$\pi gg\zeta^*$-CLOSED SETS AND QUASI $g\zeta^*$-NORMAL SPACES

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Abstract: In this paper, we introduce a new class of sets called $\pi gg\zeta^*$-closed sets in topological spaces. Also we study and investigate the relationship with other existing closed sets. Moreover, we introduce some functions such as $g\zeta^*$-closed, $\pi gg\zeta^*$-closed, almost $g\zeta^*$-closed, almost $\pi gg\zeta^*$-closed, $\pi gg\zeta^*$-continuous and almost $\pi gg\zeta^*$-continuous. We also study a new class of normal space called, quasi $g\zeta^*$-normal space. The relationships among normal, π-normal, quasi normal, softly normal, mildly normal, α-normal, $\pi\alpha$-normal, quasi α-normal, softly α-normal, mildly α-normal, $g\zeta^*$-normal, $\pi g\zeta^*$-normal, quasi $g\zeta^*$-normal, softly $g\zeta^*$-normal and mildly $g\zeta^*$-normal spaces are investigated. Further we show that this property is a topological property and it is a hereditary property only with respect to closed domain subspaces. Utilizing $\pi gg\zeta^*$-closed sets and some functions, we obtained some characterizations and preservation theorems for quasi $g\zeta^*$-normal spaces.

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1. Introduction

In 1965, Njastad [**13**] introduced the concept of α-open sets in topological spaces. In 1968, the notion of quasi normal space was introduced by Zaitsev [**21**]. In 1970, Levine [**11**] initiated the study of so called generalized closed (briefly g-closed) sets in order to extend many of the most important properties of closed sets to a large family. In 1973, Singal and Singal [**20**] introduced the notion of mildly normal spaces in topological spaces. In 1990, Lal and Rahman [**10**] have further studied notions of quasi normal and mildly normal spaces. In 1994, H. Maki et al. [**12**] introduced the notion of αg-closed sets. In 2000, Dontchev and Noiri [**4**] introduced the notion of πg-closed sets and by using these sets, obtained a new characterization of quasi normal space. In 2001, A. V. Arhangel'skii and Ludwig [**1**] introduced the concepts of α-normal and β-normal spaces. In 2004, Nono et al. [**15**] introduced the notion of $g^{\#}\alpha$-closed sets in topological spaces. In 2007, Arockiarani and C. Janaki [**2**] introduced the notion of $\pi g\alpha$-closed sets in topological spaces and by using $\pi g\alpha$-closed sets, obtained a new characterization of quasi α-normal spaces. In 2008, Kalantan [**6**] introduced a weaker version of normality called π-normality and proved that π-normality is a property which lies between normality and almost normality. In 2009, R. Devi et al. [**3**] introduced the notion of $^{\#}g\alpha$-closed sets in topological spaces. In 2013, Kokilavani [**7**] introduced the notion of $g\zeta^*$-closed sets in topological spaces and investigated some of their properties. In 2015, T. C. K. Raman [**16**] introduced the concepts of $\pi\alpha$-normal spaces. In 2018, Hamant Kumar [**5**] introduced some normal spaces such as $g\zeta^*$-normal, $\pi g\zeta^*$-normal, quasi $g\zeta^*$-normal and mildly $g\zeta^*$-normal, and the relationships among these normal spaces are investigated.

2. Preliminaries

Throughout this paper, spaces $(X, \Im)$, (Y, σ), and (Z, γ) (or simply X, Y and Z) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. The closure of A and interior of A are denoted by cl(A) and int(A) respectively. A subset A is said to be **regular open** (resp. **regular closed**) if A = int(cl(A)) (resp. A = cl(int(A)). The finite union of regular open sets is said to be **π-open**. The complement of a π-open set is said to be **π-closed**. A is said to be **α-open** [**13**] if $A \subset int(cl(int(A)))$. The complement of a α-open set is said to be **α-closed**. The intersection of all α-closed sets containing A is called **α-closure** [**13**] of A, and is denoted by α-cl(A)). The **α-interior** [**13**] of A, denoted by α-int(A), is defined as union of all α-open sets contained in A.

2.1 Definition. A subset A of a space $(X, \Im)$ is said to be

(1) **generalized closed** (briefly **g-closed**) [**11**] if $cl(A) \subset U$ whenever $A \subset U$ and $U \in \Im$.

(2) **πg-closed** [**4**] if $cl(A) \subset U$ whenever $A \subset U$ and U is π-open in X.

(3) **α-generalized closed** (briefly **αg-closed**) [**12**] if $\alpha\text{-}cl(A) \subset U$ whenever $A \subset U$ and $U \in \Im$.

(4) **πgα-closed** [**2**] if $\alpha cl(A) \subset U$ whenever $A \subset U$ and U is π-open in X.

(5) **generalized $^{\#}$α-closed** (briefly **$g^{\#}\alpha$-closed**) [**15**] if $\alpha\text{-}cl(A) \subset U$ whenever $A \subset U$ and is g-open in X.

(6) **$^{\#}$generalized α-closed** (briefly **$^{\#}g\alpha$-closed**) [**3**] if $\alpha\text{-}cl(A) \subset U$ whenever $A \subset U$ and U is $g^{\#}\alpha$-open in X.

(7) **$g\zeta^*$-closed** [**7**] if $\alpha\text{-}cl(A) \subset U$ whenever $A \subset U$ and U is $^{\#}g\alpha$-open in X.

(8) **g-open** (resp**. πg-open**, **αg-open, πgα-open, $g^{\#}\alpha$-open, $^{\#}g\alpha$-open, $g\zeta^*$-open**) set if the complement of A is g-closed (resp. πg-closed, αg-closed, πgα-closed, $g^{\#}\alpha$-closed, $^{\#}g\alpha$-closed, $g\zeta^*$-closed).

The intersection of all $g\zeta^*$-closed sets containing A is called **$g\zeta^*$-closure of A**, and is denoted by **$g\zeta^*$-cl(A)**. The **$g\zeta^*$-interior of A**, denoted by **$g\zeta^*$-int(A)**, is defined as union of all $g\zeta^*$-open sets contained in A. The family of all $g\zeta^*$-closed (resp. $g\zeta^*$-open) sets of a space X is denoted by $g\zeta^*$-C(X) (resp. $g\zeta^*$-O(X)).

2.2 Definition. A subset A of a space $(X, \Im)$ is said to be

(1) **generalized $g\zeta^*$-closed** [**9**] (briefly **$gg\zeta^*$-closed**) if $g\zeta^*\text{-}cl(A) \subset U$ whenever $A \subset U$ and $U \in \Im$.

(2) **π-generalized $g\zeta^*$-closed** (briefly **$\pi gg\zeta^*$-closed**) if $g\zeta^*\text{-}cl(A) \subset U$ whenever $A \subset U$ and U is π-open in X.

2.3 Remark. We have the following implications for the properties of subsets:

closed	⇒	g-closed	⇒	πg-closed
⇓		⇓		⇓
α-closed	⇒	αg-closed	⇒	πgα-closed
⇓		⇓		⇓
$g\zeta^*$-closed	⇒	$gg\zeta^*$-closed	⇒	$\pi gg\zeta^*$-closed

Where none of the implications is reversible as can be seen from the following examples:

2.4 Example. Let X = {a, b, c} and $\Im$ = {ϕ, {a}, X}. Then A = {b} is g-closed as well as αg-closed. Hence A is ggζ^*-closed. But it is not closed.

2.5 Example. Let X = {a, b, c, d} and $\Im$ = {ϕ, {a}, {d}, {a, d}, {c, d}, {a, c, d}, X}. Then A = {c} is πgα-closed as well as πggζ^*-closed but not g-closed.

2.6 Example. Let X = {a, b, c, d} and $\Im$ = {ϕ, {a}, {c}, {a, b}, {a, c}, {a, d}, {a, b, c}, {a, b, d}, {a, c, d}, X}. Then A = {a} is πgα-closed as well as πggζ^*-closed but not closed.

2.7 Example. Let X = {a, b, c, d} and $\Im$ = {ϕ, {c}, {d}, {a, c}, {c, d}, {a, c, d}, X}. Then A = {a} is α-closed as well as αg-closed. Hence A is gζ^*-closed. But it is not closed.

2.8 Example. Let X = {a, b, c} and $\Im$ = {ϕ, {a}, {b, c}, X}. Then A = {b} is αg-closed as well as ggζ^*-closed. But it is not gζ^*-closed.

2.9 Theorem. For πggζ^*-closed sets of a space X, the following properties hold:
(a) Every finite union of πggζ^*-closed sets is always a πggζ^*-closed set.
(b) Even a countable union of πggζ^*-closed sets need not be a πggζ^*-closed set.
(c) Even a finite intersection of πggζ^*-closed sets may fail to be a πggζ^*-closed set.
Proof.
(a) Let A and B be any two πggζ^*-closed sets. Therefore gζ^*-cl(A) $\subset$ U and gζ^*-cl(B) $\subset$ U whenever A $\subset$ U, B $\subset$ U and U is π-open. Let A $\cup$ B $\subset$ U where U is π-open.

Since, gζ^*-cl(A $\cup$ B) $\subset$ gζ^*-cl(A) $\cup$ gζ^*-cl(B) $\subset$ U, we have A $\cup$ B is πggζ^*-closed.

(b) Let R be the real line with the usual topology. Every singleton is πggζ^*-closed. However, A = {1 / i : i = 2, 3, …….} is not πggζ^*-closed, since A $\subset$ (0, 1) which is π-open but gζ^*-cl(A) $\not\subset$ (0, 1).

(c) Let X = {a, b, c, d} and let $\Im$ = {$\varnothing$, {a}, {b}, {a, b}, {a, b, c}, {a, b, d}, X}. Let A = {a, b, c} and B = {a, b, d} are πggζ^*-closed sets. But A $\cap$ B = {a, b} $\subset$ {a, b} which is π-open. gζ^*-cl(A $\cap$ B) $\not\subset$ {a, b}. Hence A $\cap$ B is not πggζ^*-closed.

2.10 Theorem: If A is πggζ^*-closed and A $\subset$ B $\subset$ gζ^*-cl(A) then B is πggζ^*-closed.
Proof: Since A is πggζ^*-closed, gζ^*-cl(A) $\subset$ U whenever A $\subset$ U and U is π-open. Let B $\subset$ U and U is π-open. Since B $\subset$ gζ^*-cl(A), gζ^*-cl(B) $\subset$ gζ^*-cl(A) $\subset$ U. Hence B is πggζ^*-closed.

2.11 Theorem. Let A be a πggζ^*-closed set in X. Then gζ^*-cl(A) – A does not contain any nonempty π-closed set.
Proof. Let F be a nonempty π-closed set such that F $\subset$ gζ^*-cl(A) – A. Then F $\subset$ gζ^*-cl(A) $\cap$ (X – A) $\subset$ (X – A) implies A $\subset$ X – F where X – F is π-open. Therefore gζ^*-cl(A) $\subset$ X – F implies F $\subset$ (gζ^*-cl(A)^c. Now F $\subset$ gζ^*-cl(A) $\cap$ (gζ^*-cl(A))^c implies F is empty.

Reverse implication does not hold.

2.12 Example. Let X = {a, b, c, d, e} and let $\Im$ = {∅, {a, b}, {c, d}, {a, b, c, d}, X}. Let A = {c} then $g\zeta^*$-cl(A) = {c, d, e}, $g\zeta^*$-cl(A) – A = {d, e} does not contain any nonempty regular closed set but A is not $\pi gg\zeta^*$-closed set.

2.13 Corollary. Let A be $\pi gg\zeta^*$-closed. A is $g\zeta^*$-closed iff $g\zeta^*$-cl(A) – A is π-closed.
Proof. Let A be $g\zeta^*$-closed set then A = $g\zeta^*$-cl(A) implies $g\zeta^*$-cl(A) – A = ∅ which is π-closed.

Conversely, if $g\zeta^*$-cl(A) – A is π-closed then A is $g\zeta^*$-closed.

2.14 Theorem. If A is π-open and $\pi gg\zeta^*$-closed. Then A is $g\zeta^*$-closed and hence clopen.

Proof. Let A be regular open. Since A is $\pi gg\zeta^*$-closed, $g\zeta^*$-cl(A) ⊂ A implies A is $g\zeta^*$-closed. Hence A is closed. (Since every π-open $g\zeta^*$-closed set is closed). Therefore A is clopen.

2.15 Theorem. For a space X, the following are equivalent:
(a) X is extremally disconnected,
(b) Every subset of X is $\pi gg\zeta^*$-closed
(c) The topology on X generated by $\pi gg\zeta^*$-closed set is the discrete one.
Proof. (a) ⇒ (b).
Assume that X is extremally disconnected. Let A ⊂ U where U is π-open in X. Since U is π-open, it is the finite union of regular open sets and X is extremally disconnected, U is finite union of clopen sets and hence U is clopen. Therefore $g\zeta^*$-cl(A) ⊂ cl(A) ⊂ cl(U) ⊂ U implies A is $\pi gg\zeta^*$-closed.

(b) ⇒ (a)
Let A be a regular open set of X. Since A is $\pi gg\zeta^*$-closed by **Theorem 2.14**, A is clopen. Hence X is extremally disconnected.

(b) ⇔ (c) is obvious.

3. $\pi gg\zeta^*$-open sets

3.1 Definition. A subset A of a space X is called **π-generalized $g\zeta^*$-open** (briefly **$\pi gg\zeta^*$-open**) iff its complement is $\pi gg\zeta^*$-closed set.

3.2 Lemma. If A be a subset of X, then
(a) $g\zeta^*$-cl(X – A) = X – $g\zeta^*$-int(A).
(b) $g\zeta^*$-int(X – A) = X – $g\zeta^*$-cl(A).

3.3 Theorem. A subset A of a space X is $\pi gg\zeta^*$-open iff F ⊂ $g\zeta^*$-int(A) whenever F is π-closed and F ⊂ A.
Proof. Let F be π-closed set such that F ⊂ A. Since X – A is $\pi gg\zeta^*$-closed and X – A ⊂ X – F where F ⊂ $g\zeta^*$-int(A). Conversely.

Let F ⊂ $g\zeta^*$-int(A) where F is π-closed and F ⊂ A. Since F ⊂ A and X – F is π-open, $g\zeta^*$-cl(X – A) = X –

gζ*-int(A) ⊂ X – F. Therefore A is πggζ*-open.

3.4 Theorem. If gζ*-int(A) ⊂ B ⊂ A and A πggζ*-open then B is πggζ*-open.
Proof: Since gζ*-int(A) ⊂ B ⊂ A, by **Theorem 2.10**, gζ*-cl(X – A) ⊃ (X – B) implies B is πggζ*-open.

3.5 Remark. For any A ⊂ X, gζ*-int(gζ*-cl(A) – A) = ∅.

3.6 Theorem. If A ⊂ X is πggζ*-closed then gζ*-cl(A) – A is πggζ*-open.
Proof. Let A be πggζ*-closed and F be a π-closed set such that F ⊂ gζ*-cl(A) – A. By **Theorem 2.11**, F = ∅

implies F ⊂ gζ*-int(gζ*-cl(A) – A). By **Theorem 3.3**, gζ*-cl(A) – A is πggζ*-open.

Converse of the above theorem is not true.

3.7 Example. Let X = {a, b, c} and let ℑ = {∅, {a}, {b}, {a, b}, X}. Let A = {b}. Then A is not πggζ*-closed but gζ*-cl(A) – A = {a, b} is πggζ*-open.

4. Quasi gζ*-normal spaces

4.1 Definition. A space X is said to be **gζ*-normal** [18] (resp. **α-normal** [**1**]) if for every pair of disjoint closed subsets A, B of X, there exist disjoint gζ*-open (resp. α-open) sets U, V of X such that A ⊂ U and B ⊂ V.

4.2 Definition. A space X is said to be **πgζ*–normal** [5] (resp. **π-normal** [6], **πα-normal** [16]) if for every pair of disjoint closed subsets A, B of X, one of which is π-closed, there exist disjoint gζ*-open (resp. open, α-open) sets U, V of X such that A ⊂ U and B ⊂ V.

4.3 Definition. A space X is said to be **quasi gζ*-normal** [5] (resp. **quasi normal** [21], **quasi α-normal [2]**) if for every pair of disjoint π-closed subsets H, K of X, there exist disjoint gζ*-open (resp. open, α-open) sets U, V of X such that H ⊂ U and K ⊂ V.

4.4 Definition. A space X is said to be **softly gζ*–normal** [5] (resp. **softly normal** [17], **softly α-normal**) if for every pair of disjoint subsets A, B of X, one of which is π-closed and the other is regularly closed, there exist disjoint gζ*-open (resp. open, α-open) sets U, V of X such that A ⊂ U and B ⊂ V.

4.5 Definition. A space X is said to be **mildly gζ*-normal** [18] (resp. **mildly-normal** [20], **mildly α-normal [2]**) if for every pair of disjoint regular closed subsets H, K of X, there exist disjoint gζ*-open (resp. open, α-open) sets U, V of X such that H ⊂ U and K ⊂ V.

By the definitions stated above, we have the following diagram:

normal ⇒ π-normal ⇒ quasi-normal ⇒ softly normal ⇒ mildly-normal

$$\begin{array}{ccccccccc} \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ \alpha\text{-normal} & \Rightarrow & \pi\alpha\text{-normal} & \Rightarrow & \text{quasi } \alpha\text{-normal} & \Rightarrow & \text{softly } \alpha\text{-normal} & \Rightarrow & \text{mildly } \alpha\text{-normal} \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ g\zeta^*\text{-normal} & \Rightarrow & \pi g\zeta^*\text{-normal} & \Rightarrow & \text{quasi } g\zeta^*\text{-normal} & \Rightarrow & \text{softly } g\zeta^*\text{-normal} & \Rightarrow & \text{mildly } g\zeta^*\text{-normal} \end{array}$$

Where none of the implications is reversible as can be seen from the following examples:

4.6 Example. Let X = {a, b, c, d} and $\Im$ = {∅, {a, b}, {c, d}, X}. The pair of disjoint closed subsets of X are A = {a, b} and B = {c, d}. Also U = {a, b} and V = {c, d} are open sets such that A ⊂ U and B ⊂ V. Hence the space X is normal as well as α-normal. It is also $g\zeta^*$-normal.

4.7 Example. Let X = {a, b, c, d} and $\Im$ = {∅, {a}, {c}, {a, c}, {a, b, d}, {b, c, d}, X}. The pair of disjoint closed subsets of X are A = {a} and B = {c}. Also U = {a} and V = {c} are open sets such that A ⊂ U and B ⊂ V. Hence the space X is normal as well as α-normal, since every open set is α-open.

4.8 Theorem. For a space X, the following are equivalent:
(a) X is quasi $g\zeta^*$-normal.
(b) For every pair of π-open subsets U and V of X whose union is X, there exist $g\zeta^*$-closed subsets G and H of X such that G ⊂ U, H ⊂ V and G ∪ H = X.
(c) For any π-closed set A and every π-open set B in X such that A ⊂ B, there exists a $g\zeta^*$-open subset U of X such that A ⊂ U ⊂ $g\zeta^*$-cl(U) ⊂ B.
(d) For every pair of disjoint π-closed subsets A and B of X, there exist $g\zeta^*$-open subsets U and V of X such that A ⊂ U, B ⊂ V and $g\zeta^*$-cl(U) ∩ $g\zeta^*$-cl(V) = ∅.
Proof. (a) ⇒ (b), (b) ⇒ (c), (c) ⇒ (d) and (d) ⇒ (a).

(a)⇒ (b).Let U and V be any π-open subsets of a quasi $g\zeta^*$-normal space X such that U ∪ V = X. Then, X – U and X – V are disjoint π-closed subsets of X. By quasi $g\zeta^*$-normality of X, there exist disjoint $g\zeta^*$-open subsets U_1 and V_1 of X such that X – U ⊂ U_1 and X – V ⊂ V_1. Let G = X – U_1 and H = X – V_1. Then, G and H are $g\zeta^*$-closed subsets of X such that G ⊂ U, H ⊂ V and G ∪ H = X.

(b) ⇒ (c). Let A be a π-closed and B is a π-open subsets of X such that A ⊂ B. Then, X – A and B are π-open subsets of X such that (X – A) ∪ B = X. Then, by part (b), there exist $g\zeta^*$-closed sets G and H of X such that G ⊂ (X – A), H ⊂ B and G ∪ H = X. Then, A ⊂ (X – G), (X – B) ⊂ (X – H) and (X – G) ∩ (X – H) = ∅. Let U = X – G and V = (X – H). Then U and V are disjoint $g\zeta^*$-open sets such that A ⊂ U ⊂ X – V ⊂ B. Since X – V is $g\zeta^*$-closed, then we have $g\zeta^*$-cl(U) ⊂ (X – V). Thus, A ⊂ U ⊂ $g\zeta^*$-cl(U) ⊂ B.

(c) ⇒ (d). Let A and B be any disjoint π-closed subset of X. Then A ⊂ X – B, where X – B is π-open. By the part (c), there exists a $g\zeta^*$-open subset U of X such that A ⊂ U ⊂ $g\zeta^*$-cl(U) ⊂ X – B. Let V = X – $g\zeta^*$-cl(U). Then, V is a $g\zeta^*$-open subset of X. Thus, we obtain A ⊂ U, B ⊂ V and $g\zeta^*$-cl(U) ∩ $g\zeta^*$-cl(V) = ∅.

(d) $\Rightarrow$ (a). It is obvious.

4.9 Proposition. Let f : X $\to$ Y be a function, then:
(a) The image of $g\zeta^*$-open subset under an open continuous function is $g\zeta^*$-open.
(b) The inverse image of $g\zeta^*$-open (resp. $g\zeta^*$-closed) subset under an open continuous function is $g\zeta^*$-open (resp. $g\zeta^*$-closed) subset.
(c) The image of $g\zeta^*$-closed subset under an open and a closed continuous surjective function is $g\zeta^*$-open.

4.10 Theorem. The image of a quasi $g\zeta^*$-normal space under an open continuous injective function is a quasi $g\zeta^*$-normal.
Proof. Let X be a quasi $g\zeta^*$-normal space and let f : X $\to$ Y be an open continuous injective function. We need to show that f(X) is a quasi $g\zeta^*$-normal. Let A and B be any two disjoint π-closed sets in f(X). Since the inverse image of a π-closed set under an open continuous function is a π-closed. Then, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint π-closed sets in X. By quasi $g\zeta^*$-normality of X, there exist $g\zeta^*$-open subsets U and V of X such that $f^{-1}(A) \subset U$, $f^{-1}(B) \subset V$ and $U \cap V = \emptyset$. Since f is an open continuous injective function, we have $A \subset f(U)$, $B \subset f(V)$ and $f(U) \cap f(V) = \emptyset$. By **Proposition 4.9**, we obtain f(U) and f(V) are disjoint $g\zeta^*$-open sets in f(X) such that $A \subset f(U)$ and $B \subset f(V)$. Hence f(X) is quasi $g\zeta^*$-normal.

From the above theorem, we have the following corollary.

4.11 Corollary. Quasi $g\zeta^*$-normality is a topological property.

The following lemma helps us to show that quasi $g\zeta^*$-normality is a hereditary with respect to closed domain subspaces.

4.12 Lemma. Let M be a closed domain subspace of a space X. If A is a $g\zeta^*$-open set in X, then $A \cap M$ is $g\zeta^*$-open set in M.

4.13 Theorem. Quasi $g\zeta^*$-normality is a hereditary with respect to closed domain subspaces.
Proof. Let M be a closed domain subspace of a quasi $g\zeta^*$-normal space X. Let A and B be any disjoint π-closed sets in M. Since M is a closed domain subspace of X, then we have A and B be any disjoint π-closed sets of X. By quasi $g\zeta^*$-normal of X, there exist disjoint $g\zeta^*$-open subsets U and V of X such that $A \subset U$ and $B \subset V$. By the **Lemma 4.12,** we obtain $U \cap M$ and $V \cap M$ are disjoint $g\zeta^*$-open sets in M such that $A \subset U \cap M$ and $B \subset V \cap M$. Hence, M is quasi $g\zeta^*$-normal subspace.

Since every closed and open (clopen) subset is a closed domain, then we have the following corollary.

4.14 Corollary. Quasi $g\zeta^*$-normality is a hereditary with respect to clopen subspaces.

The following result is useful for giving some other characterizations of quasi $g\zeta^*$-normal spaces.

4.15 Theorem. For a space X, the following are equivalent:
(a) X is quasi $g\zeta^*$-normal.

(b) For any disjoint π-closed sets H and K, there exist disjoint ggζ^*-open sets U and V such that $H \subset U$ and $K \subset V$

(c) For any disjoint π-closed sets H and K, there exist disjoint πggζ^*-open sets U and V such that $H \subset U$ and $K \subset V$.

(d) For any π-closed set H and any π-open set V containing H, there exists a ggζ^*-open set U of X such that $H \subset U \subset g\zeta^*\text{-cl}(U) \subset V$.

(e) For any π-closed set H and any π-open set V containing H, there exists a πggζ^*-open set U of X such that $H \subset U \subset g\zeta^*\text{-cl}(U) \subset V$.

Proof. (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c), (c) $\Rightarrow$ (d), (d) $\Rightarrow$ (e) and (e) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b). Let X be quasi gζ^*-normal space. Let H, K be disjoint π-closed sets of X. By assumption, there exist disjoint gζ^*-open sets U, V such that $H \subset U$ and $K \subset V$. Since every gζ^*-open set is ggζ^*-open, U and V are ggζ^*-open sets such that $H \subset U$ and $K \subset V$.

(b) $\Rightarrow$ (c). Let H, K be two disjoint π-closed sets. By assumption, there exist disjoint ggζ^*-open sets U and V such that $H \subset U$ and $K \subset V$. Since ggζ^*-open set is πggζ^*-open, U and V are πggζ^*-open sets such that $H \subset U$ and $K \subset V$.

(c) $\Rightarrow$ (d). Let H be any π-closed set and V be any π-open set containing H. By assumption, there exist disjoint πggζ^*-open sets U and W such that $H \subset U$ and $X - V \subset W$. By **Theorem 3.3**, we get $X - V \subset g\zeta^*\text{-int}(W)$ and $g\zeta^*\text{-cl}(U) \cap g\zeta^*\text{-int}(W) = \emptyset$. Hence $H \subset U \subset g\zeta^*\text{-cl}(U) \subset X - g\zeta^*\text{-int}(W) \subset V$.

(d) $\Rightarrow$ (e). Let H be any π-closed set and V be any π-open set containing H. By assumption, there exist ggζ^*-open set U of X such that $H \subset U \subset g\zeta^*\text{-cl}(U) \subset V$. Since, every gg$\zeta^*$-open set is πggζ^*-open, there exists πggζ^*-open sets U of X such that $H \subset U \subset g\zeta^*\text{-cl}(U) \subset V$.

(e) $\Rightarrow$ (a). Let H, K be any two disjoint π-closed sets of X. Then $H \subset X - K$ and $X - K$ is π-open. By assumption, there exists πggζ^*-open set G of X such that $H \subset G \subset g\zeta^*\text{-cl}(G) \subset X - K$. Put $U = g\zeta^*\text{-int}(G)$, $V = X - g\zeta^*\text{-cl}(G)$. Then U and V are disjoint gζ^*-open sets of X such that $H \subset U$ and $K \subset V$.

5. Preservation Theorems

5.1 Definition. A function $f : X \to Y$ is said to be

(a) **gζ^*-closed** [8] (resp. **πggζ^*-closed**) if f(F) is gζ^*-closed (resp. πggζ^*-closed) in Y for every closed set F of X.

(b) **rc-preserving** [14] (resp. **almost closed** [19], **almost gζ^*-closed**, **almost πggζ^*-closed**) if f(F) is regular closed (resp. closed, gζ^*-closed, πggζ^*-closed) in Y for every $F \in RC(X)$.

(c) **π-continuous** [4] (resp. **almost π-continuous** [4]) if $f^{-1}(F)$ is π-closed in X for every closed (resp. regular closed) set F of Y.

(d) **almost continuous** [19] if $f^{-1}(V)$ is open in X for every regular open set V of Y.

(e) **πggζ^*-continuous** (resp. **almost πggζ^*-continuous**) if $f^{-1}(F)$ is πggζ^*-closed in X for every closed (resp. regular closed) set F of Y.

5.2 Theorem. If $f : X \to Y$ is an almost π-continuous and $\pi gg\zeta^*$-closed function, then f(A) is $\pi gg\zeta^*$-closed in Y for every $\pi gg\zeta^*$-closed set A of X.

Proof. Let A be any $\pi gg\zeta^*$-closed set of X and V be any π-open set of Y containing f(A). Since f is almost π-continuous, $f^{-1}(V)$ is π-open in X and $A \subset f^{-1}(V)$. Therefore, we have $g\zeta^*$-cl(A) $\subset f^{-1}(V)$ and hence f($g\zeta^*$-cl(A)) $\subset$ V. Since f is $\pi gg\zeta^*$-closed, f($g\zeta^*$-cl(A)) is $\pi gg\zeta^*$-closed in Y and hence we obtain $g\zeta^*$-cl(f(A)) $\subset$ $g\zeta^*$-cl(f($g\zeta^*$-cl(A))) $\subset$ V. Hence f(A) is $\pi gg\zeta^*$-closed in Y.

5.3 Theorem. A surjection $f : X \to Y$ is almost $\pi gg\zeta^*$-closed if and only if for each subset S of Y and each $U \in RO(X)$ containing $f^{-1}(S)$, there exists a $\pi gg\zeta^*$-open set V of Y such that $S \subset V$ and $f^{-1}(V) \subset U$.

Proof. Necessity. Suppose that f is almost $\pi gg\zeta^*$-closed. Let S be a subset of Y and $U \in RO(X)$ containing $f^{-1}(S)$. If $V = Y - f(X - U)$, then V is a $\pi gg\zeta^*$-open set of Y such that $S \subset V$ and $f^{-1}(V) \subset U$.

Sufficiency. Let F be any regular closed set of X. Then $f^{-1}(Y - f(F)) \subset (X - F)$ and $(X - F) \in RO(X)$. There exists a $\pi gg\zeta^*$-open set V of Y such that $Y - f(F) \subset V$ and $f^{-1}(V) \subset (X - F)$. Therefore, we have $f(F) \supset (Y - V)$ and $F \subset X - f^{-1}(V) \subset f^{-1}(Y - V)$. Hence we obtain $f(F) = Y - V$ and f(F) is $\pi gg\zeta^*$-closed in Y, which shows that f is almost $\pi gg\zeta^*$-closed.

5.4 Theorem. If $f : X \to Y$ is an almost $\pi gg\zeta^*$-continuous, rc-preserving injection and Y is quasi $g\zeta^*$-normal, then X is quasi $g\zeta^*$-normal.

Proof. Let A and B be any disjoint π-closed sets of X. Since f is an rc-preserving injection, f(A) and f(B) are disjoint π-closed sets of Y. Since Y is quasi $g\zeta^*$-normal, there exist disjoint $g\zeta^*$-open sets U and V of Y such that $f(A) \subset U$ and $f(B) \subset V$.

Now if G = int(cl(U)) and H = int(cl(V)). Then G and H are regular open sets such that $f(A) \subset G$ and $f(B) \subset H$. Since f is almost $\pi gg\zeta^*$-continuous, $f^{-1}(G)$ and $f^{-1}(H)$ are disjoint $\pi gg\zeta^*$-open sets containing A and B, respectively. It follows from **Theorem 4.15**, that X is quasi $g\zeta^*$-normal.

5.5 Theorem. If $f : X \to Y$ is a π-continuous, almost $g\zeta^*$-closed surjection and X is quasi $g\zeta^*$-normal space then Y is $g\zeta^*$-normal .

Proof. Let A and B be any two disjoint closed sets of Y. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint π-closed sets of X. Since X is quasi $g\zeta^*$-normal, there exist disjoint $g\zeta^*$-open sets U and V such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$.

Let G = int(cl(U)) and H = int(cl(V)). Then G and H are disjoint regular open sets of X such that $f^{-1}(A) \subset G$ and $f^{-1}(B) \subset H$. Now, we set $K = Y - f(X - G)$ and $L = Y - f(X - H)$. Then K and L are $g\zeta^*$-open sets of Y such that $A \subset K$, $B \subset L$, $f^{-1}(K) \subset G$ and $f^{-1}(L) \subset H$. Since G and H are disjoint, so are K and L. Since K and L are $g\zeta^*$-open and we obtain $A \subset g\zeta^*$-int(K), $B \subset g\zeta^*$-int(L) and $g\zeta^*$-int(K) $\cap$ $g\zeta^*$-int(L) = $\varnothing$. Therefore, Y is $g\zeta^*$-normal.

5.6 Theorem. Let $f : X \to Y$ be an almost π-continuous and almost $\pi gg\zeta^*$-closed surjection. If X is quasi $g\zeta^*$-normal space then Y is quasi $g\zeta^*$-normal.

Proof. Let A and B be any disjoint π-closed sets of Y. Since f is almost π-continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint π-closed sets of X. Since X is quasi $g\zeta^*$-normal, there exist disjoint $g\zeta^*$-open sets U and V of X such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$.

Put G = int(cl(U)) and H = int(cl(V)). Then G and H are disjoint regular open sets of X such that $f^{-1}(A) \subset G$ and $f^{-1}(B) \subset H$. By **Theorem 5.3,** there exist $\pi gg\zeta^*$-open sets K and L of Y such that $A \subset K$, $B \subset L$, $f^{-1}(K) \subset G$ and $f^{-1}(L) \subset H$. Since G and H are disjoint, so are K and L by **Theorem 3.3,** $A \subset g\zeta^*\text{-int}(K)$, $B \subset g\zeta^*\text{-int}(L)$ and $g\zeta^*\text{-int}(K) \cap g\zeta^*\text{-int}(L) = \emptyset$. Therefore, Y is quasi $g\zeta^*$-normal.

5.7 Corollary. If $f : X \to Y$ is an almost continuous and almost closed surjection and X is a normal space, then Y is quasi $g\zeta^*$-normal.

Proof. Since every almost closed function is almost $\pi gg\zeta^*$-closed. Therefore, by **Theorem 5.6,** Y is quasi $g\zeta^*$-normal.

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