CONCEPT OF COHOMOLOGY GROUP AND ITS PROPERTIES

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Abstract – The paper explores the basic ideas of Cohomology groups and its properties. We have given some interpretation of Cohomology groups and some computations are necessary. We have also mentioned its future prospects. In mathematics, Cohomology is a general term for a sequence of abelian groups associated to a topological space, often defined from a co-chain complex. Cohomology can be viewed as a method of assigning richer algebraic invariants to a space than homology.

Keywords: Cohomology groups, Homology groups, Abelian groups, algebraic invariants.

I. INTRODUCTION

In mathematics, homology is a general way of associating a sequence of algebraic objects such as abelian groups or modules to other mathematical objects such as topological spaces. It is defined as an extension of A by G is an exact sequence O $\rightarrow$ A $\rightarrow$ E $\rightarrow$ G $\rightarrow$ 1. Where A is abelian through out. Homology groups were originally defined in algebraic topology and Cohomology is a general term for a sequence of abelian groups associated to a topological space, often defined from a co-chain complex. Cohomology can be viewed as a method of assigning richer algebraic invariants to a space than homology.

There are many different cohomology theories. A particular type of mathematical object, such as a topological space or a group, may have one or more associated Cohomology theories. When the underlying object has a geometric interpretation as topological spaces do, the nth homology group represents behavior in dimension. Most Cohomology and homology groups or modules may be formulated as derived functions on appropriate abelian categories, measuring the failure of a function to be exact. From this abstract perspective, homology groups are determined by objects of a derived category. Cohomology idea can be stated first of all the Euler polyhedron formula, or Euler function. This was followed by Riemann's definition of genus and n-fold connectedness numerical invariants in 1857 and Betti's proof in 1871 of the independence of "homology numbers" from the choice of basis.

In this paper we have take two sections. First section based on cohomology groups and last one illustrates its properties.

II. COHOMOLOGY GROUPS

Definition 1 : A normalized factor set is a function f : G $\times$ G $\rightarrow$ A with

(i) $x f(y, z) - f(xy, z) + f(x, yz) - f(x, y) = 0;$

(ii) $f(x, 1) = 0 = f(1, x)$;

a normalized coboundary is a function g : G $\times$ G $\rightarrow$ A such that

(iii) $g(x, y) = xh(y) - h(xy) + h(x),$

where h:G $\rightarrow$ A is a function with

(iv) $h(1) = 0.$

The gap in obtaining an isomorphism $H^2(G, A) \cong e(G, A)$ was that the resolution of Z in previous result gave cocycles satisfying condition (i) but not (ii), and coboundaries satisfying condition (iii) but not (iv).

With these formulas before our eyes, let us return to stem covers. Representation theory deals with homomorphisms G $\rightarrow$ GL (n, k), where *k* is a field. If $k^{\#}$ is the multiplicative group of nonzero elements of k, then $k^{\#}$ is isomorphic to the centre of GL (n, k), all nonzero n $\times$ n scalar matrices.

Definition 2 : The projective general linear group is

PGL (n, k = GL (n, k) / scalars.

A projective representation of G is a homomorphism G $\rightarrow$ PGL (n, k).

Now one really prefers a representation of G, but he may only have a projective representation. The next theorem says, when k is algebraically closed, that one can exchange a projective representation of G if he pays a price : G must be replaced by a stem over (i.e., a representation group) S of G. (A more complete account of this material can be fund in [Isaacs, 1976].)

Definition 3 : For $n > 0$, define Q_n as the free G-module with basis all n-tuples $[x_1, \ldots\ldots, x_n]$ of elements of G; define Q_0 as the free G-module on the single generator [ ].

For each $n \geq 0$, $P_n \cong Q_n$, for both are free G-modules on a set in one to one correspondence with $G^{(n)}$; we give a specific isomorphism.

Definition 4 : A topological space X is acyclic if $H_0(X) \cong Z$ and $H_n(X) = 0$ for $n > 0$.

Theorem 5 : If a group G operates without fixed points on an acyclic space X, then the singular complex of X is a deleted G-free resolution of Z.

Proof : We already know the singular complex is a complex of G-free modules. The condition that X is acyclic gives an exact sequence $\ldots\ldots \to S_1(X) \to S_0(X) \to H_0(X) \to 0$, and $H_0(X) \cong Z$.

Let A be a G-module. Now $H^n(X; A)$ is the homology of the complex $\mathrm{Hom}_z(S(X), A)$. If X a cyclic and G operates on X without fixed points, then $H^n(G, A)$ is the homology of the complex $\mathrm{Hom}_G(S(X), A)$, To complete this discussion, we assume some knowledge of topology. Suppose G operates properly on X, i.e., each $x \in X$ lies in an open set U with $gU \cap U = \varnothing$, all $g \in G$, $g \neq 1$ (this implies that G operates without fixed points). One can give an isomorphism of complexes.

$$\mathrm{Hom}_z(S(X/G), A) \cong \mathrm{Hom}_G(S(X), A),$$

where A is G-trivial and X/G is the orbit space of X, and this induces isomorphisms for all $n > 0$

$$H^n(X/G\,;\, A) \cong H^n(G, A).$$

The next step is to exhibit a space X as above. Given a group G, there exists an Eilenberg–MacLane space $Y = K(G, 1)$: a path connected, "aspherical" space (i.e., the nth homotopy groups $\pi_n(Y) = 0$ for $n > 1$) having fundamental group $\pi_1(Y) \cong G$. If one defines X = Y, the universal covering space of Y, then X is acyclic, G acts properly on X, and $X/G \cong Y$. It follows that

$$H^n(K(G, 1); A) \cong H^n(G, A):$$

the cohomology of an abstract group G (with G-trivial coefficients) coincides with the cohomology of a certain topological space $Y = K(G, 1)$.

It is also true that there are isomorphisms in homololgy : If $Y = K(G, 1)$ and A is G-trivial, then

$$H_n(K(G, 1); A) \cong H_n(G, A)$$

Theorem 6 (Universal Coefficient Theorem) : If G is a group and A is G-trivial, then

$$H^n(G, A) \cong \mathrm{Hom}_z(H_n(G, Z), A) \oplus \mathrm{Ext}_Z^1(H_{n-1}(G, Z)\, A)$$

Proof : The Universal Coefficient gives such an isomorphism for any topological space Y. Choose $Y = K(G, 1)$ and use the fact that $H^n(Y\,;\, A) \cong H^n(G, A)$ and $H_n(Y;A) \cong H_n(G, A)$ for all n.

Remark : A purely algebraic proof of may be found in [Gruenberg, 1970, p. 49].

There is also a universal coefficient theorem for homology, using if A is G-trivial,

$$H_n(G, A) \cong H_n(G, Z) \otimes_Z A \oplus \mathrm{Tor}_1^Z(H_{n-1}(G, Z), A).$$

The fact that one may realize homology groups of G as homology groups of a topological space "explains". If Y is a path connected space having fundamental group π, then the Hurewicz theorem states that

$$H_1(Y; Z) \cong \pi/\pi'.$$

Setting $Y = K(\pi, 1)$ shows $H_1(Y; Z) \cong H_1(\pi, Z)$, which gives a topological.

We complete to algebra, still seeking to prove $H^2(G, A) \cong e(G, A)$.

Definition 7 : The bar resolution (or standard resolution or normalized resolution) is

$$B = \ldots. \to B_1 \xrightarrow{d_1} B_0 \xrightarrow{\varepsilon} Z \to 0,$$

where B_n is the free G-module on all $[x_1, \ldots.., x_n]$ with $x_i \in G$ and $x_i \neq 1$, and the formula for $d_n : B_n \to B_{n-1}$ is the same as in Q. (B_0 is free on the single generator [ ] :)

Remarks : 1. In order that d_n be defined, we agree that $[x_1, \ldots.., x_n] = 0$ whenever some $x_i = 1$.

2. It is not obvious that B is a complex (this does not follow immediately from $d_{\pi-1} d_n = 0$ in Q, for here we are making some of the terms in the formula equal to 0).

3. B is called the bar resolution because the original notation for $[x_1 \ldots\ldots x_n]$ was $[x_1 \mid x_2 \mid \ldots. \mid x_n]$

Theorem 8 : The bar resolution B is a G-free resolution of Z.

Proof : Again, we do not yet know B is even a complex (so far, all we know is that each B_n is G-free and the d's are G-maps). First, we construct a contracting homotopy

$$.... \leftarrow B_1 \xrightarrow{S_0} B_0 \xrightarrow{S_{-1}} Z,$$

where each s_n is a Z-map. Define

$$s_{-1} : Z \to B_0 \qquad \text{by} \quad 1 \mapsto [\ \ ]$$

and

$$s_n : B_n \to B_{n+1} \qquad \text{by} \quad x\,[x_1, ..., x_n] \mapsto [x, x_1, ..., x_n]$$

(since s_n is only a Z-map, it must be defined on a Z-free set of generators, not merely on the G-free generators $[x_1, ..., x_n]$). It is easy to check that this is a contracting homotopy, i.e.,

$$d_{n-1} S_n + S_{n-1} d_n = 1_{Bn} \quad (\text{where } d_o = \varepsilon)$$

and

$$\varepsilon s_{-1} = 1_z.$$

If we show B is a complex, then the theorem is completes. Now B_{n+1} is generated as a G-module by the subgroup im S_n, so that it suffices to show $d_n d_{n+1} = 0$ on this subgroup. We do an induction on n, noting that $0 = \varepsilon\, d_1 = s_{-1}d_1$. For the inductive step.

$$\begin{aligned} d_n dn_{+1} s_n &= d_n (1 - s_{n-1} d_n) && \text{(contracting homotopy)} \\ &= d_n - (1 - s_{n-2} d_{n-1}) d_n && \text{(ditto)} \\ &= d_n - d_n - s_{n-2} d_{n-1}\, dn = 0. \end{aligned}$$

III. COMPUTATION AND APPLICATIONS

We have given some interpretation of cohomology groups, but are we in a stronger position having done so? In order to use cohomology and homology, some computations are necessary.

Theorem 9 : Let G be a finite group of order m. For every G-module A and every n > 0,

$$mH^n(G, A) = 0 = mH_n (G, A).$$

Proof : We use the unnormalized standard resolution Q. If $f : Q_n \to A$.

defined $g:Q_{n-1} \to A$ by

$$g\,(x_1, ..., x_{n-1}) = \sum_{x \in G} f(x_1, ... x_{n-1}, x).$$

Now sum the coboundary formula

$$(df)\,(x_1, ..., x_{n+1}) = x_1 f\,(x_2, ... x) + \sum_{i=1}^{n-2} (-1)^i f(x_1, ... x_i x_i + 1, ..., x)$$

$$+ (-1)^{n-1} f(x_1 ..... x_n x) + (-1)^n f\,(x_1, ..., x_n)$$

over all $x = x_{n+1}$ in G. In the next to last term, as x varies over G, so does $x_n x$. Therefore, if f is cycle, then $dj = 0$ and

$$0 = x_1\, g(x_2, ..., x_n) + \sum_{i=1}^{n-2} (-1)^i\ g(x_1, ..., x_i x_{i+1}, ...x_n)$$

$$+ (-1)^{n-1} g(x_1, ..., x^{n-2}) + m\,(-1)^n f\,(x_1, ... x_n)$$

(the last term is independent of x). Hence

$$0 = dg + m\,(-1)^n f,$$

and mf is a coboundary.

The same proof works for homology, and we merely set up notation. If $f(x_1, ... x_n, a) = [x_1, ..., x_n] \otimes a$, where $a \in A$, then define

The proof proceeds as above, but remember that one begins with an element of the form $\sum {}_j^p{=}1\, f\,(x_1^i, ..., x_n^j, a^j)$.

Definition : A group G has cohomological dimension ≤ n, denoted cd (G) ≤ n, if H^k (G, A) = 0 for all G-modules A and all k > n. If n is the least such integer, one defines cd (G) = n; if no such integer n exists, define cd(G) = ∝.

Exercises : 9.1 Prove that cd(G) = 0 if and only if G = {1}.

Since we know that if G is free of rank > 0, then cd (G) = 1.

Also if H^{n+1} (G, A) = 0 for all G-modules A, then H^k (G, A) = 0 for all k > n and all G-modules A.

Let G be free abelian with basis S = $\{x_1, ..., x_n\}$. Prove that ZG ≅ S^{-1} $(x_1, ..., x_n]$; use Hilbert's Syzygy Theorem to prove cd (G) ≤ n + 1.

Remarks : 1. The ring of Laurent polynomials in x, coefficients in Z, consists of all formal sums

$$\sum_{i=k}^{n} m_i x^i, \qquad m_i \in Z, \ \text{yaha se karna hai}$$

k, n (possibility negative) integers, with obvious addition and multiplication. One may easily generalize this definition to several (commuting) variables, and observe, using, that Laurent polynomials in n variables, coefficients in Z, in ZG, where G is free abelian of rank n.

2. If G is free abelian of rank n, then cd(G) = n. The reason the bound in just above exercise is too high is that global dimension ≤ n demands vanishing of Ext_{ZG}^{k} (B, A), all k > n and all pairs of G-modules B, A, whereas cd(G) ≤ n only demands such vanishing in the special case B = Z.

Is there a relation between cd(S) and cd(G) when S is a subgroup of G ?

Theorem 11 (Shapiro's Lemma) : If S is a subgroup of G and A is an S-module, then, for all n ≥ 0.

$$H^n(S, A) \cong H^n(G, \mathrm{Hom}_s(ZG, A)).$$

Proof : First of all, the right side makes sense, for Hom_s(ZG, A) may be regarded as a G-module, (i.e., as in the adjoint isomorphism). A mixed identity, arising from the adjoint isomorphism, gives

$$\mathrm{Ext}_{ZS}^{n}(ZG \otimes_G Z, A) \cong \mathrm{Ext}_{ZG}^{n}(Z, \mathrm{Hom}_s(ZG, A)).$$

Since $ZG \otimes_G Z \cong Z$, this is the desired isomorphism

$$H^n(S, A) \cong H^n(G, \mathrm{Hom}_s(ZG, A)).$$

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