=192z^3 \quad (1)$$

Introducing the linear transformations

$$x=u+v, y=u-v \quad (2)$$

In (1), it is written as

$$(u+3)^2+23v^2=192z^3 \quad (3)$$

We employ different ways of solving (3) and thus, different pattern of integer solutions to (1) are illustrated below:

2.1 Pattern I:

Write 192 as

$$192 = \left(13 + i\sqrt{23}\right)\left(13 - i\sqrt{23}\right) \quad (4)$$

Assume, $z = a^2 + 23b^2$ (5)

Where a and b are non-zero distinct integers.

Using (4) and (5) in (3) and employing the method of factorization, define

$$(u+3) + i\sqrt{23}v = (13 + i\sqrt{23})(a + i\sqrt{23}b)^3$$

Equating the real and imaginary parts of the above equation, we get,

$$u + 3 = 13a^3 - 69a^2b - 897ab^2 + 529b^3$$
$$v = a^3 + 39a^2b - 69ab^2 - 299b^3$$

Substituting the values of u, v in (2), we have

$$x(a,b) = 14a^3 - 30a^2b - 966ab^2 + 230b^3 - 3$$
$$y(a,b) = 12a^3 - 108a^2b - 828ab^2 + 828b^3 - 3 \quad (6)$$

Thus (5) and (6) represent the non-zero distinct integral solution of (1).

2.1.1 Properties

1. $x(1,b) + y(1,b) + z(1,b) - 529SO_b + 1771Pr_b \equiv 21(mod\ 2162)$
2. $y(a,1) + x(a,1) + 2P_a^5 + 138Pr_a + 3312t_{3,a} - 1657t_{4,a} - 1052$ is a cubical integer.
3. Each of the following expression represent the nasty number.
   (i). $12x(1,b) - 14y(1,b) + 1152SO_b + 13056P_b^5 - 6528t_{4,b}$
   (ii). $y(1,b) - 3z(1,b) - 414SO_b - 306Pr_b + 1203t_{4,b}$
   (iii). $x(a,1) - y(a,1) - SO_a + 137Pr_a - 137t_{4,a} + 598$
4. $y(1,b) - z(1,b) - 414SO_b - 306Pr_b + 1157t_{4,b} = 8$
5. $z(1,2^n) - 22Ky_n + 461$ is a perfect square.

2.1.2 Note:

In (4), 192 may also be written as

$$192 = \left(-13 + i\sqrt{23}\right)\left(-13 - i\sqrt{23}\right) \quad (7)$$

For this case the corresponding integer solutions are given by,

$$x(a,b) = -12a^3 - 108a^2b + 828ab^2 + 828b^3 - 3$$
$$y(a,b) = -14a^3 - 30a^2b + 966ab^2 + 230b^3 - 3$$
$$z(a,b) = a^2 + 23b^2$$

2.2 Pattern II:

Instead of (4) write 192 as

$$192 = (10 + i2\sqrt{23})(10 - i2\sqrt{23}) \quad (8)$$

Substituting (8) in (3) and proceeding as in pattern I, the non-zero distinct integral solutions to (1) are given by,

$$x(a,b) = 12a^3 - 108a^2b + 828ab^2 + 828b^3 - 3$$
$$y(a,b) = 8a^3 - 168a^2b - 552ab^2 + 1288b^3 - 3$$
$$z(a,b) = a^2 + 23b^2$$

2.2.1 Properties

1. $x(a,1) + y(a,1) - 40P_a^5 + 2760t_{3,a} - 1084t_{4,a} = 2110$
2. $x(1,b) - y(1,b) + 920P_b^5 - 124t_{4,b} - 60Pr_b$ is a perfect square.
3. $x(a,1) + y(a,1) + 276Pr_a + 6SO_a + 555Gno_a - 1555$ is a cubical integer.
4. $x(a,1) + y(a,1) + 138CS_a + 823Gno_a - 10SO_a = 1425$
5. $z(a,1) - y(a,1) + 4SO_a - 274Gno_a + 988$ is a perfect square.

2.2.2 Note:

In (8), 192 may also be consider as

$$192 = \left(-10 + i2\sqrt{23}\right)\left(-10 - i2\sqrt{23}\right) \quad (9)$$

For this case the corresponding integer solutions are given by,

$$x(a,b) = -8a^3 - 168a^2b + 552ab^2 + 1288b^3 - 3$$
$$y(a,b) = -12a^3 - 108a^2b + 828ab^2 + 828b^3 - 3$$
$$z(a,b) = a^2 + 23b^2$$

2.3 Pattern III:

Write (3) as,

$$(u+3)^2 + 23v^2 = 192 * z^3 * 1 \quad (10)$$

1 can be written as

$$1 = \frac{(11 + i\sqrt{23})(11 - i\sqrt{23})}{(12)^2} \quad (11)$$

Using (4), (5) and (11) in (10) and proceeding as above, the non-zero distinct integer solution to (1) are obtained as

$$x(a,b) = 144(144a^3 - 1296a^2b - 9936ab^2 + 9576b^3) - 3$$
$$y(a,b) = 144(96a^3 - 2016a^2b - 6624ab^2 + 15456b^3) - 3$$
$$z(a,b) = 144(a^2 + 23b^2)$$

2.3.1 Properties

1. $x(a,1) - y(a,1) - 3456SO_a + 4t_{4,a} + 236736Gno_a + 1083456$ is a perfect square.
2. $x(a,1) - y(a,1) - 13824PP_a - 96768t_{4,a} + 238464Gno_a + 1085184 = 0$
3. $x(1,b) + 3312z(1,b) - 689472SO_b + 502848t_{2,b} = 497661$
4. $y(a,1) + 2016z(a,1) - 27648CP_a^3 + 483840Gno_a = 2612733$
5. $x(a,1) + y(a,1) + 476928Pr_a + 2384640t_{2,a} - 34560CP_a^6 = 3604602$

2.3.2 Note:

In (11), 1 may also be considered as

$$1 = \frac{(-11+i\sqrt{23})(-11-i\sqrt{23})}{(12)^2} \qquad (12)$$

For this case, the corresponding integer solutions are given by,

$$x(a,b) = 144(-164a^3 - 636a^2b + 11316ab^2 + 4516b^3) - 3$$
$$y(a,b) = 144(-168a^3 + 360a^2b + 11592ab^2 - 2760b^3) - 3$$
$$z(a,b) = 144(a^2 + 23b^2)$$

2.4 Pattern IV:

1 may be written as

$$1 = \frac{(7+i3\sqrt{23})(7-i3\sqrt{23})}{(16)^2} \qquad (13)$$

Using (4), (5) and (13) in (10) the non-zero distinct integer solution of (1) are given by

$$x(a,b) = 256(68a^3 - 3108a^2b - 4692ab^2 + 23828b^3) - 3$$
$$y(a,b) = 256(-24a^3 - 3240a^2b + 1656ab^2 + 24840b^3) - 3$$
$$z(a,b) = 256(a^2 + 23b^2)$$

2.4.1 Properties

1. $z(a,1) - T_{108} - 2$ is a perfect square.
2. $x(a,1) - y(a,1) - 47104PP_a - 10240Pr_a + 817664Gno_a + Tha_{19} = 496127$
3. $x(a,1) + 84SO_a + 795648Pr_a + 202794Gno_a - T_{3434} + 724$ is a cubical integer.
4. $z(a,1) + 5888Gno_a - 11776t_{2,a}$ is a perfect square.

2.4.2 Note:

In (13), 1 may also be considered as

$$1 = \frac{(-7+i3\sqrt{23})(-7-i3\sqrt{23})}{(16)^2} \qquad (14)$$

For this case, the corresponding non-zero integer solutions are given by,

$$x(a,b) = 256(-128a^3 - 2688a^2b + 8832ab^2 + 20608b^3) - 3$$
$$y(a,b) = 256(-192a^3 - 1728a^2b + 13248ab^2 + 13248b^3) - 3$$
$$z(a,b) = 256(a^2 + 23b^2)$$

2.5. Pattern V:

192 may be taken as

$$192 = \left(-13 + i\sqrt{23}\right)\left(-13 - i\sqrt{23}\right) \qquad (15)$$

Using (15), (5) and (11) in (10) the corresponding non-zero integer solutions of (1) are given by

$$x(a,b) = 144(-168a^3 - 360a^2b + 11592ab^2 + 2760b^3) - 3$$
$$y(a,b) = 144(-164a^3 + 636a^2b + 11316ab^2 - 4876b^3) - 3$$
$$z(a,b) = 144(a^2 + 23b^2)$$

2.5.1 Properties

1. $z(a,1) - T_{81} + 9$ is a perfect square.
2. $x(1,1) + y(1,1) + OH_2$ is a cubical integer.
3. $y(a,1) - x(a,1) - 1728sqP_a - 71280t_{6,a} - 15624Gno_a + 1083960 = 0$
4. $864z(1,1) - x(1,1) - y(1,1)$ is a nasty number.
5. $x(a,1) - y(a,1) + z(a,1) + 864HXP_a + 71424Hex_a + 15912Gno_a = 1086984$

3. Generalization

1 can be written as ,

$$1 = \frac{[(2n^2-2n-11)+i\sqrt{23}(2n-1)][(2n^2-2n-1)-i\sqrt{23}(2n-1)]}{(2n^2-2n+12)^2} \qquad (16)$$

Using (4), (5) and (16) in (10) and proceeding as above, the non-zero distinct integer solutions to (1) are obtained as,

$$x(a,b) = (2n^2 - 2n + 12)^2[(2n^2 - 2n - 11)(14a^3 - 30a^2b - 966ab^2 + 230b^3) - (2n - 1)(10a^3 + 966a^2b - 690ab^2 - 7406b^3)] - 3$$

$$y(a,b) = (2n^2 - 2n + 12)^2[(2n^2 - 2n - 11)(12a^3 - 108a^2b - 828ab^2 + 828b^3) - (2n - 1)(36a^3 + 828a^2b - 2484ab^2 - 6348b^3)] - 3$$

$$z(a,b) = (2n^2 - 2n + 12)^2[a^2 + 23b^2]$$

4. conclusion

As the cubic equations are rich in variety, one may search for integer solutions to other choices of ternary cubic equations as well as cubic equations with multi variables (>3).

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