NEW LABELINGS ON CYCLES WITH PARALLEL P_3 CHORDS

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Abstract**:** One of the growing research area in Graph theory is graph labeling which is defined by assigning non negative integer values to the vertices or edges of a graph subject to certain conditions. Labeling finds its application in social psychology, astronomy, electrical circuit theory, communication network, channel assignment problems and X-ray crystallography. In this paper different labelings are applied on cycles with parallel P_3 chords. It is proved that cycles with parallel P_3 chords admit square difference labeling, cube sum labeling, cube difference labeling and absolute difference of cube sum and square sum labeling.

Keywords**:** Square difference labeling, Cube sum labeling, Cube difference labeling, absolute difference of cube sum and square sum labeling.

AMS Subject Classification: 05C78

1.INTRODUCTION

In 1967 Rosa [10] introduced the notion of graph labeling. An extensive survey on graph labeling can be seen in Gallian [4]. In the past five decades researchers have introduced many labeling techniques. Shiama introduced square difference labeling and proved that P_n, C_n, cycle cactus, wheels, comb, star graphs, crown, dragon, quadrilateral snakes admit square difference labeling [12]. Some results on square difference labeling can also be seen in [1], [12], [13], [15], [18]. Cube sum labeling was also studied by researchers in the literature of graph labeling [16]. The notion of cube difference labeling was also introduced by Shiama and proved in [14] that paths, cycles, stars, fan graphs, crown graphs, coconut trees and shell graphs admit cube difference labeling. Cube difference labeling of various classes of graphs are discussed in [2], [3] and [19]. Based on the definition of square sum labeling and cube sum labeling, Mathew Varkey T.K. et al [5] introduced absolute difference of cube sum and square sum labeling abbreviated as ADCSS labeling and proved that banana tree, coconut tree and bamboo tree admit ADCSS labeling. Many families of graphs that admit ADCSS labeling are found in [5], [6], [7], [8] and [9]. The main aim of this paper is to show that cycles with parallel P_3 chords admit these labelings. Labelings on cycles with parallel chords and parallel P_3 chords are discussed in [17], [20] and [21]. The basic definitions needed for this work are given below.

Definition 1.1: [1]

Let G be a graph with p vertices and q edges. G is said to be a square difference graph if there is a bijection $f : V(G) \rightarrow \{0,1,...,p\text{-}1\}$ such that the induced mapping $f^* : E(G) \rightarrow Z^+$ given by $f^*(uv) = |\,[f(u)]^2 - [f(v)]^2\,|$ for every edge $uv \in E(G)$ are all distinct.

Definition 1.2: [16]

Let G be a graph with p vertices and q edges. G *is* said to be a cube sum graph if there is a bijective mapping $f : V(G) \rightarrow \{0,1,...,p\text{-}1\}$ such that the induced mapping $f^* : E(G) \rightarrow Z^+$ defined by $f^*(uv) = [f(u)]^3 + [f(v)]^3$ for every edge $uv \in E(G)$ are all distinct.

Definition 1.3: [3]

Let G be a graph with p vertices and q edges. G is said to be a cube difference graph if there is a bijection $f : V(G) \rightarrow \{0,1, ..., p\text{-}1\}$ such that the induced mapping $f^*: E(G) \rightarrow Z^+$ given by $f^*(uv) = |[f(u)]^3 - [f(v)]^3|$ for every edge $uv \in E(G)$ are all distinct.

Definition 1.4: [10]

A graph G with p vertices is said to admit absolute difference of cubic and square sum labeling-ADCSS labeling if there is a bijective mapping $f : V(G) \rightarrow \{1,2, ..., p\}$ such that the induced mapping $f^*: E(G) \rightarrow 2Z^+$ defined by $f^*(uv) = |[f(u)]^3 + [f(v)]^3 - ([f(u)]^2 + [f(v)]^2)|$ is injective.

Definition 1.5: [11]

A cycle with parallel P_3 chords is a graph G obtained from the cycle C_n: $u_0u_1u_2\ldots..u_{n-1}u_0$ by adding disjoint paths P_3's (chords) between the pair of vertices $u_1u_{n-1}, u_2u_{n-2},\ldots u_\alpha u_\beta$ of C_n where $\alpha = \lfloor\frac{n}{2}\rfloor$ -1, $\beta = \lfloor\frac{n}{2}\rfloor + 2$ if n is odd (or) $\beta = \lfloor\frac{n}{2}\rfloor + 1$ if n is even. Then $|V(G)| = N = \frac{3n-2}{2}$ and $|E(G)| = M = 2n - 2$ if n is even and $N = \frac{3n-3}{2}$ and $M = 2n - 3$ if n is odd as shown in Fig.1a and Fig.1b.

Note that throughout this paper the word 'chord' is used to denote a path connecting two non-adjacent vertices of the cycle. C_n denotes a cycle of length n and P_k is a path of order k.

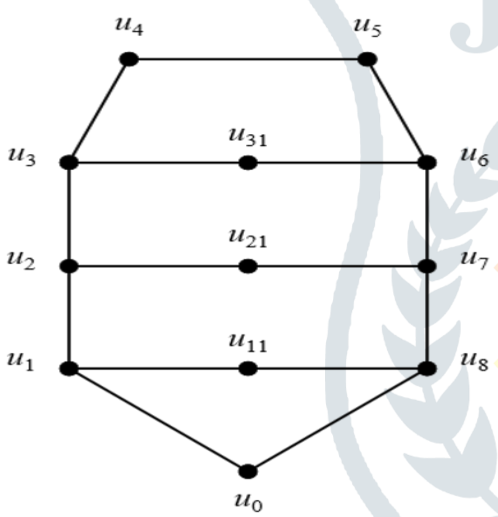

Fig. 1a Cycle C_9 with Parallel P_3 chords

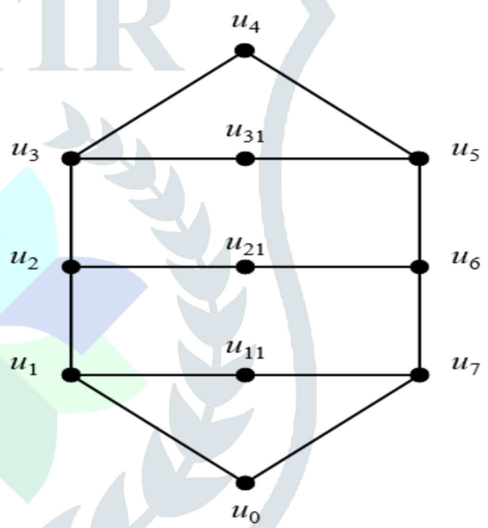

Fig. 1b Cycle C_8 with Parallel P_3 chords

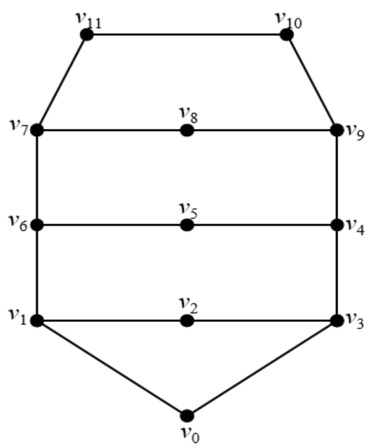

Fig. 2a: Cycle C_9 with Parallel P_3 chords having Hamiltonian path starting at v_0 and ending at v_{11}

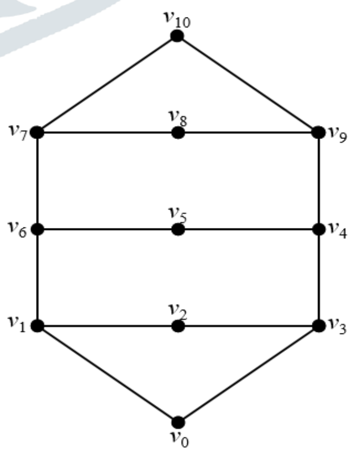

Fig. 2b: Cycle C_8 with Parallel P_3 chords having Hamiltonian path starting at v_0 and ending at v_{10}

2. MAIN RESULTS

Theorem 2.1: *Every cycle C_n ($n \geq 6$) with parallel P_3 chords is a square difference graph.*

Proof: Consider G as a cycle C_n with parallel P_3 chords having vertices $v_0, v_1, v_2 \dots, v_{N-1}$ where $N = |V(G)| = \frac{3n-2}{2}$ if n is even and $\frac{3n-3}{2}$ if n is odd. $v_0 v_1 v_2 \dots v_{N-1}$ be the Hamiltonian path of G with $v_0 = u_0$ and $v_{N-1} = u_{\lfloor \frac{n}{2} \rfloor}$ of C_n considered in definition 1.5 which is shown in fig. 2a and fig. 2b. The vertex labeling and the edge labeling for the two cases depending on n are given in case 1 and case 2 respectively.

Case 1: Let n be even.

Define a bijection $f: V(G) \to \{0,1,2,\dots N\text{-}1\}$as follows:

$$f(v_{2i}) = i, 0 \le i \le \frac{N-1}{2} \; if\; n \equiv 0 \; mod\; 4 \; and \; 0 \le i \le \frac{N-2}{2} \; if \; n \equiv 2 \; mod \; 4$$

$$f(v_{2i+1}) = \frac{3n-2i-4}{2}, 0 \le i \le \frac{N-3}{2} \; if \; n \equiv 0 \; mod \; 4 \; and \; 0 \le i \le \frac{N-2}{2} \; if \; n \equiv 2 \; mod \; 4$$

Edge set $E(G)$ is given by $E(G) = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$ where

$E_1 = \{v_{2i}v_{2i+1}, 0 \le i \le \left\lfloor \frac{N-3}{2} \right\rfloor \; if \; n \; \equiv 0 \; mod \; 4 \; and \; 0 \le i \le \left\lfloor \frac{N-2}{2} \right\rfloor \; if \; n \; \equiv 2 \; mod \; 4\}$

$E_2 = \{v_{2i-1}v_{2i}, 1 \le i \le \left\lfloor \frac{N-1}{2} \right\rfloor \; if \; n \; \equiv 0 \; mod \; 4 \; and \; 1 \le i \le \left\lfloor \frac{N-2}{2} \right\rfloor \; if \; n \; \equiv 2 \; mod \; 4\}$ along the Hamiltonian path and

$E_3 = \{v_{6i-5}v_{6i}, 1 \le i \le \left\lceil \frac{N-2}{6} \right\rceil\}$

$E_4 = \left\{v_{6i-2}v_{6i+3}, 1 \le i \le \left\lfloor \frac{N-4}{6} \right\rfloor\right\}$ and

$E_5 = \{v_0v_3, v_{N-4}v_{N-1}\}$ along the non-Hamiltonian path.

Define the injective function $f^*: E(G) \to Z^+$ as follows:

$$f^*(v_{2i}\, v_{2i+1}) = \frac{3n-4}{4}(3n-4-4i), 0 \le \frac{N-3}{2} \; if \; n \equiv 0 \; mod \; 4 \; and \; 0 \le i \le \frac{N-2}{2} \; if \; n \equiv 2 \; mod \; 4$$

$$f^*(v_{2i-1}\, v_{2i}) = \frac{3n-2}{4}(3n-2-4i), 1 \le \frac{N-1}{2} \; if \; n \equiv 0 \; mod \; 4 \; and \; 1 \le i \le \frac{N-2}{2} \; if \; n \equiv 2 \; mod \; 4$$

$$f^*(v_{6i-5}v_{6i}) = \frac{3n+2}{4}(3n-12i+2), 1 \le i \le \left\lceil \frac{N-2}{6} \right\rceil$$

$$f^*(v_{6i-2}v_{6i+3}) = \frac{3n-2}{4}(3n-12i-10) + 3(6i+1), 1 \le i \le \left\lfloor \frac{N-4}{6} \right\rfloor$$

$$f^*(v_0v_3) = (N-2)^2$$

$$f^*(v_{N-4}v_{N-1}) = \begin{cases} 3n, if \; n \equiv 0 \; mod \; 4 \\ 3n-6, if \; n \equiv 2 mod \; 4 \end{cases}$$

All the edge labels are distinct and G is a square difference graph.

Case 2: Let n be odd

Define a bijection $f: V(G) \to \{0,1,2,\dots N\text{-}1\}$as follows:

$$f(v_{2i}) = i, 0 \le \frac{N-2}{2} \; if \; n \equiv 1 \; mod \; 4 \; and \; 0 \le i \le \frac{N-1}{2} \; if \; n \equiv 3 \; mod \; 4$$

$$f(v_{2i+1}) = \frac{3n-2i-5}{2}, 0 \le \frac{N-2}{2} \; if \; n \equiv 1 \; mod \; 4 \; and \; 0 \le i \le \frac{N-3}{2} \; if \; n \equiv 3 \; mod \; 4$$

Edge set $E(G)$ is given by $E(G) = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$ where

$E_1 = \{v_{2i}v_{2i+1}, 0 \le i \le \left\lfloor \frac{N-2}{2} \right\rfloor \; if \; n \; \equiv 1 \; mod \; 4 \; and \; 0 \le i \le \left\lfloor \frac{N-3}{2} \right\rfloor \; if \; n \; \equiv 3 \; mod \; 4\}$

$E_2 = \{v_{2i-1}v_{2i}, 1 \le i \le \left\lfloor \frac{N-2}{2} \right\rfloor \; if \; n \; \equiv 1 \; mod \; 4 \; and \; 1 \le i \le \left\lfloor \frac{N-1}{2} \right\rfloor \; if \; n \; \equiv 3 \; mod \; 4\}$ along the Hamiltonian path and

$E_3 = \{v_{6i-5}v_{6i}, 1 \le i \le \left\lceil \frac{N-2}{6} \right\rceil\}$

$E_4 = \left\{v_{6i-2}v_{6i+3}, 1 \le i \le \left\lfloor \frac{N-4}{6} \right\rfloor\right\}$ and

$E_5 = \{v_0v_3, v_{N-5}v_{N-1}\}$ along the non-Hamiltonian path.

Define the induced function $f^*: E(G) \to Z^+$ as follows:

$$f^*(v_{2i}\, v_{2i+1}) = \frac{3n-5}{4}(3n-5-4i), 0 \le \frac{N-2}{2} \; if \; n \equiv 1 \; mod \; 4 \; and \; 0 \le i \le \frac{N-3}{2} \; if \; n \equiv 3 \; mod \; 4$$

$$f^*(v_{2i-1}\, v_{2i}) = \frac{3n-3}{4}(3n-3-4i), 1 \le \frac{N-2}{2} \; if \; n \equiv 1 \; mod \; 4 \; and \; 1 \le i \le \frac{N-1}{2} \; if \; n \equiv 3 \; mod \; 4$$

$f^*(v_{6i-5}v_{6i}) = \frac{3n+1}{4}(3n-12i+1), 1 \le i \le \left\lfloor\frac{N-2}{6}\right\rfloor$

$f^*(v_{6i-2}v_{6i+3}) = \frac{3n-3}{4}(3n-12i-11)+3(6i+1), 1 \le i \le \left\lfloor\frac{N-4}{6}\right\rfloor$

$f^*(v_0v_3) = (N-2)^2$

$f^*(v_{N-5}v_{N-1}) = \begin{cases} 3n+1, if\ n \equiv 1\ mod\ 4 \\ 3n-9, if\ n \equiv 3\ mod\ 4 \end{cases}$

All the edge labels are distinct and G is a square difference graph.

Theorem 2.2: *Every cycle C_n $(n \ge 6)$ with parallel P_3 chords is a cube sum graph.*

Proof: Consider the graph G as a cycle C_n with parallel P_3 chords. The Hamiltonian path of G is $v_0\, v_1 v_2 \ldots v_{N-1}$ where $N = |V(G)|$ by definition 1.5. The vertex labeling and the induced edge labeling for the two cases depending on n are given below.

Define a bijection $f : V(G) \to \{0,1,2,\ldots N\text{-}1\}$ as follows:

$f(v_i) = i$, $0 \le i \le N\text{-}1$

The edge set $E(G)$ is given by

$E(G) = E_1 \cup E_2 \cup E_3$ where

$E_1 = \{v_i\, v_{i+1},\ 0 \le i \le N\text{-}2\}$ is the edge set along the Hamiltonian path.

$E_2 = \{v_{3i-2}\, v_{3i+3},\ 1 \le i \le \left\lfloor\frac{N}{4}\right\rfloor$ if n is even & $1 \le i \le \left\lfloor\frac{N-3}{4}\right\rfloor$ if n is odd$\}$ and $E_3 = \{v_0v_3,\ v_{N-4}\, v_{N-1}$ if n is even & $v_0v_3,\ v_{N-5}\, v_{N-1}$ if n is odd$\}$ are the edge sets that are not in the Hamiltonian path.

Define the injective function $f^* : E(G) \to Z^+$ as follows

$f^*(v_i\, v_{i+1}) = 2i^3 + 3i^2 + 3i + 1,\ 0 \le i \le N\text{-}2$

$f^*(v_{3i-2}\, v_{3i+3}) = 54i^3 + 27i^2 + 117i + 19,\ 1 \le i \le \left\lfloor\frac{N}{4}\right\rfloor$ if n is even and $1 \le i \le \left\lfloor\frac{N-3}{4}\right\rfloor$ if n is odd

$f^*(v_0\, v_3) = 27$

$f^*(v_{N-4}\, v_{N-1}) = [N\text{-}1]^3 + [N\text{-}4]^3$

$f^*(v_{N-5}\, v_{N-1}) = [N\text{-}1]^3 + [N\text{-}5]^3$

The eldge labels along the Hamiltonian path are in an increasing sequence and are distinct. Along the non-Hamiltonian path, the adjacent vertex of v_{3i-2} is v_{3i+3} whereas in the Hamiltonian path the adjacent vertices of v_{3i-2} is either v_{3i-1} or v_{3i-3}. When $i \ne j$, $f^*(v_i\, v_{i+1}) \ne f^*(v_{3j-2}\, v_{3j+3})$. Hence G is a cube sum graph.

Theorem 2.3: *Every cycle C_n $(n \ge 6)$ with parallel P_3 chords is a cube difference graph.*

Proof: Consider the graph G as a cycle C_n with parallel P_3 chords. The Hamiltonian path of G is $v_0\, v_1 v_2 \ldots v_{N-1}$ where $N = |V(G)|$ by definition 1.5. The vertex labeling and the induced edge labeling for the two cases depending on n are given below.

Define a bijection $f : V(G) \to \{0,1,2,\ldots N\text{-}1\}$ as follows:

$f(v_i) = i$, $0 \le i \le N\text{-}1$

The edge set $E(G)$ is given by

$E(G) = E_1 \cup E_2 \cup E_3$ where

$E_1 = \{v_i\, v_{i+1},\ 0 \le i \le N\text{-}2\}$ is the edge set along the Hamiltonian path.

$E_2 = \{v_{3i-2}\, v_{3i+3},\ 1 \le i \le \left\lfloor\frac{N}{4}\right\rfloor$ if n is even & $1 \le i \le \left\lfloor\frac{N-3}{4}\right\rfloor$ if n is odd$\}$ and

$E_3 = \{v_0v_3\ v_{N-4}\, v_{N-1}$ if n is even & $v_0v_3,\ v_{N-5}\, v_{N-1}$ if n is odd$\}$are the edge sets that are not in the Hamiltonian path.

Define the injective function $f^* : E(G) \to Z^+$ as follows:

$f^*(v_i\, v_{i+1}) = 3i^2 + 3i + 1,\ 0 \le i \le N\text{-}2$

$f^*(v_{3i-2}\, v_{3i+3}) = 135i^2 + 45i + 35,\ 1 \le i \le \left\lfloor\frac{N}{4}\right\rfloor$ if n is even and $1 \le i \le \left\lfloor\frac{N-3}{4}\right\rfloor$ if n is odd

$f^*(v_0\, v_3) = 27$

$f^*(v_{N-4}\, v_{N-1}) = [N-1]^3 - [N-4]^3$

$f^*(v_{N-5}\, v_{N-1}) = [N-1]^3 - [N-5]^3$

The eldge labels along the Hamiltonian path are odd and not a multiple of 5 whereas the edge labels along the non-Hamiltonian path are multiples of 5. Hence G is a cube sum graph.

Theorem 2.4: *Every cycle C_n ($n \geq 6$) with parallel P_3 chords admits ADCSS labeling.*

Proof: Consider the graph G as a cycle C_n with parallel P_3 chords having vertices $v_1, v_2, v_3, \ldots\ldots v_N$. The Hamiltonian path of G is $v_1\, v_2 v_3 \ldots v_N$ where $N = |V(G)|$ by definition 1.5. The vertex labeling and the induced edge labeling for the two cases depending on n are given below.

Define a bijection $f : V(G) \rightarrow \{1,2,\ldots N\}$ as follows:

$f(v_i) = i,\ 1 \leq i \leq N$

The edge set $E(G)$ is given by

$E(G) = E_1 \cup E_2 \cup E_3$ where

$E_1 = \{v_i\, v_{i+1},\ 1 \leq i \leq N-1\}$ is the edge set along the Hamiltonian path.

$E_2 = \{v_{3i-1}\, v_{3i+4},\ 1 \leq i \leq \left\lfloor \frac{N}{4} \right\rfloor$ if n is even & $1 \leq i \leq \left\lfloor \frac{N-3}{4} \right\rfloor$ if n is odd$\}$ and

$E_3 = \{v_1 v_4,\ v_N\, v_{N-3}$ if n is even & $v_1 v_4,\ v_{N-4}\, v_N$ if n is odd$\}$are the edge sets that are not in the Hamiltonian path.

Define the injective mapping $f^* : E(G) \rightarrow 2\,Z^+$ as follows

$f^*(v_i\, v_{i+1}) = 2i^3 + i^2 + i,\ 1 \leq i \leq N-1$

$f^*(v_{3i-1}\, v_{3i+4}) = 54i^3 + 63i^2 + 135i + 46,\ 1 \leq i \leq \left\lfloor \frac{N}{4} \right\rfloor$ if n is even and $1 \leq i \leq \left\lfloor \frac{N-3}{4} \right\rfloor$ if n is odd

$f^*(v_1\, v_4) = 48$

$f^*(v_{N-3}\, v_N) = [27n^3 - 153n^2 + 366n - 328] / 4$

$f^*(v_{N-4}\, v_N) = [27n^3 - 207n^2 + 669n - 809] / 4$

The eldge labels of E_1 along the Hamiltonian path are in an increasing sequence and distinct as the labels of the vertices are also in an increasing sequence and distinct. Along the non-Hamiltonian path the adjacent vertex of v_{3i-1} is v_{3i+4} whereas in the Hamiltonian path the adjacent vertices of v_{3i-1} is either v_{3i} or v_{3i-2}. Hence $f^*(v_i\, v_{i+1}) \neq f^*(v_{3j-1}\, v_{3j+4})$ when $i \neq j$. The graph under consideration admits ADCSS labeling.

CONCLUSION

In this paper it is proved that cycles C_n ($n \geq 6$) with parallel P_3 chords admit square difference labeling, cube sum labeling, cube difference labeling, ADCSS labeling. Our future research work is on extending this result for constructing new graph families by applying graph operations on cycles with parallel P_3 chords.

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