GENERALIZATION OF R_0 SPACES VIA IDEALS

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Abstract: In this paper, we introduce and study a generalization of R_0 space called pre-I-R_0 space.

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1. INTRODUCTION

The concept of ideals in topological spaces has been introduced and studied by Kuratowski [6] and Vaidyanathaswamy, [11]. An ideal I on an ideal topological space (X,τ) is a nonempty collection of subsets of X which satisfies (i) $A \in I$ and $B \subset A$ implies $B \in I$ and (ii) $A \in I$ and $B \in I$ implies $A \cup B \in I$. Given an ideal topological space (X, τ) with an ideal I on X and if $\mathcal{P}(X)$ is the set of all subsets of X, a set operator $(.)^{\star} : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$, called the local function [11] of A with respect to τ and I, is defined as follows: for $A \subset X$, $A^{\star}(\tau, I) = \{x \in X | U \cap A \notin I \text{ for every } U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau \mid x \in U\}$. A Kuratowski closure operator $\mathrm{Cl}^{\star}(.)$ for a topology $\tau^{\star}(\tau, I)$ called the *-topology, finer than τ is defined by $\mathrm{Cl}^{\star}(A) = A \cup A^{\star}(\tau, I)$ when there is no chance of confusion, $A^{\star}(I)$ is denoted by $A^{\star}$. If I is an ideal on X, then (X, τ, I) is called an ideal topological space. In this paper, we introduce and study a generalization of R_0 space called pre-I-R_0 space.

2. PRELIMINARIES

For a subset A of (X, τ), Cl(A) and Int(A) denote the closure of A with respect to τ and the interior of A with respect to τ , respectively. A subset S of an ideal topological space (X, τ, I) is pre-I-open [1] if $S \subset \mathrm{Int}(\mathrm{Cl}^*(S)$. The complement of a pre-I-open set is called a pre-I-closed set. The intersection of all pre-I-closed sets containing S is called the pre-I-closure of S and is denoted by pICl(S). The family of all pre-I-open (resp. pre-I-closed) sets of (X, τ, I) is denoted by PIO(X) (resp. PIC(X)). The family of all pre-I-open (resp. pre-I-closed) sets of (X, τ, I) containing a point $x \in X$ is denoted by PIO(X, x) (resp. PIC(X, x)). Recall, that a subset B_x of an ideal topological space (X,τ) is said to be a pre-I-neighbourhood of a point $x \in X$ [9] if there exists a pre-I-open set U such that $x \in U \subseteq B_x$. A subset of an ideal topological space X is pre-I-open in X if and only if it is pre-I-neighbourhood of each of its points.

3. Pre-I-R_0 SPACES

Definition 3.1. Let (X,τ, I) be an ideal topological space and A ⊆ X. Then the pre-I-kernel of A, denoted by pIker (A) is defined to be the set pIKer (A) = ∩ {G ϵ PIO(X) | A ⊆ G}.

Lemma 3.2. Let (X,τ, I) be an ideal topological space and x ϵ X. Then, y ϵ pIKer ({x}) if and only if x ϵ pICl({y}).

Proof. Suppose that y ϵ pIKer({x}). Then there exists a pre-I-open set V of X containing x such that y ϵ V. Then we have x ϵ pICl({y}). The proof of the converse case can be done similarly.

Lemma 3.3. Let (X,τ, I) be an ideal topological space and A a subset of X. Then, pIKer (A) = {x ϵ X | pICl({x}) ∩ A ≠ ϕ}.

Proof. Let x ϵ pIKer(A) and pICl({x}) ∩ A ≠ ϕ. Hence x ϵ X\ pICl({x}) which is a pre-I-open set containing A. This is impossible, since x ϵ pIKer(A). Consequently, pICl({x}) ∩ A ≠ ϕ. Next, let x ϵ X such that pICl({x}) ∩ A ≠ ϕ and suppose that x ϵ pIKer(A). Then, there exists a pre-I-open set U containing A and x ϵ U. Let y ϵ pICl({x}) ∩ A. Hence, U is a pre-I-neighbourhood of y which does not contain x. By this contradiction x ϵ pIKer(A) and the claim.

Definition 3.4. An ideal topological space (X,τ, I) is said to be a pre-I-R_0 space if every pre-I-open set contains the pre-I-closure of each of its singletons.

Proposition 3.5. For an ideal topological space (X,τ, I), the following properties are equivalent:

(1) (X,τ, I) is pre-I-R_0 space.
(2) For any F ϵ PIC(X), x ϵ F implies F ⊆ U and x ϵ U for some U ϵ PIO(X);
(3) For any F ϵ PIC(X), x ϵ F implies F ∩ pICl({x}) = φ;
(4) For any distinct points x and y of X, either pICl({x}) = C1({y}) or pICl({x}) ∩ pICl({y}) = φ.

Proof. (1) => (2): Let F ϵ PIC(X), and x ϵ F. Then by (1) pICl({x}) ⊆ X \ F. Set U=X \ pICl({x}), then U ϵ PIO(X), F ⊆ U and x ϵ U.

(2) => (3): Let F ϵ PIC(X) and x ϵ F. There exists U ϵ PIO(X) such that F ⊆ U and x ϵ U. Since U ϵ PIO(X), U ∩ pICl({x}) = φ and F ∩ pICl({x}) = φ.

(3) => (4): Suppose that pICl({x}) ≠ pICl({y}) for distinct points x,y ϵ X. There exists z ϵ pICl({x}) such that z ϵ pICl({y}) (or z ϵ pICl({y}) such that z ϵ pICl({x})). There exists V ϵ PIO(X), such that y ϵ V and z ϵ V; hence x ϵ V. Therefore, we have x ϵ pICl({y}). By (3), we obtain pICl({x}) ∩ pICl({y}) = φ. The proof for otherwise is similar.

(4) => (1): Let V ϵ PIO(X) and x ϵ V. For each y ϵ V, x ≠ y and x ϵ pICl({y}). Then pICl({x}) ≠ pICl({y}). By (4), pICl({x}) ∩ pICl({y}) = φ for each y ϵ X \ V, and hence pICl({x}) ∩(U_y ϵ X \ V pICl({y})) = φ. On

other hand, since V ϵ PIO(X) and y ϵ X \ V, we have pICl({y}) ⊆ X \ V and hence X \ V = U_y ϵ X \ V pICl({y}). Therefore, we obtain (X \ V) ∩ pICl({x}) = φ and pICl({x}) ⊆ V. This shows that (X,τ, I) is a pre-I-R_0 space.

Theorem 3.6. An ideal topological space (X,τ, I) is a pre-I-R_0 space if and only if for any x and y in X, pICl({x}) ≠ pICl({y}) implies pICl({x}) ∩ pICl({y}) = φ.

Proof. Suppose that (X,τ, I) is pre-I-R_0 and x,y ϵ X such that pICl({x}) ≠ pICl({y}). Then, there exist z ϵ pICl({x}) such that z ϵ pICl({y}) (or z ϵ pICl({y}) such that z ϵ pICl({x}). There exists V ϵ PIO(X) such that y ϵ V and z ϵ V; hence x ϵ V. Therefore, we have x ϵ pICl({y}). Thus x ϵ X \ pICl({y}) ϵ PIO(X), which implies pICl({x}) ⊆ X \ pICl({y}) and pICl({x}) ∩ pICl({y}) = φ. The proof for otherwise is similar. Conversely let V ϵ PIO(X) and let x ϵ V. We will show that pICl({x}) ⊆ V. Let y ϵ X \ V. Then x ≠ y and x not in pICl({y}). This shows that pICl({x}) ≠ pICl({y}) By assumption, pICl({x}) ∩ pICl({y}) = φ. Hence y # pICl({x}) and therefore pICl({x}) ⊆ V.

Lemma 3.7. The following statements are equivalent for any points x and y in an ideal topological space (X,τ, I):

(1) pIKer({x}) ≠ pIKer({y});
(2) pICl({x}) ≠ pICl({y}).

Proof. (1) => (2): Suppose that pIKer({x}) ≠ pIKer({y}), then there exists a point z in X such that z ϵ pIKer({x}) and z ϵ pIKer({y}). It follows from z ϵ pIKer({x}) that {x} ∩ pICl({z}) = ϕ. This implies that x ϵ pICl({z}). By z ϵ pIKer({y}), we have {y} ∩ pICl({z}) = ϕ. Since x ϵ pICl({z}), pICl({x}) ⊆ pICl({z}) and {y} ∩ pICl({x}) = ϕ. Therefore, it follows that pICl({x}) ≠ pICl({y}). Now pIKer({x}) ≠ pIKer({y}) implies that pICl({x}) ≠ pICl({y}).

(2) => (1): Suppose that pICl({x}) ≠ pICl({y}). Then there exists a point z in X such that z ϵ pICl({x}) and z ϵ pICl({y}). Then, there exists a pre-I-open set containing z and therefore x but not y, namely, y ϵ pIKer({x}) and thus pIKer({x}) ≠ pIKer({y}).

Theorem 3.8. An ideal topological space (X,τ, I) is a pre-I-R_0 space if and only if for any point x and y in X, pIKer({x}) ≠ pIKer({y}) implies pIKer({x}) ∩ pIKer({y}) = ϕ.

Proof. Suppose that (X,τ, I) is a pre-I-R_0 space. Thus by Lemma 3.7, for any points x and y in X if pIKer({x}) ≠ pIKer({y}), then pICl({x}) ≠ pICl({y}). Now we prove that pIKer({x}) ∩ pIKer({y}) = ϕ. Assume that z ϵ pIKer({x}) ∩ pIKer({y}). By z ϵ pIKer({x}) and Lemma 3.2, it follows that x ϵ pICl({z}). Since x ϵ pICl({x}), by Theorem 3.6 pICl({x}) = pICl({z}). Similarly, we have pICl({y}) = pICl({z}) = pICl({x}). This is a contradiction. Therefore, we have pIKer({x}) ∩ pIKer({y}) = ϕ. Conversely, let (X, τ, I) be an ideal topological space such that for any points x and y in X, pIKer({x}) ≠ pIKer({y}) implies pIKer({x}) ∩ pIKer({y})= ϕ. If pICl({x}) ≠ pICl({y}), then by Lemma 3.2, pIKer({x}) ≠ pIKer({y}).

Hence, pIKer({x}) ∩ pIKer({y})= ϕ which implies pICl({x}) ∩ pICl({y}) = ϕ. Because z ϵ pICl({x}) implies that x ϵ pIKer({z}) and therefore pIKer({x}) ∩ pIKer({y})= ϕ. By hypothesis, we have pIKer({x}) = pIKer({z}). Then z ϵ pICl({x}) ∩ pICl({y}) implies that pIKer({x}) = pIKer({z}) = pIKer({y}). This is a contradiction. Therefore, pICl({x}) ∩ pICl({y}) = ϕ and by Therorem 3.6 (X,τ, I) is a pre-I-R_0 space.

Theorem 3.9. For an ideal topological space (X,τ, I), the following properties are equivalent:

(1) (X,τ, I) is a pre-I-R_0 space;

(2) For any nonempty set A and G ϵ PIO(X) such that A ∩ G ≠ ϕ, there exists F ϵ PIC(X) such that A ∩ F ≠0 and F ⊆ G;

(3) Any G ϵ PIO(X), G = U { F ϵ PIC(X) | F ⊆ G};

(4) Any F ϵ PIC(X), F = ∩ { G ϵ PIO(X) | F ⊆ G};

(5) For any x ϵ X, pICl({x}) # pIKer({x}).

Proof. (1) => (2): Let A be a nonempty set of X and G ϵ PIO(X) such that A ∩ G ≠ ϕ. There exists x ϵ A ∩ G. Since x ϵ G ϵ PIO(X), pICl({x}) ⊆ G. Set F = pICl({x}), then F ϵ PIC(X), F ⊆ G and A ∩ F ≠ ϕ.

(2) => (3): Let G ϵ PIO(X), then G ⊆U { F ϵ PIC(X) | F ⊆ G}. Let x be any point of G. There exists F ϵ PIC(X) such that x ϵ F # and F ⊆ G. Therefore, we have x ϵ F ⊆ u { F ϵ PIC(X) | F ⊆ G} and hence G = U { F ϵ PIC(X) | F ⊆ G}.

(3) => (4): This is obvious.

(4) => (5): Let x be any point of X and y ϵ pIKer({x}). There exists V ϵ PIO(X) such that x ϵ V and y ϵ V; hence pICl({y}) ∩V = ϕ. By (4) (∩ { G ϵ PIO(X) | pICl({y}) ⊆ G})∩V = ϕ and there exists G ϵ PIO(X) such that x ϵ G and pICl({y}) ⊆ G. Therefore, pICl({x}) ∩G≠ ϕ and y ϵ pICl({x}). Consequently, we obtain pICl({x}) ⊆ pIKer({x}).

(5) =>(1): Let G ϵ PIO(X) and x ϵ G. Let y ϵ pIKer({x}), then x ϵ pICl({y}) and y ϵ G. This implies that pIKer({x}) ⊆ G. Therefore, we obtain x ϵ pICl({x}) ⊆ pIKer({x}) ⊆ G. This shows that (X,τ, I), is pre-I-R_0 space.

Corollary 3.10. For an ideal topological space (X,τ, I), the following properties are equivalent:

(1) (X,τ, I) is pre-I-R_0 space;

(2) pICl({x}) = pIKer({x}) for all x ϵ X.

Proof. (1) => (2): Suppose that (X, τ, I) is pre-I-R_0 space. By Theorem 3.9, pICl({x}) ⊆ pIKer({x}) for each x ϵ X. Let y ϵ pIKer({x}), then x ϵ pICl({y}) and by Theorem 3.6 pICl({x}) = pICl({y}). Therefore, y ϵ pICl({x}) and hence pIKer({x}) ⊆ pICl({x}). This shows that pICl({x}) = pIKer({x}).

(2) => (1): This is obvious by Theorem 3.9.

Theorem 3.11. For an ideal topological space (X,τ, I), the following properties are equivalent:

(1) (X,τ, I) is pre-I-R_0 space;

(2) x ϵ pICl({y}) if and only if y ϵ pICl({x}) for any points x and y in X.

Proof. (1) => (2): Assume that X is pre-I-R_0. Let x ϵ pICl({y}) and D be any pre-I-open set such that y ϵ D. Now by hypothesis, x ϵ D. Therefore, every pre-I-open set containing y contains x. Hence y ϵ pICl({x}).

(2) => (1): Let U be a pre-I-open set and x ϵ U . If y ϵ U , then x ϵ pICl({y}) and hence y ϵ pICl({x}). This implies that pICl({x}) ⊆ U. Hence (X,τ, I) is pre-I-R_0.

Theorem 3.12. For an ideal topological space (X,τ, I), the following properties are equivalent:

(1) (X,τ, I) is pre-I-R_0 space;

(2) If F is a pre-I-closed subset of X, then F = pIKer(F);

(3) If F is a pre-I-closed subset of X and x ϵ F, then pIKer({x}) ⊆ F;

(4) If x ϵ X, then pIKer({x}) ⊆ pICl({x}).

Proof. (1) => (2): Let F be pre-I-closed subset of X and x ϵ F. Thus X \ F is pre-I-open and contains x. Since (X, τ, I) is pre-I-R_0, pICl({x}) ⊆ X \ F. Thus pICl({x}) ∩ F = ϕ and Lemma 3.3 x ϵ pIKer(F). Therefore, pIKer(F) = F.

(2) => (3): In general, A⊆B implies pIKer(A) ⊆ pIKer(B). Therefore, it follows from (2) that pIKer({x}) ⊆ pIKer(F) = F.

(3) => (4): Since x ϵ pICl({x}) and pICl({x}) is pre-I-closed, by (3) pIKer({x}) and pICl({x}).

(4) => (1): We show the implication by using Theorem 3.11. Let x ϵ pICl({y}). Then by Lemma 3.2 y ϵ pIKer({x}). Since x ϵ pICl({x}) and pICl({x}) is pre-I-closed, by (4) we obtain y ϵ pIKer({x}) ⊆ preC1({x}). Therefore, x ϵ pICl({x}) implies y ϵ pICl({x}). The converse is obvious and (X,τ, I) is pre-I-R_0.

Definition 3.13. A filterbase F is called pre-I-convergent to a point x in X, if for any pre-I-open set U of X containing x, there exists B in F such that B is a subset of U.

Lemma 3.14. Let (X,τ, I) be an ideal topological space and let x and y be any two points in X such that every net in X pre-I-converging to y pre-I-converge to x. Then x ϵ pICl({y}).

Proof. Suppose that x_n = y for each n ϵ N. Then { x_n }$_{n}$ ϵ N is a net in preC1({y}). Since { x_n }$_{n\epsilon N}$ pre-I-converges to y, then { x_n }$_{n\epsilon N}$ pre-I-converges to x and this implies that x ϵ pICl({y}).

Theorem 3.15. For an ideal topological space (X,τ, I), the following properties are equivalent:

(1) (X,τ, I) is pre-I-R_0 space;

(2) If y ϵ X, then y ϵ pICl({x}) if and only if every net in X pre-I-converging to y pre-I-converges to x.

Proof. (1) => (2): Let x,y ϵ X such that y ϵ pICl({x}). Suppose that $\{ x_\alpha \}_{\alpha \varepsilon N}$ be a net in X such that $\{ x_\alpha \}_{\alpha \varepsilon N}$ pre-I-converges to y. Since y ϵ pICl({x}), by Theorem 3.6 we have pICl({x}) = pICl({y}). Therefore x ϵ pICl({y}). This means that $\{x_\alpha\}_{\alpha \varepsilon N}$ pre-I-converges to y. Conversely, let x,y ϵ X such that every net in X pre-I-converging to y pre-I-converges to x. Then x ϵ pICl({y}) by Lemma 3.3. By Theorem 3.6, we have pICl({x}) = pICl({y}). Therefore y ϵ pICl({x}).

(2) => (1): Assume that x and y are any two points of X such that pICl({x}) ∩ pICl({y}) ≠ ϕ. Let z ϵ pICl({x}) ∩ pICl({y}). So there exists a net $\{x_\alpha\}_{\alpha \varepsilon N}$ in pICl({x}) such that $\{x_\alpha\}_{\alpha \varepsilon N}$ pre-I-converges to z. Since z ϵ pICl({y}), then $\{ x_\alpha \}_{\alpha \varepsilon N}$ pre-I-converges to y. It follows that y ϵ pICl({x}). By the same manner we obtain x ϵ pICl({y}). Therefore pICl({x}) = pICl({y}) and by Theorem 3.6 (X,τ, I) is pre-I-R_0.

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