A BEST PROXIMITY POINT THEOREM FOR ALTERNATE CONVEXICALLY NONEXPANSIVE MAP

[1]S. Sujith, [2]P. S. Srinivasan

[1,2]Assistant Professor
[1,2] Department of Mathematics,
[1]St.Xavier's College, Palayamkottai, Tamil Nadu, India.
[2]Bharathidasan University, Tiruchirappalli, Tamil Nadu, India.

Abstract : Let X be a Banach Space and A,B be any two nonempty subsets of X. In this article, a new class of mapping, $T{:}A\cup B \rightarrow A\cup B$, satisfying the condition $T(A) \subset A$ and $T(B) \subset B$, called relatively k-alternate convexically nonexpansive map is introduced and proved that if X is strictly convex and A,B are any two nonempty weakly compact convex subset of X then there exists $x\in A$, $y\in B$ such that $Tx=x$, $Ty=y$ and $\|x-y\|=d(A,B)$, called fixed points and best proximity point respectively. If $A=B$, then our result proves the existence of fixed point of an alternate convexically nonexpansive map proved by Amini-Harandi and also proves the existence of fixed point of an k-alternate convexically nonexpansive map proved by Dowling.

AMS Subject Classification: 47H09, 47H10.
Key Words and Phrases: Best proximity Points, relatively k-alternate convexically nonexpansive map, strictly convex space.

I.Introduction

Let X be a nonempty normed linear space. A mapping T: $X\rightarrow X$ is said to be nonexpansive mapping if $\|Tx- Ty\|\leq \|x- y\|$, for all $x, y\in X$. It is easy to see that the translation map T: $\mathbb{R}\rightarrow\mathbb{R}$ defined by $Tx=x+a$, where a is a non-zero real number, is an example of nonexpansive mapping having no fixed points. In 1965, Kirk[7], Browder[4], Gohde[5] independently provided sufficient conditions for the existence of fixed point for nonexpansive mappings. Among these results, Kirk used a geometric notion called "normal structure" which was introduced by Brodskii and Milman [9]. A nonempty convex subset K is said to have normal structure if every nonempty bounded convex subset H of K containing more than one point must contains a point $x_0 \in H$, satisfying $\sup\{\|x- x_0\| : x\in H\}< \sup \{\|x-y\|: x,y \in H\}=\text{diam}(H)$. This point $x_0 \in H$ is called a non-diametral point of H. Kirk proved that every non-expansive self mapping on a weakly compact convex subset of K of a normed linear space has at least one fixed point provided K has normal structure. In [1], Alspach showed that the normal structure property is necessary for the existence of a fixed point for nonexpansive mapping, by providing an example of a fixed point free nonexpansive mapping on a weakly compact convex subset of $L^1[0,1]$. In [2], Amini-Harandi introduced a subclass of nonexpansive mappings called alternate convexically nonexpansive mapping and investigate the existence of fixed points in absence of normal structure property. The main result of [2], showed that every alternate convexically nonexpansive mapping on a weakly compact convex subset of a strictly convex Banach space has atleast one fixed point. Later, Dowling [6], introduced a weaker version of alternate convexically nonexpansive mapping called k-alternate convexically nonexpansive mapping and obtained the same result. On the other hand, in [3], Eldred et.al., introduced a class of mappings called relatively nonexpansive mapping which extend the notion of usual nonexpansive mapping. Let A, B be any two nonempty subsets of a normed linear space X. Then a mapping T: $A \cup B \rightarrow A \cup B$ is said to be relatively nonexpansive map if it satisfies (1) $T(A)\subset B$ and $T(B) \subset A$ and (2) $\|Tx- Ty\|\leq \|x- y\|$ for all $x\in A$ and $y\in B$. Note that if $A = B$, then the relatively nonexpansive mapping is nothing but the usual nonexpansive mapping. It is worth to mention that a relatively nonexpansive mapping need not be continuous, whereas the nonexpansive mappings are uniformly continuous. In [3], the authors established a notion called proximal normal structure which generalize the notion of normal structure. In [3], the authors considered a relatively nonexpansive mapping T: $A \cup B \rightarrow A \cup B$ where A,B are nonempty weakly compact convex subset of a normed linear space X. One of the main results in [3], showed that if the pair (A, B) has proximal normal structure property, then there exists a point $(x_0,y_0)\in A\times B$ satisfying $\|x_0 - Tx_0\| = \|Ty_0 -y_0\| = \inf \{\|x- y\| : x \in A, y\in B\} = d(A, B)$. The elements x_0, y_0 are said to be best proximity points of T in A and B respectively. In [12] Sankar Raj et.al, introduced a subclass of relatively nonexpansive mappings called relatively k- alternate convexically nonexpansive and provided sufficient conditions for the existence of best proximity points with out using proximal normal structure property. In [12], the map they have considered is cyclic, that is the map T: $A \cup B \rightarrow A \cup B$ satisfies the condition $T(A)\subset B$ and $T(B)\subset A$. In this article, we will consider the map T: $A \cup B \rightarrow A \cup B$ such that $T(A)\subset A$ and $T(B)\subset B$. Our result extends the result of Harandi and Dowling for noncontinuous k-alternate convexically nonexpansive mapping defined on a strictly convex Banach space X.

II.Preliminaries

In this section, we discuss some of the basic notations and terminologies which we will use in our main results. Let A, B be nonempty subsets of a normed linear space X. We denote

$A_0 =\{x \in A : \|x-y\|= d(A,B) \text{ for some } y \in B\}$ and
$B_0 =\{y \in B : \|x-y\|= d(A,B) \text{ for some } x \in A\}$,

In [8], Kirk et al. showed that A_0 and B_0 are nonempty weakly compact and convex subsets provided A and B are nonempty weakly compact and convex . For each $x \in X$ and $r > 0$; we define $B[x, r] := \{y \in X : \|x- y\| \leqslant r\}$. A Banach space X is said to be strictly convex if for each $x, y \in X$ with $\|x\| = \|y\| = 1$ and $x \neq y$ then $\left\|\frac{x+y}{2}\right\| < 1$.

Definition 1.[10]. A pair (A, B) of nonempty subsets of a normed linear space X is said to have P -property if and only if $\|x_1 - y_1\| = d(A, B)$ and $\|x_2 - y_2\| = d(A, B)$ implies $\|x_1 - x_2\| = \| y_1 - y_2\|$ whenever $x_1, x_2 \in A$ and $y_1, y_2 \in B$.

Definiton 2. [10]. A normed linear space X is said to have the P -property if and only if every pair (A,B) of nonempty and closed convex subsets of X has the P -property.

In [11], Anthony et.al. proved that a normed linear space X is strictly convex if and only if X has the P -property. In this article, we say that a map $T: A\cup B \to A\cup B$ is noncyclic if $T(A) \subset A$ and $T(B) \subset B$. Now we define a new class of noncyclic mapping $T: A\cup B \to A\cup B$ called relatively alternate convexically nonexpansive as follows:

Definition 3. Let A and B be two nonempty subset of a Banach space X. A noncyclic map $T: A\cup B \to A\cup B$ is called alternate convexically nonexpansive with respect to A (respectively with respect to B) if $\left\|\sum_{i=1}^{n} \frac{(-1)^{i+1}}{n} Tx_i - Ty\right\| \leq \left\|\sum_{i=1}^{n} \frac{(-1)^{i+1}}{n} x_i - y\right\|$ for each $n \in \mathbb{N}$, $x_i \in A$ and $y \in B$ (respectively for each $x_i \in B$ and $y \in A$).

Definition 4. Let A and B be two nonempty subset of a Banach space X and let $k \in \mathbb{N}$. A noncyclic map $T: A\cup B \to A\cup B$ is called k-alternate convexically nonexpansive with respect to A (respectively with respect to B) if $\left\|\sum_{i=1}^{n} \frac{(-1)^{i+1}}{n} Tx_i - Ty\right\| \leq \left\|\sum_{i=1}^{n} \frac{(-1)^{i+1}}{n} x_i - y\right\|$ for each $1 \leqslant n \leqslant k$, $x_i \in A$ and $y \in B$ (respectively for each $x_i \in B$ and $y \in A$).

Definition 5. A cyclic map $T: A\cup B \to A\cup B$ is called relatively alternate (respectively k-alternate) convexically nonexpansive if it is an alternate (k-alternate) convexically nonexpansive with respect to both A and B.

Remark 6. If $T: A\cup B \to A\cup B$ is a relatively alternate (or k-alternate, where $k \geqslant 2$) convexically nonexpansive then for $n = 2$ and for $x_1 = x_2$ we get $\|T x\| \leqslant \|x\|$, for all $x \in A$ and $\|T y\| \leqslant \|y\|$, for all $y \in B$. Hence if $0 \in A \cup B$ then clearly $0 \in A \cap B$ and it will be a fixed point of T.

III.Main Result

Theorem 7. *Let A and B be two nonempty weakly compact convex subsets of a strictly convex Banach space X. Let* $T: A \cup B \to A \cup B$ *be a relatively 2-alternate convexically nonexpansive. Then there exists an element* $x \in A$ *such that* $\|x- T x\| = d(A, B)$.

Proof: By Remark 6, if $0 \in A \cup B$ then clearly $0 \in A\cap B$ and it will be a fixed point of T. This completes the proof. Hence, we assume that $0 \notin A \cup B$. Let $d=\inf\{\|y\| : y \in A_0 \cup B_0\}$. Since $A_0 \cup B_0$ is weakly compact, there exists $y_0 \in A_0 \cup B_0$ such that $\|y_0\| = d > 0$. Let $R=\inf\{\delta>0: d(A_0 \cap B[0,\delta], B_0 \cap B[0,\delta]) = d(A,B)\}$. Since $A \cup B$ is bounded, R is nonempty and bounded below by $\|y_0\|$. Let $r = \inf\{\delta: \delta \in R\}$ then for each n, there exists $x_n \in A_0 \cap B[0, r+\frac{1}{n}]$ and $y_n \in B_0 \cap B[0, r + \frac{1}{n}]$ such that $\| x_n - y_n\| = d(A,B)$. Since A_0 and B_0 are weakly compact there exists weakly convergent subsequence $\{x_{n_k}\}$ and $\{y_{n_k}\}$ such that $\{x_{n_k}\}$ converges to x^* and $\{y_{n_k}\}$ converges to y^* weakly as $k \to \infty$. Then by weak lower semicontinuity of the norm, $\|x^*\| \leq \liminf \|x_{n_k}\|$ and $\|y^*\| \leq \liminf \|y_{n_k}\|$. That is $\|x^*\| \leq r$ and $\|y^*\| \leq r$. Also, $d(A,B) \leq d(A_0 \cap B[0,r], B_0 \cap B[0,r]) \leq \|x^*-y^*\| \leq \liminf \|x_{n_k} - y_{n_k}\| = d(A,B)$. Let us now complete the proof by showing that $Tx^*=x^*$ and $Ty^*=y^*$. By P-property it is enough to show that $Tx^* = x^*$ or $Ty^*=y^*$. Suppose not then consider the elements, $a=\frac{x*+Tx*}{2}$ and $b = \frac{y*+Ty*}{2}$. Since $\|x^*\| \leq r$ and $\|y^*\| \leq r$, by Remark 6 both $\|Tx^*\| \leq r$ and $\|Ty^*\| \leq r$. By strictly convex property of X, we get both $\|a\|$ and $\|b\|$ are strictly less than r. Let $s = \max\{\|a\|,\|b\|\}$ then $s<r$ and $\|a-b\|=d(A,B)$ which implies $d(A_0 \cap B[0,s], B_0 \cap B[0,s])=d(A,B)$. That is $s \in R$ but r is the infimum of the set R. This contradiction shows that either $Tx^*=x^*$ or $Tx^*=y^*$. Hence by P-property we get both $Tx^*=x^*$ and $Ty^*=y^*$.

The following fixed point theorem due to Amini-Harandi [2] for alternate convexically nonexpansive map and the fixed point theorem due to Dowling [6] for k-alternate convexically non-expansive can be obtained from Theorem 7. by taking $A=B$.

Corollary 10. Let C be a weakly compact convex subset of a strictly convex Banach space X. Then every alternate convexically nonexpansive map $T:C \to C$ has a fixed point.

Corollary 11. Let X be a Banach space which is strictly convex. Let C be a nonempty weakly compact convex subset of X. Then every 2-alternate convexically nonexpansive mapping $T:C \to C$ has a fixed point.

REFERENCES

[1] Alspach, D. A fixed point free nonexpansive mapping, *Proc. Amer. Math. Soc.*, 82 (1981), 423-424.

[2] Amini-Harandi, A. A fixed point result in strictly convex Ba-nach spaces, *Acta Math.Hungar.*, 105 (2004), 139-143.

[3] Anthony Eldred, A. Kirk, W.A. and Veeramani, P. Proximal nor-mal structure and relatively nonexpansive mappings , *Studia Mathematica.*, 171 3 (2005): 283-293.

[4] Felix E. Browder, Nonexpansive nonlinear operators in a Banach space, *Proc. Nat. Acad. Sci. U.S.A.*, 54 (1965), 10411044. MR MR0187120 (32 #4574)

[5] Dietrich Gohde, Zum Prinzip der kontraktiven Abbildung, *Math. Nachr.*, 30 (1965), 251258. MR MR0190718 (32 #8129)

[6] Dowling, P. N. On a fixed point result of Amini-Harandi in strictly convex Banach spaces, *Acta Math.Hungar.*, 112 (2006), 85-88.

[7] Kirk, W. A. A fixed point theorem for mappings which do not increase distances, *Amer. Math. Monthly.*, 72 (1965), 1004-1006.

[8] Kirk, W. A. Simeon Reich, Veeramani, P. Proximinal retracts and best proximity pair theorems, *Numer. Funct. Anal. Optim.*, 24 (2003) 851862.

[9] Milman, D. P and Brodskii, M. S. On the center of a convex set, *Dokl. Akad. Nauk. SSSR (N.S).*, 59 (1948), 837840.

[10] Sankar Raj, V. A best proximity point theorem for weakly contractive non-self-mappings, *Nonlinear Anal.*, 74 (2011), 48044808.

[11] Sankar Raj, V. and Anthony Eldred, A. A Characterization of strictly convex spaces and applications, *J Optim Theory Appl.*, 160.2 (2014): 703-710.

[12] Sankar Raj, V. and Sujith, S. Best proximity point theorem without proximal normal structure, Global Journal of Pure and Applied Mathematics, 13(5), 36-41.