ACYCLIC GRAPHOIDAL COVERS OF ZERO DIVISOR GRAPH

V. RAJAKUMARAN[1] and N. SELVI[2]
[1]Part – Time Research Scholar, Bharathidasan University, Trichy, India.
Department of Mathematics, ADM College for Women, Nagappattinam – 611001, India.

Abstract:

S. Arumugam and J. Suresh Suseela introduced the concept of acyclic graphoidal covering of graphs[5]. S. Arumugam and C. Pakkiam [3] have determined the graphoidal covering number of several families of graphs. A complete survey of the various results in graphoidal covers and graphoidal graphs is given in [4]. In this paper, we study the properties of acyclic graphoidal coverings of zero divisor graph and we also obtain a characterization of acyclic graphoidal coverings of complete and complete bipartite zero divisor graphs.

Mathematics Subject Classification: 05C25, 05C69.

Key Words: Commutative Ring, Zero Divisor Graphs, Graphoidal Coverings, Acyclic Graphoidal Coverings.

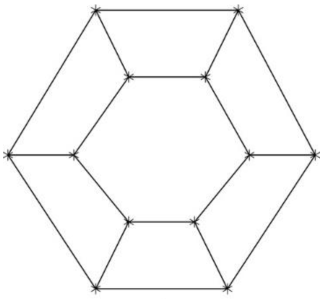
1. Introduction:

A related study of this parameter was initiated by B. D. Acharya and E. Sampathkumar [1], where the definition of graphoidal covering and the graphoidal covering number are being introduced. This concept further developed by Acharya. The later results of this concept were developed by S. Arumugam and C. Pakkiam [3]. S. Arumugam and J. Suresh Suseela introduced the concept of acyclic graphoidal covering of graphs [5]. Further while studying about graphoidal cover it will be more realistic if two cycles do not have same start and end nodes, since in traffic two mobile traffics that are forming a cycle cannot get the same junction point at a time. Algebraic Graph theory has close links with the study of Graph invariants. The Zero Divisor Graph is very useful to find the Algebraic structures and properties of Rings[9, 10]. This paves way to the investigation of the interplay between the ring theoretic properties of ring R and the Graph theoretic properties of certain Graphs obtained from R. In 1988, the idea of Zero Divisor Graph of a Commutative Ring was introduced by I. Beck [6]. Let R be a commutative ring with unity and let $Z(R)$ be its set of zero divisors. The zero divisor graph of R denoted by $\Gamma(R)$ is a graph which is undirected with vertices $Z^*(R) = Z(R) - \{0\}$, the set of non-zero divisors of R, and for distinct $y, z \in Z^*(R)$, the vertices y and z are adjacent if and only if $yz = 0$. D. F. Anderson and P. S. Livingston often focus on the case when R is finite, as these Rings yield finite Graphs [2].

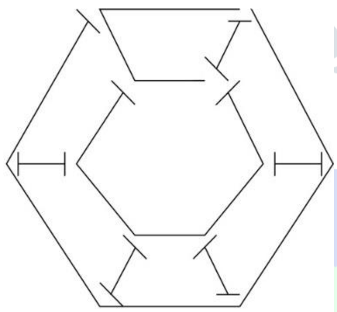
2. Acyclic Graphoidal Covers of Zero Divisor Graph

Let G = (V, E) be a graph. A graphoidal cover of G is a set ψ of (not necessarily open) paths in G satisfying the following conditions. (i) Every path in ψ has atleast two vertices. (ii) Every vertex of G is an internal vertex of atmost one path in ψ. (iii) Every edge of G is in exactly one path in ψ. The minimum cardinality of a graphoidal cover of G is called the graphoidal covering number of G and is denoted by η(G). A graphoidal cover ψ of a graph G is called an acyclic graphoidal cover if every member of ψ is a path. The minimum cardinality of an acyclic graphoidal cover of G is called the acyclic graphoidal covering number of G and is denoted by $\eta_a\,(G)$. A path partition or a path cover of a graph G is a collection P of paths in G such that every edge of G is in exactly one path in P. The path partition number π of G is also called the path covering number of G. A graph G is called a path cover graph if there exists a graph K and an acyclic graphoidal cover ψ of k such that G is isomorphic to $\Omega(\psi)$. That is G≅ $\Omega(\psi)$. Throughout this chapter, we assume that $\Gamma(Zn)$ is a connected graph without loops and multiple edges.

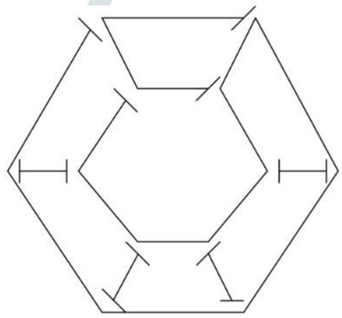
2.1 Example: consider the graph G,



G



G1



G2

Two acyclic graphoidal covers ψ_1 and ψ_2 of G with $|\psi_1|$=7 and $|\psi_2|$=5. ψ_1 is not a minimum acyclic graphoidal cover of G. In fact ψ_2 is a minimum acyclic graphoidal cover of G. Let β denote be set of all acyclic graphoidal covers in G. Since, E(G) is trivially an acyclic graphoidal cover of G, we have $\beta \neq \emptyset$.

$$\text{Let } \eta_a(G) = \min_{\psi \notin \beta} |\psi|.$$

Then $\eta_a(G)$ is called the acyclic grapoidal covering number of G. So, in the above graph, $\eta_a(G) = 5$. Clearly, $\eta(G) \leq \eta_a(G)$. Let ψ be a collection of internally disjoint paths in $\Gamma(Zn)$. A vertex of $\Gamma(Zn)$ is said to be in the interior of ψ if it is an internal vertex of some path in ψ. Any vertex which is not in the interior of ψ is said to be in the exterior of ψ.

2.1 Theorem: For any $\Gamma(Z_n)$, $\eta_a\big(\,\Gamma(Z_n)\big)$=$|E(\Gamma(Z_n)|$ if and only if $\Gamma(Z_n)$= $\Gamma(Z_9)$.

Proof: If $\Gamma(Z_n)$= $\Gamma(Z_9)$ trivially $\eta_a\big(\,\Gamma(Z_n)\big)$=$|E(\Gamma(Z_9)|$=1,

We know that $\Gamma(Z_9)$ is isomorphic to K_2. Suppose, $\Gamma(Z_n) \neq \Gamma(Z_9)$. Let P be a path in $\Gamma(Z_n)$ such that $|E(\Gamma(Z_9)|$ >1. Then, $\psi = \Big\{\{P\} \cup \big\{E\big(\Gamma(Z_n)\backslash E(P)\big)\big\}\Big\}$is a acyclic graphidal cover of $\Gamma(Z_n)$ and $|\psi| < \big|E\big(\Gamma(Z_n)\big)\big|$. Hence $\eta_a\Gamma(Z_n) <$ $\big|E\big(\Gamma(Z_n)\big)\big|$, which is a contradiction. Therefore, $\eta_a\Gamma(Z_n) = \big|E\big(\Gamma(Z_n)\big)\big|$ if and only if $\Gamma(Z_n) = \Gamma(Z_9)$. Hence proved.

2.2 Theorem: Let $\Gamma(Z_{2p})$ be a tree with p vertices, then $\eta_a\Gamma(Z_{2p}) = p-1$.

Proof: Since, $\Gamma(Z_{2p})$ contains no cycles. Therefore, $\eta_a\left(\Gamma(Z_{2p})\right) = \eta(\Gamma(Z_n)) = p-1$.

Theorem: *For any graph* $\Gamma(Z_n)$ *with* $\delta \geq 3, \eta_a = \mathrm{e} - \mathrm{p}$, where e is the number of edges, p is the number of vertices and δ is the minimum degree.

Proof: Let $P_1 = (v_1, v_2, \ldots v_n)$ be a longest path in $\Gamma(Z_n)$. So that all vertices adjacent to v_1 and v_n one already in P_1. Let v_i, v_j, v_k and v_s be the vertices on P_1 such that v_i and v_j are distinct, different from v_2 and are adjacent to v_1 and v_k ,v_s are distinct, different from v_{n-1} and are adjacent to v_n.

If v_1 and v_n are non-adjacent, Let P_2=(v_i,v_1,v_j) and P_3=(v_k,v_n,v_s). If v_1 and v_n are adjacent, we may assume that v_i=v_n and v_k=v_1.If $v_j \neq v_s$, Let P_4=(v_j,v_1,v_n,v_s). Suppose v_j=v_s , Now since, $\delta \geq 3$, there exist a vertex v such that v≠ $v_{j.}$ and v≠ v_{j+2} and v is adjacent to v_{j+1}. If v does not lie on P_1, then (v, v_{j+1},v_j,v_1,v_n,v_{n-1},…,v_{j+2}) is a path which is longer then p_1, so that v=v_m for some, m≠ j and m≠ $j+2$.

Now let,

$P_1' = (v_{j,}, v_{j-1}, \ldots, v_1, v_n, v_{n-1}, \ldots, v_{j+2}, v_{j+1})$

$P_2' = (v_1, v_j, v_n)$

$P_2' = (v_j, v_{j+1}, v_m)$

Thus, we get a collection ψ_1 of paths $\{P_1, P_2, P_3\}$ or $\{P_1, P_4\}$ or $\{P'_1, P'_2, P'_3\}$ such that all the paths in ψ_1 are edge disjoint and internally disjoint and all the vertices of P_1 are interior to ψ_1.

If $V(G) \neq \{v_1, v_2, \ldots, v_n\}$, we choose a longest path R_1 such that R_1 is edge disjoint and internally disjoint with all the path in ψ_1. If the end vertices of R_1 are not in P_1, we make them internal vertices of some path as before. By continuing this process, we obtain a collection of ψ of edges disjoint and internally disjoint path without exterior vertices in $\Gamma(Z_n)$. Hence we know that, for any graph $\Gamma(Z_n)$, $\eta_a = \mathrm{e} - \mathrm{p}$, then $\eta_a$is equivalent to e − p, where e is number of edges and p is number of vertices. Therefore, $\eta_a = \mathrm{e} - \mathrm{p}$.

2.3 Corollary: Let $\Gamma(Z_n)$ be any r-regular graph, then $\eta_a\left(\Gamma(Z_{2p})\right) = \begin{cases} 1 \text{ if } r=1 \\ 2 \text{ if } r=2 \\ \text{e-p if } r\geq 3 \end{cases}$, where e is the number of edges and p is the number of vertices.

Proof: If $r = 1$ then $\Gamma(Z_n) = \Gamma(Z_9) \simeq K_2$, so that $\eta_a = 1$.

If $r = 2$ then $\Gamma(Z_n)$ is either isomorphic with K_3 (or) C_n. But, there is no graph of $\Gamma(Z_n)$ is isomorphic to K_3 and C_n.

If $r = 3$ then $\Gamma(Z_n)$ is isomorphic to $\Gamma(Z_{25}) = K_4$. Then, $\eta_a = e - n$, where e is the number of edges and n is the number of vertices.

2.4 Remark: Using the above corollary, we get the following results $\eta_a\left(\Gamma(Z_{p^2})\right) = \eta_a(K_{p-1}) = \begin{cases} 1 \; if \; r = 3 \\ \frac{(p-1)(p-4)}{2} \; if \; p \geq 5, \end{cases}$ where p is prime number.

2.5 Theorem: For any graph $\Gamma(Z_n)$, $\eta_a(\Gamma(Z_n)) \geq \Delta - 1$, where Δ is maximum degree of $\Gamma(Z_n)$.

Proof: Let v be a vertex with maximum degree in $\Gamma(Z_n)$, and ψ be an acyclic graphoidal cover of $\Gamma(Z_n)$. Since, v is an internal vertex of at most one path in ψ it follows that $v$is terminal vertex of at least $\Delta - 2$ paths in ψ. Hence $|\psi| \geq \Delta - 1$ so that $\eta_a(\Gamma(Z_n)) \geq \Delta - 1$.

2.6 Observation:

Based on the above theorems, we conclude the following observations,

i). $\eta_a(\Gamma(Z_9)) = 1$.

ii). $\eta_a\left(\Gamma(Z_{2p})\right) = p - 2$.

2.7 Theorem: For any graph $\eta_a\left(\Gamma(Z_{pq})\right)$, where p and q are distinct prime number, then the following conditions are holds.

i). $\eta_a\left(\Gamma(Z_{3p})\right) = 1$.

ii). $\eta_a\left(\Gamma(Z_{pq})\right) = (p-1)(q-1) - (p-1)(q-1)$.

Proof: We prove (i) by the method of induction on number of vertices. Let $V_1 = \{v_1, v_2\}$

And $V_2 = \{u_1, u_2, u_3, u_4\}$ be a bipartition of $K_{2,4}$, where $q = 5$. Then, $\{(v_1, u_1, v_2, u_2)(u_2, v_1, u_3, v_2)(v_2, u_4, v_1, u_3)\}$ is an acyclic graphoidal cover of $\Gamma(Z_{15})$, which is isomorphic to $K_{2,4}$. Hence, $\eta_o(\Gamma(Z_{15})) \leq 3$. Also, we know that for any graph $\Gamma(Z_n)$, $\eta_a(\Gamma(Z_n)) \geq \Delta - 1$, where Δ is the maximum degree in $\Gamma(Z_n)$. Therefore, $\eta_o(\Gamma(Z_{15})) \leq 3$ and $\eta_o(\Gamma(Z_{15})) \geq 3$,

which implies that $\eta_o(\Gamma(Z_{15})) = 3$. We now assume that the result is true for $\Gamma(Z_{3p})$, where $p > 3$.

Let $V_1 = \{v_1, v_2\}$ and $V_2 = \{u_1, u_2, u_3, \ldots, u_{p-1}\}$ be a bipartition of $\Gamma(Z_{3p})$.

By induction hypothesis, $\eta_a(\Gamma(Z_{3p}) - u_{p-1}) = p - 3$. Let ψ_1 be a minimum acyclic graphoidal cover of $\Gamma(Z_{3p}) - u_{p-1}$, then $\psi_1 \cup \{x_1, y_n, x_2\}$ is acyclic graphoidal cover of $\Gamma(Z_{3p})$, so that $\eta_a\left(\Gamma(Z_{3p})\right) \leq p - 2$. Also we know that $\eta_a\left(\Gamma(Z_{3p})\right) \geq p - 2$ and hence, $\eta_a\left(\Gamma(Z_{3p})\right) = p - 2$.

ii). We know that for any graph $\Gamma(Z_n)$ with $\delta \geq 3$, then $\eta_a = e - n$, where e is the nu,ber of edges and n is the number of vertices. Clearly, we know that the zero divisor graph $\Gamma(Z_{pq})$ is isomorphic to $K_{p-1,q-1}$, Therefore, $\eta_a\left(\Gamma(Z_{pq})\right)$= no. of edges , no. of vertices.

$$= (p-1)(q-1) - [p-1+q-1]$$

$$= (p-1)(q-1) - (p-1) - (q-1)$$

3. Graphoidal Covering of Cartesian product of $\Gamma(Z_9)$ and P_n

For any two graphs G and H, $G \times H$ denotes the Cartesian product or simply the product of G and H. The following theorem is the Cartesian product of $\Gamma(Z_9)$ and P_n , where P_n is a path with n vertices.

3.1 Theorem: $\eta(\Gamma(Z_9) \times P_n) = \eta_a(\Gamma(Z_9) \times P_n)$ for all $n \geq 3$.

Proof: We know that $\Gamma(Z_9)$ is isomorphic to P_2. Let $P_2 = \{v_1, v_2\}$ and $P_n = \{u_1, u_2, u_3, \dots, u_n\}$ where n=3, let $w_{ij} = (v_i, u_j)$, $1 \leq i \leq 2; 1 \leq j \leq 3$. Then $\{(w_{12}, w_{11}, w_{21}, w_{22}, w_{33}, w_{13}), (w_{13}, w_{12}, w_{22})\}$ is a minimum acyclic graphoidal cover of $\Gamma(Z_9) \times P_3$ and $\Gamma(Z_9) \times P_3$ is not a acyclic and hence, $\eta(\Gamma(Z_9) \times P_3) = \eta_a(\Gamma(Z_9) \times P_3) = 2$. Let $n \geq 3$ and let $w_{ij} = (v_i, u_j)$, $1 \leq i \leq 2; 1 \leq j \leq n$, then $\{(w_{12}, w_{11}, w_{21}, w_{22}, \dots, w_{2n}, w_{1n}, w_{1(n-1)}, \dots, w_{13}), (w_{22}, w_{12}, w_{13}, w_{23})\}$ is a set of internally disjoint and edge disjoint paths without exterior points in $\Gamma(Z_q) \times P_n$ and hence, $\eta(\Gamma(Z_9) \times P_n) = \eta_a(\Gamma(Z_9) \times P_n) = q - p$.

3.2 Remark: i). Any two isomorphic acyclic graphoidal covers of G give rise to the same partition of q, but the converse is not true.

ii). A tree $\Gamma(Z_n)$ has unique minimum acyclic graphoidal cover if and only if there exists at most one vertex u with degree $u \geq 2$ and all the perdant vertices are at the same distance form v.

iii). There is no unicycle graph $\Gamma(Z_n)$ has a unique minimum acyclic graphoidal cover if and only if $\Gamma(Z_n) = C_3$.

3.3 Theorem: Let p denotes the number of vertices of degree Δ. If $\eta_a = \Delta - 1$, then, $p \leq \frac{2(\Delta-1)}{\Delta-2}$, where Δ *is maximum degree.*

Proof: Let ψ be a minimum acyclic graphoidal cover of $\Gamma(Z_n)$ so that $|\psi| = \Delta - 1$. Since each vertex of degree Δ appears as pm external vertex in a path of ψ at leat $\Delta - 2$ times and the total number of external vertices is at most $2(\Delta - 1)$, we have $p(\Delta - 2) \leq 2(\Delta - 1)$. Hence, $p \leq \frac{2(\Delta-1)}{(\Delta-2)}$.

The concept of path partition and path partition number π of a graph was introduced by F. Harary [7]. B. Peroche [8] obtained the path partition number π for the some multipartite graphs. Since every acyclic graphoidal cover is a path partitions, we have $\pi \leq \eta_a$.

3.4 Theorem: The path partition number of $\Gamma(Z_{p^2}) = \frac{p-1}{2}$, where p is any prime numbers.

Proof: Since, $\Gamma(Z_{p^2})$ is isomorphic to K_{p-1}, we know that , $\Gamma(Z_{p^2})$ has a decomposition into $\frac{p-1}{2}$ Hamilton paths and hence, $\pi(\Gamma(Z_{p^2})) \leq \frac{p-1}{2}$. Since, we know that the path partition number of $\Gamma(Z_{p^2}) \geq \frac{p-1}{2}$ and hence, $\pi\left(\Gamma(Z_{p^2})\right) = \frac{p-1}{2}$.

The concept of unrestricted path number π^* was introduced by B. Peroche [8]. We now proceed to determine the unrestricted path number of complete graphs and complete bipartite graphs of $\Gamma(Z_n)$.

3.5 Theorem: The unrestricted path number of the complete graph of $\Gamma(Z_n)$ is $\left[\frac{p-1}{2}\right]$, where p is a prime number.

Proof: Since, we know that the complete graphs of $\Gamma(Z_n)$ are $\Gamma\left(Z_{p^2}\right)$, where p is any prime number. Clearly, we know that $\Gamma\left(Z_{p^2}\right)$ is isomorphic to K_{p-1}, so, each path can cover at most $p-2$ edges, we have $\pi^*\left(\Gamma\left(Z_{p^2}\right)\right) \geq \left[\frac{p-1}{2}\right]$, we know that, the unrestricted path number of a graph is less than or equal to the path partition number of a graph. Therefore, $\pi^*\left(\Gamma\left(Z_{p^2}\right)\right) \leq \pi\left(\Gamma\left(Z_{p^2}\right)\right) = \left[\frac{p-1}{2}\right]$ and hence $\pi^*\left(\Gamma\left(Z_{p^2}\right)\right) = \left[\frac{p-1}{2}\right]$.

3.6 Theorem: The unrestricted path number of $\Gamma\left(Z_{pq}\right)$is $\pi^*\left(\Gamma\left(Z_{pq}\right)\right) = \left[\frac{(p-1)(q-1)}{2(q-1)-\delta(p-1,q-1)}\right]$, where p and q are distinct prime numbers and δ is the conversional kronecker delta function.

Proof: We know that $\Gamma\left(Z_{pq}\right)$ is a complete bipartite graph namely, $K_{p-1,\ q-1}$. That is, $\Gamma\left(Z_{pq}\right)$ is isomorphic to $KK_{p-1,\ q-1}$, where p and q are distinct prime numbers with $p > q$.

Since, $2(q-1)-\delta(p-1,q-1)$ is the length of a longest path in $\Gamma\left(Z_{pq}\right)$ and there are $(p-1)(q-1)$ edges to be covered, so, we have $\pi^*\left(\Gamma\left(Z_{pq}\right)\right) \geq \left[\frac{(p-1)(q-1)}{2(q-1)-\delta(p-1,q-1)}\right]$. Clearly the product $(p-1)(q-1)$ is even. So, $\pi^*\left(\Gamma\left(Z_{pq}\right)\right) \leq \pi\left(\Gamma\left(Z_{pq}\right)\right) = \left[\frac{(p-1)(q-1)}{2(q-1)-\delta(p-1,q-1)}\right]$. Hence, $\pi^*\left(\Gamma\left(Z_{pq}\right)\right) = \left[\frac{(p-1)(q-1)}{2(q-1)-\delta(p-1,q-1)}\right]$

3.7 Theorem: Let $\Gamma(Z_n)$ be a tree then there exists an acyclic graphoidal cover which is also a paring on odd vertices if and only if $\Delta \leq 3$, where Δ is the maximum degree of $\Gamma(Z_n)$.

Proof: If there exists an acyclic graphoidal cover in $\Gamma(Z_n)$ which is also a pairing on odd vertices then $\Delta \leq 3$. By the method of induction on r, where r is the number of end vertices of $\Gamma(Z_n)$. When $r = 2$, $\Gamma(Z_n)$ is a path and the result is trivial. Suppose the result is true for all the tree with $r = 1$ end vertices, where $r \geq 3$.

Let $\Gamma(Z_n)$ be a tree with r pendant vertices. Let x_0 be a pendant vertex of $\Gamma(Z_n)$. Let $P = (x_0, x_1, x_2, \dots, x_t)$ be a path of $\Gamma(Z_n)$ such that $deg\ x_1 = deg\ x_2 = \cdots = deg\ x_{t-1} = 2$ and $deg\ x_t = 3$. Deleting $x_0, x_1, x_2, \dots, x_{t-1}$ from $\Gamma(Z_n)$. We get a tree namely $\Gamma(Z_n)_1$ with $r-1$ pendant vertices and the degree of x_{t-1} in $\Gamma(Z_n)_1$ is 2. By induction hypothesis $\Gamma(Z_n)_1$ has an acyclic graphoidal cover ψ_1 which I a pairing on odd vertices of $\Gamma(Z_n)_1$. Now $\psi = \psi_1 \cup \{P\}$ is an acyclic graphoidal cover of $\Gamma(Z_n)$ which is a pairing an odd vertices of $\Gamma(Z_n)_1$. Hence proved the theorem.

3.8 Observation: If $\Gamma(Z_n)$ is a 3-regular graph, then every minimum acyclic graphoidal cover is a pairing on odd vertices of $\Gamma(Z_n)$. Let ψ be a minimum acyclic graphoidal cover of $\Gamma(Z_n)$. Since, $\Gamma(Z_n)$ is a 3-regular, then we know that from theorem, any graph $\Gamma(Z_n)$ with $\delta \geq 3$ then $\eta_a = e - p$. Therefore, $|\psi| = e - p = 3\left(\frac{p}{2}\right) - p = \frac{p}{2}$ and every vertex is interior to ψ. Hence every vertex appears as an external vertex of exactly one path in ψ so that ψ is a pairing on odd vertices of $\Gamma(Z_n)$.

4. References:

1. B. D. Acharya and E. Sampathkumar, Graphoidal covers and graphoidal covering number of a graph, Indian Journal of pure and applied mathematics, 18 (10), (1987), 882 – 890.
2. D. F. Anderson and P. S. Livingston, The zero-divisor graph of a commutative ring, J.Algebra,217, (1999), No-2, 434 - 447.
3. S. Arumugam and C. Pakkiam, The graphoidal covering number of unicyclic graphs, Indian journal of pure and applied mathematics, 23, (1992), 141 – 143.
4. S. Arumugam, B.D. Acharya and E. Sampathkumar, Graphoidal covers of a graph – A creative review in : Proceedings of the national workshop on graph theory and its applications, Manonmanium Sundaranar University, Tirunelveli, 1996, Eds S. Arumugam, B. D. Acharya and E. Sampathkumar, Tata McGraw Hill, New Delhi (1997), 1- 28.
5. S. Arumugam and J. Suresh Suseela, Acyclic graphoidal covers and path partitions in a graph, Discrete Mathematics, 190, (1998), 67 – 77.
6. I. Beck, Coloring of Commutative Rings, J. Algebra, 116, (1988),208 - 226.
7. F. Harary, Covering and packing in graphs – I, Ann. N. Y. Acad. Sci., 195 (1970), 198 – 205.
8. B. Peroche, The path number of some multipartite graphs, Annals of Discrete Mathematics, 23 (2), (1982), 193 – 197.
9. J. Ravi Sankar and S. Meena, Changing and unchanging the Domination Number of a commutative ring, International Journal of Algebra, 6, (2012), No-27, 1343 - 1352.
10. J. Ravi Sankar and S. Meena, Connected Domination Number of a commutative ring, International Journal of Mathematical Research, 5, (2012), No-1, 5 - 11