Results in the theory of perturbed Differential equations and Integral equations with non-linearity conditions

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Abstract: In present paper, some new results related to the existence, strict and non-strict inequalities and existence of the maximal and minimal solutions are proved for perturbed differential and integral equations with non-linearity conditions.

Index Terms: Perturbed differential equation; Existence theorem, Differential and Integral inequalities, maximal and minimal solutions.

1. Introduction :

Given a bounded interval $I = [t_0, t_0 + p]$ in connected set R for some fixed $t_0, p \in R$ *with* $p > 0$. Consider the initial value problems for perturbed hybrid differential equations (in short PHDE),

$$\frac{d}{dt}\left[\frac{x(t) - g(t, x(\eta(t)))}{f(t, x(\eta(t)))}\right] = q(t, x(\eta(t))) \quad a.e.\, t \in I \qquad (1.1)$$

$$x(t_0) = x_0 \in R$$

Where $f : I \times R \to R_+ - \{0\}$, $g, q : I \times R \to R$ *and* $\eta : I \to I$.

By a solution of PHDE (1.1) we mean a function $x \in C(I, R)$ such that

i) the function $t \to \dfrac{x - g(t, x(\eta(t)))}{f(t, x(\eta(t)))}$ is absolutely continuous function for each $x \in R$, and

ii) x satisfies the equations in (1.1)

The significance of the study of Hybrid differential equation lies in the fact that they cover diverse dynamic systems as a special cases. The forethought of Hybrid Differential equations is absolute in the work of Krasnoselskii[3] and extensively treated in the various papers on Hybrid differential equations with different perturbations. See Burton[4], Bellale[7] and the references therein. This class of Hybrid differential equations includes the perturbations of original differential equations in different ways. In this paper, we commence the new results in the theory of perturbed differential and integral equations with non-linearity conditions and prove the basic results such as existence theorem, maximal and minimal solutions etc.

We pretense that the results of this paper are new and important contribution to the theory of non-linear ordinary differential equations.

2. Strict and Non-Strict Inequalities :

We need regularly the following hypothesis in what follows:

(A_0) The function $x \to \dfrac{x - g(t, x(\eta(t)))}{f(t, x(\eta(t)))}$ is increasing in R for all $t \in I$.

We begin by proving the basic results dealing with hybrid differential inequalities.

Theorem 2.1 : Assume that hypothesis (A_0) hold. Suppose that there exist $y, z \in C(I, R)$ such that

$$\frac{d}{dt}\left[\frac{y(t) - g(t, y(\eta(t)))}{f(t, y(\eta(t)))}\right] \le q(t, y(\eta(t))) \quad a.e. \, t \in I \quad (2.1)$$

and

$$\frac{d}{dt}\left[\frac{z(t) - g(t, z(\eta(t)))}{f(t, z(\eta(t)))}\right] \ge q(t, z(\eta(t))) \quad a.e. \, t \in I \quad (2.2)$$

If one of the inequalities (2.1) and (2.2) is strict and

$$y(t_0) < z(t_0) \quad (2.3)$$

Then $\quad y(t) < z(t) \quad (2.4)$

For all $t \in I$.

Proof:- We prove this result by contradiction method.

Suppose that the inequality (2.4) is false. Then the set Z* defined by

$$Z^* = \{t \in I \setminus Z \mid y(t) \ge z(t)\} \quad (2.5)$$

Is non-empty, when Z is a set of measure zero in I.

Denote $t_1 = \inf Z^*$.

Without loss of generality we may assume that $y(t_1) = z(t_1) \; and \; y(t) < z(t) \; for \; all \; t < t_1$.

Assume that

$$\frac{d}{dt}\left[\frac{z(t) - g(t, z(\eta(t)))}{f(t, z(\eta(t)))}\right] > q(t, z(\eta(t))) \quad for \; all \; t \in I.$$

Denote

$$Y(t) = \left[\frac{y(t) - g(t, y(\eta(t)))}{f(t, y(\eta(t)))}\right]$$

and

$$Z(t) = \left[\frac{z(t) - g(t, z(\eta(t)))}{f(t, z(\eta(t)))}\right] \quad for \; all \; t \in I.$$

Now by definition of continuity of y and z together with the inequality (2.3) implies that, there exists a point $t_1 > t_0$ such that

$$y(t_1) = z(t_1) \; and \; y(t) < z(t) \quad (2.6)$$

$for \; all \; t_0 \le t < t_1$

As hypothesis (A_0) holds, it follows from (2.5) that

$$Y(t_1) = \left[\frac{y(t_1) - g(t_1, y(\eta(t_1)))}{f(t_1, y(\eta(t_1)))}\right]$$

$$= \left[\frac{z(t_1) - g(t_1, z(\eta(t_1)))}{f(t_1, z(\eta(t_1)))}\right]$$

$$Y(t_1) = Z(t_1). \quad \{\because t_1 > t_0\}$$

and

$$Y(t)=\left[\frac{y(t)-g(t,y(\eta(t)))}{f(t,y(\eta(t)))}\right]$$
$$<\left[\frac{z(t)-g(t,z(\eta(t)))}{f(t,z(\eta(t)))}\right]$$
$$Y(t)<Z(t) \qquad (2.7)$$

for all $t_0 \le t < t_1$.

The above relation (2.7) further yields

$$\frac{Y(t_1+h)-Y(t_1)}{h} > \frac{Z(t_1+h)-Z(t_1)}{h}$$

For small h<0. Taking the limit as $h \to 0$, we obtain

$$Y'(t_1) \ge Z'(t_1). \qquad (2.8)$$

Hence, from (2.7) and (2.8), we get

$$g(t_1, y(\eta(t_1))) \ge Y'(t_1) \ge Z'(t_1) > g(t_1, z(\eta(t_1))).$$

This is a contradiction and the proof is complete.

The next result deals with the nonstrict inequality for the PHDE (1.1) on I which requires a one sided Lipschitz condition.

Theorem 2.2 :- Assume that the hypotheses of Theorem 2.1 hold. Suppose that there exists a real number L>0 such that $q(t,y(\eta(t)))-q(t,z(\eta(t))) \le L \sup_{t_0\le s\le t}\left[\frac{y(s)-g(s,y(\eta(t)))}{f(s,(\eta(t)))}-\frac{z(s)-g(s,z(\eta(t)))}{f(s,(\eta(t)))}\right]$ (2.9)

Whenever

$y(s) \ge z(s), t_0 \le q \le t.$ *Then,*

$$y(t_0) \le z(t_0) \qquad (2.10)$$

implies $$y(t) \le z(t) \qquad (2.11)$$

for all $t \in I$.

Proof:- Let $\in > 0$ and let a real number $L > 0$ be given. Set

$$\frac{z_\in(t)-g(t,z_\in(\eta(t)))}{f(t,z_\in(\eta(t)))} = \frac{z(t)-g(t,z(\eta(t)))}{f(t,z(\eta(t)))} + \in e^{2L(t-t_0)} \qquad (2.12)$$

So that

$$\frac{z_\in(t)-g(t,z_\in(\eta(t)))}{f(t,z_\in(\eta(t)))} > \frac{z(t)-g(t,z(\eta(t)))}{f(t,z(\eta(t)))}.$$

Define

$$Z_\in(t) = \frac{z_\in(t)-g(t,z_\in(\eta(t)))}{f(t,z_\in(\eta(t)))} \quad and \quad Z(t) = \frac{z(t)-g(t,z(\eta(t)))}{f(t,z(\eta(t)))}$$

for $t \in I$.

Now using the one-sided Lipschitz condition (2.9), we obtain

$$q(t,z_\in(\eta(t)))-q(t,z(\eta(t))) \le L \, \underset{t_0\le s\le t}{Sup}\left[Z_\in(s)-Z(s)\right] = L \in e^{2L(t-t_0)}.$$

Now,

$$\begin{aligned}
Z_\in'(t) &= Z'(t)+2L \in e^{2L(t-t_0)} \\
&\ge q(t,z(\eta(t)))+2L \in e^{2L(t-t_0)} \\
&\ge q(t,z_\in(\eta(t)))+2L \in e^{2L(t-t_0)} - L \in e^{2L(t-t_0)} \\
&= q(t,z_\in(\eta(t)))+L \in e^{2L(t-t_0)} \\
&> q(t,z_\in(\eta(t)))
\end{aligned}$$

For all $t \in I$. Also, we have

Also we have $Z_\in(t_0) > Z(t_0) \ge Y(t_0)$.

Now we apply Theorem 2.1 with $z = z_{\in}$ to yield

$$Y(t) < Z_{\in}(t)$$

For all $t \in I$. On taking $\in \to 0$ in the above inequality, we get

$$Y(t) \leq Z(t)$$

Which is in view of hypothesis (A_0) implies that (2.11) holds on I. This completes the proof.

Remark 2.1 The conclusion of Theorem 2.1 and 2.2 also remain true if we replace the derivatives in the inequalities (2.1) and (2.2) by Dini-derivative D_ of the function $\dfrac{x(t) - g(t, x(\eta(t)))}{f(t, x(\eta(t)))}$ on the bounded interval I.

3. Existence Result:

In this part we give a proof of an existence result for the perturbed hybrid differential equation (1.1) on a closed and bounded interval $I = [t_0, t_0 + p]$ under the mixed Lipschitz and Compactness conditions on the nonlinearities involved in it.

We put the perturbed hybrid differential equation (1.1) in the space C(I,R) of continuous real valued functions defined on $[t_0, t_0 + p]$.first we define a supremum norm $\| \square \|$ in C(I,R) as

$$\|x\| = \sup_{t \in I} |x(t)|$$

and a multiplication " . " in C(I,R) by

$$(x \cdot y)(t) = (xy)(t) = x(t)y(t)$$

$for\ x, y \in C(I, R)$.Clearly C(I,R) is a Banach algebra with respect to the above supremum norm and Multiplication in it. By L^1(I,R) we denote the space of Lebesgue integrable real valued functions on I equipped with the norm $\| \cdot \|_{L^1}$ defined by

$$\| x \|_{L^1} = \int_{t_0}^{t_0 + p} |x(s)| ds.$$

We prove the existence of solutions for the PHDE (1.1) via the following hybrid fixed point theorem in the Banach algebras [7].

Theorem 3.1 Suppose that V is closed, convex and bounded subset of the Banach algebra E and let $A, C : E \to E$ and $B : V \to E$ be three operators such that

a) A and C are Lipschitz with Lipschitz constants α *and* β respectively,

b) B is compact and continuous,

c) $x = AxBy + Cx \quad for \quad all \quad y \in V \Rightarrow x \in V, and$

d) $\alpha M + \beta < 1,\ where\ M = \|B(V)\| = \sup\{\|Bx\| : x \in V\}$.

Then the operator equation $AxBy + Cx = x$ has a solution in V.

We consider the following hypothesis in what follows.

(A_1) There exists a constant $L_1 > 0\ and\ L_2 > 0$ such that

$$|f(t, x) - f(t, y)| \leq L_1 |x - y|$$

and

$$|g(t,x)-g(t,y)|\le L_2|x-y|$$

for all $t\in I$ *and* $x,y\in R.$

(A_2) There exists a continuous function $h\in L^1(I,R)$ such that $|q(t,x)|\le h(t)\;\; t\in I,$ for all $x\in R.$

To prove the theorem the following lemma is useful which is step by step discussed.

Lemma 3.1 Assume that hypothesis (A_0) holds. Then for any continuous function $h\in L^1(I,R_+)$, the function $x\in C(I,R_+)$ is a solution of the PHDE

$$\frac{d}{dt}\left[\frac{x(t)-g(t,x(\eta(t)))}{f(t,x(\eta(t)))}\right]=h(t) \qquad a.e.\, t\in I \tag{3.1}$$

$$x(0)=x_0\in R$$

if and only if x satisfies the perturbed hybrid integral equation(PHIE)

$$x(t)=g(t,x(\eta(t)))+f(t,x(\eta(t)))\left(\frac{x_0-g(t_0,x_0)}{f(t_0,x_0)}+\int_{t_0}^{t}h(s)ds\right), t\in I \tag{3.2}$$

Proof :- Let $h\in L^1(I,R_+)$.First suppose that x satisfy the PHDE (3.1). Then by definition, $\frac{x(t)-g(t,x(\eta(t)))}{f(t,x(\eta(t))}$ is almost everywhere continuous on the interval $I=[t_0,t_0+p)$ and so almost everywhere differentiable there, whence $\frac{d}{dt}\left[\frac{x(t)-g(t,x(\eta(t)))}{f(t,x(\eta(t)))}\right]$ is integrable on I.

Integrating (3.1) from t_0 to t, we have

$$\int_{t_0}^{t}\frac{d}{dt}\left[\frac{x(t)-g(t,x(\eta(t)))}{f(t,x(\eta(t)))}\right]dt=\int_{t_0}^{t}h(t)dt$$

After simplifying we obtain

$$\frac{x(t_0)-g(t_0,x(\eta(t_0)))}{f(t_0,x(\eta(t_0)))}=\frac{x_0-g(t_0,x_0)}{f(t_0,x_0)} \text{ on I.}$$

Conversely suppose that x satisfies

$$x(t)=g(t,x(\eta(t)))+f(t,x(\eta(t)))\left(\frac{x_0-g(t_0,x_0)}{f(t_0,x_0)}+\int_{t_0}^{t}h(s)ds\right), t\in I$$

Differentiating above equation we get $\frac{d}{dt}\left[\frac{x(t)-k(t,x(\eta(t)))}{f(t,x(\eta(t)))}\right]=h(t) \qquad a.e.\, t\in I$

Now substituting $t=t_0$ in (3.2) ,we get

$$\frac{x(t_0)-g(t_0,x(\eta(t_0)))}{f(t_0,x(\eta(t_0)))}=\frac{x_0-g(t_0,x_0)}{f(t_0,x_0)}.$$

Since the mapping $x\mapsto\frac{x-g(t,x)}{f(t,x)}$ is an increasing in R almost everywhere for $t\in I$. Also the mapping $x\mapsto\frac{x-g(t_0,x)}{f(t_0,x)}$ is one -one in R, Whence $x(t_0)=x_0$ .This completes the proof of the lemma.

Now we are going to discuss the following existence theorem for the PHDE (1.1) on the interval I.

Theorem 3.2 : Assume that the hypothesis (A_0) – (A_2) hold. Further, if

$$L_1\left(\left|\frac{x_0-g(t_0,x_0)}{f(t_0,x_0)}\right|+\|h\|_{L^1}\right)+L_2<1 \tag{3.3}$$

then the PHDE (1.1) has a solution defined on I.

Proof : Set E=C(I,R) and define a subset V of E defined by

$$V=\{x\in E\,\|x\|\le N\} \tag{3.4}$$

Where $N=\dfrac{F_0\left(\left|\dfrac{x_0-g(t_0,x_0)}{f(t_0,x_0)}\right|+\|h\|_{L^1}\right)+K_0}{1-L_1\left(\left|\dfrac{x_0-g(t_0,x_0)}{f(t_0,x_0)}\right|+\|h\|_{L^1}\right)-L_2}$,

and $F_0=\sup_{t\in I}|f(t,0)|$ *and* $K_0=\sup_{t\in I}|g(t,0)|$.

Clearly V is a closed, convex and bounded subset of the Banach algebra E.

Now using the hypotheses (A_0) and (A_2) and application of lemma 3.1, it can be easily show that the PHDE (1.1) is equivalent to the non-linear PHIE

$$x(t)=g(t,x(\eta(t)))+[f(t,x(\eta(t)))]\left(\frac{x_0-g(t_0,x_0)}{f(t_0,x_0)}+\int_{t_0}^{t}q(s,x(\eta(s)))ds\right) \tag{3.5}$$

for $t\in I$.

We define three operators $A,C:E\to E$ *and* $B:V\to E$ by

$$Ax(t)=f(t,x(\eta(t)))\ \ ,t\in I \tag{3.6}$$

$$Bx(t)=\left(\frac{x_0-g(t_0,x_0)}{f(t_0,x_0)}+\int_{t_0}^{t}q(s,x(\eta(s)))ds\right),t\in I. \tag{3.7}$$

and

$$Cx(t)=g(t,x(\eta(t))),t\in I. \tag{3.8}$$

Then the Hybrid integral equation (3.5) is transformed into an operator equation as

$$Ax(t)+Bx(t)+Cx(t)=x(t)\ \ ,t\in I \tag{3.9}$$

Our aim is to show that the operators A,B and C satisfy all the conditions of theorem (3.1).

i.e., we first show that A is a Lipschitz operator on E with the Lipschitz constant L_1.

Let x, y be any two members in E, then by hypothesis (A_1)

$$\begin{aligned}|A(x(t))-A(y(t))|&=|f(t,x(\eta(t)))-f(t,y(\eta(t)))|\\&\le L_1|x(\eta(t))-y(\eta(t))|\\&\le L_1\|x-y\|\quad for\ \ all\ \ t\in I.\end{aligned}$$

Taking supremum over t, we obtain

$$\|Ax-Ay\|\le L_1\|x-y\|$$

for all $x,y\in E$.This shows that A is a Lipschitz operator on E with the Lipschitz constant L_1 Similarly, it can be shown that C is also a Lipschitz operator on E with the Lipschitz constant L_2.

Now we have to show that B is compact and continuous operator on V into E. First we prove, B is continuous on V.

Let $\{x_n\}$ be a sequence in V converging to a point $x\in V$.Then by dominated convergence theorem for integration, we obtain

$$\begin{aligned}\lim_{n\to\infty}Bx_n(t)&=\lim_{n\to\infty}\left(\frac{x_0-g(t_0,x_0)}{f(t_0,x_0)}+\int_{t_0}^{t}q(s,x_n(\eta(s)))ds\right)\\&=\frac{x_0-g(t_0,x_0)}{f(t_0,x_0)}+\lim_{n\to\infty}\int_{t_0}^{t}q(s,x_n(\eta(s)))ds\\&=\frac{x_0-g(t_0,x_0)}{f(t_0,x_0)}+\int_{t_0}^{t}\left[\lim_{n\to\infty}q(s,x_n(\eta(s)))\right]ds\\&=\frac{x_0-g(t_0,x_0)}{f(t_0,x_0)}+\int_{t_0}^{t}q(s,x(\eta(s)))ds\\&=Bx(t)\ \ ,\quad for\ \ t\in I.\end{aligned}$$

Moreover, it can be shown as below that $\{Bx_n\}$is an equicontinuous sequence of functions in X. Now, following the arguments similar to that given in Granas et.al[2] , it is proved that B is a continuous

operator on V.
Now we have to show B is compact operator on V.
To prove this it is sufficient to show that B(V) is a uniformly bounded and equicontinuous set in E.
Let $x \in V$ be an arbitrary element . Then by hypothesis (A_2).

$$\begin{aligned}|Bx(t)| &\le \left|\frac{x_0 - g(t_0, x_0)}{f(t_0, x_0)}\right| + \int_{t_0}^{t} |q(s, x(\eta(s)))| ds \\ &\le \left|\frac{x_0 - g(t_0, x_0)}{f(t_0, x_0)}\right| + \int_{t_0}^{t} h(s) ds \\ &\le \left|\frac{x_0 - g(t_0, x_0)}{f(t_0, x_0)}\right| + \|h\|_{L^1} \quad \textit{for all} \;\; t \in I.\end{aligned}$$

Taking supremum over t,

$$\sup_t |Bx(t)| \le \sup_t \left\{ \left|\frac{x_0 - g(t_0, x_0)}{f(t_0, x_0)}\right| + \|h\|_{L^1} \right\}$$

$$\therefore \|Bx\| \le \left|\frac{x_0 - g(t_0, x_0)}{f(t_0, x_0)}\right| + \|h\|_{L^1} \;, \textit{ for all } x \in V.$$

This shows that B is uniformly bounded on V. Again let $t_1, t_2 \in I$ .then for any $x \in V$, we have

$$\begin{aligned}|Bx(t_1) - Bx(t_2)| &= \left| \int_{t_0}^{t_1} q(s, x(\eta(s)))ds - \int_{t_0}^{t_2} q(s, x(\eta(s)))ds \right| \\ &\le \left| \int_{t_2}^{t_1} q(s, x(\eta(s)))ds \right| \\ &\le |p(t_1) - p(t_2)|\end{aligned}$$

Where $p(t) = \int_{t_0}^{t} h(s)ds.$ Since the function p is continuous on compact I, it is uniformly continuous.

Hence, for $\in > 0$, there exists a $\delta > 0$ such that $|t_1 - t_2| < \delta \Rightarrow |Bx(t_1) - Bx(t_2)| < \in$ for all $t_1, t_2 \in I$ and for all $x \in V$. This shows that B(V) is an equicontinuous set in E.
Now the set B(V) being uniformly bounded and equicontinuous set in E, so it is compact by Arzela-Ascoli theorem. This proves, B is a continuous and compact operator on V.

Now we have to show that hypothesis (c) of Theorem 3.1 is satisfied. Let $x \in E \textit{ and } y \in V$ be arbitrary such that $x = AxBy + Cx$ Then, by assumption (A_1) , we have

$0 < \in \le \in_0$.

Taking supremum over t,

$$\|x\| \le \frac{F_0 \left(\left|\frac{x_0 - g(t_0, x_0)}{f(t_0, x_0)}\right| + \|h\|_{L^1} \right) + K_0}{1 - L_1 \left(\left|\frac{x_0 - g(t_0, x_0)}{f(t_0, x_0)}\right| + \|h\|_{L^1} \right) - L_2} = N$$

This shows that hypothesis (c) of Theorem (3.1) is satisfied.
Finally we have to prove hypothesis (d) of Theorem (3.1) holds.

$$M = \|B(V)\| = Sup\{\|Bx\| : x \in V\} \le \left|\frac{x_0 - g(t_0, x_0)}{f(t_0, x_0)}\right| + \|h\|_{L^1}$$

We have *and so* ,

$$L_1 M + L_2 \le L_1 \left(\left|\frac{x_0 - g(t_0, x_0)}{f(t_0, x_0)}\right| + \|h\|_{L^1} \right) + L_2 < 1.$$

Thus all the conditions of theorem (3.1) are satisfied and hence the operator equation

$AxBy + Cx = x$ has a solution in V.As a result, the PHDE (1.1) has a solution defined on I. This completes the proof.

4. Maximal and Minimal Solutions:

In this section we shall discuss the existence of maximal and minimal solutions for PHDE (1.1) on $I = [t_0, t_0 + p]$.

Definition 4.1: - A solution r of the PHDE (1.1) is said to be Maximal if for any other solution x to the PHDE (1.1) one has $x(t) \le r(t),\ for\ all\ t \in I$ .Again, a solution ρ of the PHDE (1.1) is said to be minimal if $\rho(t) \le x(t),\ for\ all\ t \in I$, where x is any solution of the PHDE (1.1) existing on I.

We study the case of Maximal solutions only , as the case of minimal solution is similar and can be proved with the suitable and appropriate modifications.

Given a arbitrary small real number $\in > 0$, consider the following IVP of PHDE

$$\frac{d}{dt}\left[\frac{x(t) - g(t, x(\eta(t)))}{f(t, x(\eta(t)))}\right] = q(t, x(\eta(t))) + \in \quad a.e.\ t \in I \tag{4.1}$$

$$x(t_0) = x_0 + \in$$

Where $f \in C(I \times R, R \setminus \{0\}), g, q \in C(I \times R, R)\ and\ \eta : I \to I.$

An existence theorem for the PHDE (1.1) can be stated as follows:

Theorem 4.1 Assume that the hypothesis (A_0)-(A_2) hold. Suppose also that the inequality (3.3) holds. Then for every small number $\in > 0$, the PHDE (4.1) has a solution defined on I.

Proof :-By hypothesis , since

$$L_1\left(\left|\frac{x_0 - g(t_0, x_0)}{f(t_0, x_0)}\right| + \|h\|_{L^1}\right) + L_2 < 1,$$

There exists an $\in_0 > 0$ such that

$$L_1\left(\left|\frac{x_0 + \in - g(t_0, x_0 + \in)}{f(t_0, x_0 + \in)}\right| + \|h\|_{L^1} + \in p\right) + L_2 < 1 \tag{4.2}$$

for all $0 < \in \le \in_0$ .Now rest of the proof is similar to Theorem (3.2)

Our main existence theorem for maximal solution for the PHDE(1.1) can be stated as

Theorem 4.2 Assume that the hypotheses (A_0)-(A_2) hold. Further if the condition (3.3) holds, then the PHDE (1.1) has a maximal solution defined on I.

Proof:- Let $\{\in_n\}_0^\infty$ be a decreasing sequence of positive real numbers such that $\lim_{n\to\infty} \in_n = \in_0$,where $\in_0$ is a positive real number satisfying the inequality

$$L_1\left(\left|\frac{x_0 + \in_0 - g(t_0, x_0 + \in_0)}{f(t_0, x_0 + \in_0)}\right| + \|h\|_{L^1} + \in_0\ p\right) + L_2 < 1 \tag{4.3}$$

The number $\in_0$ exists in view of the inequality (3.3). Then for any solution u of the PHDE(1.1), by Theorem 2.1,one has

$$t_1, t_2 \in I \tag{4.4}$$

For all $t \in I\ and\ n \in N \bigcup \{0\}$,where $r(t, \in_n)$ is a solution of the PHDE,

$$\frac{d}{dt}\left[\frac{x(t) - g(t, x(\eta(t)))}{f(t, x(\eta(t)))}\right] = q(t, x(\eta(t))) + \in_n \quad a.e.\ t \in I \tag{4.5}$$

$$x(t_0) = x_0 + \in_n \in R$$

Defined on I.

Since, by Theorem 3.1 and 3.2 ,{ $r(t, \in_n)$ } is a decreasing sequence of positive real numbers, the limit

$$r(t) = \lim_{n\to\infty} r(t, \in_n) \tag{4.6}$$

exists. We show that the convergence in (4.6) is uniform on I. To finish ,it is enough to prove that the sequence { $r(t,\in_n)$ } is equicontinuous in C(I,R). Let $t_1,t_2 \in I$ be arbitrary. Then

$$
\begin{aligned}
&|r(t_1,\in_n)-r(t_2,\in_n)| \\
&=|g(t_1,r(t_1,\in_n))-g(t_2,r(t_2,\in_n))| \\
&\quad+\left|[f(t_1,r(t_1,\in_n))]\left(\frac{x_0+\in_n}{f(t_0,x_0+\in_n)}+\int_{t_0}^{t_1} q(s,r_{\in_n}(s))ds+\int_{t_0}^{t}\in_n ds\right)\right. \\
&\quad\left.-[f(t_2,r(t_2,\in_n))]\left(\frac{x_0+\in_n}{f(t_0,x_0+\in_n)}+\int_{t_0}^{t_2} q(s,r_{\in_n}(s))ds+\int_{t_0}^{t_2}\in_n ds\right)\right| \\
&\le|g(t_1,r(t_1,\in_n))-g(t_2,r(t_2,\in_n))| \\
&\quad+\left|[f(t_1,r(t_1,\in_n))]\left(\frac{x_0+\in_n}{f(t_0,x_0+\in_n)}+\int_{t_0}^{t_1} q(s,r_{\in_n}(s))ds+\int_{t_0}^{t}\in_n ds\right)\right. \\
&\quad\left.-[f(t_2,r(t_2,\in_n))]\left(\frac{x_0+\in_n}{f(t_0,x_0+\in_n)}+\int_{t_0}^{t_2} q(s,r_{\in_n}(s))ds+\int_{t_0}^{t_2}\in_n ds\right)\right| \\
&\quad+\left|[f(t_2,r(t_2,\in_n))]\left(\frac{x_0+\in_n}{f(t_0,x_0+\in_n)}+\int_{t_0}^{t_1} q(s,r_{\in_n}(s))ds+\int_{t_0}^{t_2}\in_n ds\right)-[f(t_2,r(t_2,\in_n))]\left(\frac{x_0+\in_n}{f(t_0,x_0+\in_n)}+\int_{t_0}^{t_2} q(s,r_{\in_n}(s))ds+\int_{t_0}^{t_2}\in_n ds\right)\right| \\
&\le|g(t_1,r(t_1,\in_n))-g(t_2,r(t_2,\in_n))| \\
&\quad+|[f(t_1,r(t_1,\in_n))]-[f(t_2,r(t_2,\in_n))]|\left(\left|\frac{x_0+\in_n}{f(t_0,x_0+\in_n)}\right|+\|h\|_{L^1}+\in_n p\right) \\
&\quad+F[|p(t_1)-p(t_2)|+|t_1-t_2|\in_n \qquad (4.7)
\end{aligned}
$$

where

$$F=\sup_{(t,x)\in I\times[-N,N]}|f(t,x)| \; and \; p(t)=\int_{t_0}^{t} h(s)ds.$$

Since f and g are continuous on compact set $I\times[-N,N]$, they are uniformly continuous there. Hence

$$|f(t_1,r(t_1\in_n))-f(t_2,r(t_2\in_n))|\to 0 \quad as \quad t_1\to t_2$$

and

$$|g(t_1,r(t_1,\in_n))-g(t_2,r(t_2,\in_n))|\to 0 \quad as \quad t_1\to t_2$$

uniformly for all $n\in N$. Similarly , since the function p is continuous on compact set I, it is uniformly continuous and hence

$$|p(t_1)-p(t_2)|\to 0 \quad as \; t_1\to t_2$$

uniformly for all $t_1,t_2\in I$.

Therefore, from the above inequality (4.7), it follows that

$$|r(t_1,\in_n)-r(t_2,\in_n)|\to 0 \; as \; t_1\to t_2$$

uniformly for all $n\in N$. Therefore ,

$$r(t,\in_n)\to r(t) \; as \; n\to\infty$$

for all $t\in I$.

Next , we show that the function $r(t)$ is a solution of the PHDE(3.1) defined on I.

Now, since $r(t,\in_n)$ is a solution of the PHDE (4.5), we have

$$r(t,\in_n)=\left[f(t,r(t,\in_n))\right]\left(\frac{x_0+\in_n}{f(t_0,x_0+\in_n)}+\int_{t_0}^{t_1}q(s,r_{\in_n}(s))ds+\int_{t_0}^{t}\in_n\,ds\right) \qquad (4.8)$$
$$+g(t,r(t,\in_n))$$

for all $t\in I$. Taking the limit as $n\to\infty$ in the above equation (4.8) yields

$$r(t,\in_n)=g(t,r(\eta(t)))+\left[f(t,r(\eta(t)))\right]\left(\frac{x_0-g(t_0,x_0)}{f(t_0,x_0)}+\int_{t_0}^{t}q(s,r(\eta(s)))ds\right)$$

for all $t\in I$. Thus the function r is a solution of the PHDE(1.1) on I. Finally from the inequality (4.4) it follows that

$$u(t)\le r(t)$$

for all $t\in I$. Hence the PHDE(1.1) has a maximal solution on I. This completes the proof.

5. Comparison Theorems:

The main purpose of differential inequalities is to find the bound for the solution set for the differential inequality related to the PHDE (1.1).In this section we prove that the maximal and minimal solutions serve the bounds for the solutions of the related differential inequality to PHDE (1.1) on $I=[t_0,t_0+p]$.

Theorem 5.1 Assume that the hypotheses (A_0)-(A_2) hold. Suppose that the condition (3.3) holds. Further, if there exists a function $u\in C(I,R)$ such that

$$\frac{d}{dt}\left[\frac{u(t)-g(t,u(\eta(t)))}{f(t,u(\eta(t)))}\right]\le q(t,u(\eta(t))),\quad t\in I \qquad (5.1)$$
$$u(t_0)\le x_0,$$

Then , $$u(t)\le r(t) \qquad (5.2)$$

for all $t\in I$,where r is a maximal solution of the PHDE (1.1) on I.

Proof:- Let $\in>0$ be arbitrary small real number. Then by theorem (4.2), $r(t,\in)$ is a Maximal solution of the PHDE (4.1) and that the limit

$$r(t)=\lim_{\in\to 0}r(t,\in) \qquad (5.3)$$

is uniform on I and the function r is a maximal solution of the PHDE (1.1) on I. Hence we obtain

$$\frac{d}{dt}\left[\frac{r(t,\in)-g(t,r(\eta(t),\in))}{f(t,r(\eta(t),\in))}\right]=q(t,r(\eta(t),\in))+\in,\ t\in I \qquad (5.4)$$
$$r(t_0,\in)=x_0+\in$$

From above inequality it follows that

$$\frac{d}{dt}\left[\frac{r(t,\in)-g(t,r(\eta(t),\in))}{f(t,r(\eta(t),\in))}\right]>q(t,r(\eta(t),\in))\quad a.e.\,t\in I \qquad (5.5)$$
$$r(t_0,\in)>x_0$$

Now we apply theorem 2.1 to the inequalities (5.1) and (5.5) and conclude that

$$u(t)<r(t,\in) \qquad (5.6)$$

For all $t\in I$. This further in view of limit (5.3) implies that inequality (5.2) holds on I.

This completes the proof.

Theorem 5.2 Assume that the hypotheses (A_0)-(A_2) hold. Suppose that the condition (3.3) holds. Further, if there exists a function $v \in C(I,R)$ such that

$$\frac{d}{dt}\left[\frac{v(t)-g(t,v(\eta(t)))}{f(t,v(\eta(t)))}\right] \geq q(t,v(\eta(t))), \quad a.e.\, t \in I \tag{5.7}$$

$$v(t_0) \geq x_0.$$

Then ,

$$\rho(t) \leq v(t) \tag{5.8}$$

for all $t \in I$, where ρ is a minimal solution of the PHDE (1.1)on I.

Note that, Theorem 5.1 is useful to prove the boundedness and uniqueness of the solutions for PHDE (1.1) on I.A result in this direction is

Theorem 5.3 : Assume that the hypotheses (A_0)-(A_2) hold and let the condition (3.3) holds. Suppose that there exists a function $G: I \times R_+ \to R_+$ such that

$$|g(t,x_1(\eta(t))-g(t,x_2(\eta(t))| \leq G\left(t, \max_{s\in[t_0,t]}\left|\frac{(x_1(s)-g(s,x_1(\eta(s))))}{f(s,x_1(\eta(s)))}-\frac{(x_2(s)-g(s,x_2(\eta(s))))}{f(s,x_2(\eta(s)))}\right|\right) \tag{5.9}$$

for all $t \in I$ and $x_1, x_2 \in E$. If identically zero function is the only solution of the differential equation

$$m'(t) = G(t,m(t)) \quad a.e.\ t \in I, \quad m(t_0)=0, \tag{5.10}$$

then the PHDE (1.1) has a unique solution defined on I.

Proof:- By theorem 3.2, the non-linear PHDE (1.1) has a solution on I.

Suppose that there are two solutions u_1 *and* u_2 of the PHDE (1.1) existing on I.

Define a function $m: I \to R_+$ by

$$m(t) = \left|\frac{(u_1(t)-g(t,u_1(\eta(t)))}{f(t,u_1(\eta((t)))}-\frac{(u_2(t)-g(t,u_2(\eta(t)))}{f(t,u_2(\eta((t)))}\right| \tag{5.11}$$

As $(|x(t)|)' \leq |x'(t)|$ *for* $t \in I$, we have that

$$\begin{aligned} m'(t) &\leq \left|\frac{d}{dt}\left[\frac{u_1(t)-g(t,u_1(\eta(t))}{f(t,u_1(\eta((t)))}\right]-\frac{d}{dt}\left[\frac{u_2(t)-g(t,u_2(\eta(t))}{f(t,u_2(\eta((t)))}\right]\right| \\ &\leq |(Qx_1)(t)-(Qx_2)(t)| \\ &\leq G\left(t, \left|\frac{u_1(t)-g(t,u_1(\eta(t))}{f(t,u_1(\eta((t)))}-\frac{u_2(t)-g(t,u_2(\eta(t))}{f(t,u_2(\eta((t)))}\right|\right) \\ &= G(t,m(t)) \end{aligned}$$

for all $t \in I$; *and that* $m(t_0)=0$

Now, we apply Theorem 5.1 with $g \equiv 0, f \equiv 1$ to get that $m(t)=0$ *for all* $t \in I$.

This gives

$$\frac{u_1(t)-g(t,u_1(\eta(t))}{f(t,u_1(\eta((t)))} = \frac{u_2(t)-g(t,u_2(\eta(t))}{f(t,u_2(\eta((t)))}$$

for all $t \in I$. Finally, in view of hypothesis (A_0) we conclude that $u_1(t) = u_2(t)$ on I. This completes the proof.

Remark 5.1 The hybrid differential equations is a rich area for variety of nonlinear ordinary as well as partial differential equations. Here we have considered a very simple hybrid differential equation involving three nonlinearities, however , a more complex hybrid differential equation can also be studied on similar lines with suitable modifications.

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