Applications of Some Fixed Point Theorems in POSET Metric Spaces to Differential and Integral Equations

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Abstract: In present paper we introduce the applications of some fixed point theorems to the existence and uniqueness of solutions to the class of nonlinear differential and integral equations in partially ordered metric spaces using different contractive mappings.

Index Terms: Fixed Point, Metric Spaces, Differential Equations, Integral Equations.

1. Introduction

It is well known that metric fixed point theory plays a very important role in functional analysis and it is very rich in applications to the branches of science in particularly in mathematics to the theory of differential and integral equations. In approach to find applications so many researchers study and gives generalizations of fixed point theorems, common fixed point theorems and hybrid fixed point theorem in complete metric space, partial ordered metric spaces, quasi metric spaces, etc. by using various contractive mappings.[1, 2, 3]
In present paper we apply some fixed point results in complete partially ordered metric space with different contractive mappings to get existence and solutions of differential and integral equations particularly nonlinear integral equations and Lebesgue integral along with nonlinear hybrid differential equations.

2. Some Fixed Point Theorems in POSET Metric Spaces.

Definition 2.1[1, 2, 3]
A self-mapping T: X → X is said to be a contraction if $\exists \lambda \in [0, 1)$ such that ∀ x, y ∈ X, d (Tx, Ty) ≤ λd (x, y) where (X, d) is a metric space.
In 1922 Banach [1] proved that a contraction mapping has a unique fixed point in a complete metric space (X, d).
In 2004 Ran and Reurings [2] introduced the Banach Fixed Point Theorem in ordered metric space as follows

Theorem 2.2[2]
Let (X, ≼) be a partially ordered set with a metric d, then (X, d) be a complete metric space. Also, every pair x, y ∈ X has a lower bound and an upper bound. If f is a continuous, monotone self-map from X into X then there exists $\lambda \in (0,1)$ such that d(fx, fy) ≤ λ d(x, y) , x ≥ y and there exists x_0 ∈ X such that $x_0 \leq fx_0$ or $x_0 \geq fx_0$ then f has a unique fixed point $\hat{x}$. Moreover, for every x∈ X, $\lim_{n\to\infty} f^n x = \hat{x}$.

Definition 2.3
Let S be the set of all functions Ψ : [0, ∞) → [0, ∞) satisfying the following conditions
(I) Ψ is continuous and monotonic increasing
(II) Ψ (x) = 0 iff x = 0

Remark: Now onwards from here we referred POSET as partially ordered set in this paper.

Theorem 2.4[2]
Let (X, ≼) be a POSET with a metric d and (X, d) be a complete metric space. Let f: X → X be a monotonic increasing self- map and T is one to one, continuous, subsequentially convergent order preserving or increasing self-map with Ψ ∈ S.
Then for all x, y ∈ X with x ≼ y, $\lambda \in [0, 1)$ and Ψ (d (Tfx, Tfy)) ≤ λ Ψ (d (Tx, Ty))
Also, suppose that either
(I) f is continuous or
(II) Assume that if any increasing sequence $\{x_n\}$ in X converges to z, then x_n ≼ z for all n ≥ 0.
If there exists x_0 ∈ X with $x_0 \preccurlyeq fx_0$, then f has a fixed point in X. Moreover, if for each x, y ∈ X there exists z ∈ X which is comparable to x and y, then the fixed point is unique.
Theorem 2.4[2] is called Banach Fixed point theorem in POSET metric space using Ψ mapping.

Theorem 2.5[11]
Let (X, ≼) be a POSET with a metric d and (X, d) be a complete metric space. Let f: X → X be a monotonic increasing self-map and T is one to one, continuous, subsequentially convergent order preserving or increasing self map with Ψ ∈ S.
Then for all x, y ∈X with x ≼ y, $\lambda_i \in [0, 1)$ and
Ψ(d(Tfx, Tfy))≤ λ_1Ψ (d(Tx , Ty)) + λ_2Ψ(d(Tx ,Tfx)) + λ_3Ψ(d(Ty, Tfy))+ λ_4Ψ (d(Tx, Tfy)) + λ_5Ψ (d(Ty, Tfx))
where $\lambda_i \geq 0$, for i = 1, 2, 3, 4, 5, such that $\lambda = \sum_{i=1}^{5} \lambda_i$
Also, suppose that either
(I) f is continuous or
(II) Assume that if any increasing sequence $\{x_n\}$ in X converges to z, then x_n ≼ z for all n ≥ 0.
If there exists x_0 ∈ X with $x_0 \preccurlyeq fx_0$, then f has a fixed point in X. Moreover, if for each x, y ∈ X there exists z ∈ X which is comparable to x and y , then the fixed point is unique.
Theorem 2.5[11] is called Hardy-Roger's Fixed Point Theorem in POSET metric space using Ψ mapping.

Theorem 2.6[11]
Let (X, ≼) be a POSET with a metric d and (X, d) be a complete metric space. Let f: X → X be a monotonic increasing self- map and T is one to one, continuous, subsequentially convergent order preserving or increasing self-map with Ψ ∈ S. For all x, y ∈ X with x ≼ y, $\alpha \in \left[0,\frac{1}{2}\right[$ and Ψ (d (Tfx, Tfy)) ≤ α [Ψ (d (Tx, Tfx)) + Ψ (d (Ty, Tfy))]
Also, suppose that either
(I) f is continuous or
(II) Assume that if any increasing sequence $\{x_n\}$ in X converges to z, then x_n ≼ z for all n ≥ 0.
If there exists x_0 ∈ X with x_0 ≼ fx_0 , then f has a fixed point in X. Moreover, if for each x, y ∈ X there exists z ∈ X which is comparable to x and y, then the fixed point is unique.
Theorem 2.6[11] is called kannan's Fixed Point Theorem in POSET metric space using Ψ mapping.

3. Applications of some Fixed Point Theorems related to POSET Metric Spaces in L-Integral Equations.

Theorem 3.1
Let (X, ≼) be a POSET with a metric d and (X, d) be a complete metric space. Let f: X → X be a monotonic increasing self- map and T is one to one, continuous, subsequentially convergent order preserving or increasing self-map with Ψ ∈ S. Then for all x, y ∈ X with x ≼ y, λ ∈ [0, 1) and

$$\int_0^{\Psi\,(\mathrm{d}\,(\mathrm{Tfx,Tfy}))} \chi(t)\,dt \le \lambda \int_0^{\Psi\,(\mathrm{d}\,(\mathrm{Tx,Ty}))} \chi(t)\ \ dt$$

where χ : [0, ∞) → [0, ∞) is a L-integrable nonnegative finite map on every compact subset of [0, ∞) such that for every ϵ > 0 we have $\int_0^{\epsilon} \chi(t)\,dt$ > 0.
Also, suppose that either
(I) f is continuous or
(II) Assume that if any increasing sequence $\{x_n\}$ in X converges to z, then x_n ≼ z for all n ≥ 0.
If there exists x_0 ∈ X with x_0 ≼ fx_0 , then f has a fixed point in X. Moreover, if for each x, y ∈ X there exists z ∈ X which is comparable to x and y, then the fixed point is unique.
Proof: If we take $\chi(t)$ = 1 in L-integrable nonnegative finite map then we get Banach Fixed point theorem in POSET metric space with Ψ mapping and theorem hold.
now we generalize this application of L-integrable map from Banach Fixed point theorem to other fixed point theorems as follows

Theorem 3.2
Let (X, ≼) be a POSET with a metric d and (X, d) be a complete metric space. Let f: X → X be a monotonic increasing self- map and T is one to one, continuous, subsequentially convergent order preserving or increasing self-map with Ψ ∈ S. Then for all x, y ∈ X with x ≼ y, λ_i ∈ [0, 1) and where λ_i ≥ 0 , for i = 1, 2, 3, 4, 5, such that $\lambda = \sum_{i=1}^{5} \lambda_i$

$$\int_0^{\Psi(\mathrm{d(Tfx,Tfy)})} \chi(t)\,dt \le \lambda \int_0^{\lambda_1\Psi\,(\mathrm{d(Tx\,,Ty)})\,+\,\lambda_2\Psi(\mathrm{d(Tx\,,Tfx)})\,+\,\lambda_3\Psi(\mathrm{d(Ty,Tfy)})+\,\lambda_4\Psi\,(\mathrm{d(Tx,Tfy)})\,+\,\lambda_5\Psi\,(\mathrm{d(Ty,Tfx)})} \chi(t)\ \ dt$$

where : [0, ∞) → [0, ∞) is a L-integrable nonnegative finite map on every compact subset of [0, ∞) such that for every ϵ > 0 we have $\int_0^{\epsilon} \chi(t)\,dt$ > 0.
Also, suppose that either
(I) f is continuous or
(II) Assume that if any increasing sequence $\{x_n\}$ in X converges to z, then x_n ≼ z for all n ≥ 0.
If there exists x_0 ∈ X with x_0 ≼ fx_0 , then f has a fixed point in X. Moreover, if for each x, y ∈ X there exists z ∈ X which is comparable to x and y, then the fixed point is unique.
Proof: If we take $\chi(t)$ = 1 in L-integrable nonnegative finite map with $\lambda = 1$ then we get Hardy-Roger fixed point theorem in POSET metric space with Ψ mapping and theorem hold.

Corollary 3.3
Let (X, ≼) be a POSET with a metric d and (X, d) be a complete metric space. Let f: X → X be a monotonic increasing self- map and T is one to one, continuous, subsequentially convergent order preserving or increasing self-map with Ψ ∈ S. For all x, y ∈ X with x ≼ y, $\lambda \in \left[0,\frac{1}{2}\right[$ and

$$\int_0^{\Psi\,(\mathrm{d}\,(\mathrm{Tfx,Tfy}))} \chi(t)\,dt \le \lambda \int_0^{\Psi\,(\mathrm{d(Tx,Tfx)})\,+\,\Psi\,(\mathrm{d\,(Ty,Tfy)})} \chi(t)\ \ dt$$

where : [0, ∞) → [0, ∞) is a L-integrable nonnegative finite map on every compact subset of [0, ∞) such that for every ϵ > 0 we have $\int_0^{\epsilon} \chi(t)\,dt$ > 0.
Also, suppose that either
(I) f is continuous or
(II) Assume that if any increasing sequence $\{x_n\}$ in X converges to z, then x_n ≼ z for all n ≥ 0.
If there exists x_0 ∈ X with x_0 ≼ fx_0 , then f has a fixed point in X. Moreover, if for each x, y ∈ X there exists z ∈ X which is comparable to x and y, then the fixed point is unique.
Proof: If we take $\chi(t)$ = 1 in L-integrable nonnegative finite map with α = $\lambda = 1$ then we get Kannan's fixed point theorem in POSET metric space with Ψ mapping and result hold.

4. Hybrid Fixed Point Theorem in POSET related Metric Spaces.
Hybrid fixed point theorem plays very important role in nonlinear differential and integral equations initially Ran and Reurings[3] study hybrid fixed point theorems in POSET related metric spaces and Nieto, Rodriguez-Lopez[9] established some results of hybrid fixed point theorems for monotonic map in POSET related metric spaces, now we apply some monotonic iterative methods with hybrid fixed point theorems study by S. Heikkila and V. Lakshmikantham[10] to the nonlinear differential and nonlinear integral equations.

Definition 4.1
A POSET related metric space (X, ≼, d) is said to be regular if $\{x_n\}$ is a monotonic sequence in X such that $x_n \to x^*$ as n → ∞ then $x_n \preccurlyeq x^*$ or $x_n \succcurlyeq x^*$ for all n ∈ ℕ.
Definition 4.2
A map T: X → X is said to be monotonic if it preserves the order relation ≼, hence if x ≼ y ⇒ Tx ≼ Ty or x ≼ y ⇒ Tx ≽ Ty for all x, y ∈ X.
Theorem 4.3[9]
Let (X, ≼) be a POSET with a metric d and (X, d) be a complete metric space. Let T: X → X be a monotonic map such that there exists a constant $\lambda \in [0, 1)$ such that d (Tx, Ty) ≤ λ d (x, y) for all elements x, y ∈ X, x ≥ y. Suppose that either T is continuous or if every convergent sequence $\{x_n\}$ in X to the point x^* whose consecutive terms are comparable then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that every term in it is comparable to the limit x^*. Further if there is an element $x_0 \in$ X satisfying $x_0 \preccurlyeq$ Tx_0 or $x_0 \succcurlyeq$ Tx_0, then T has a unique fixed point if every pair of elements in X has a lower and an upper bound. This hybrid fixed point theorem for monotonic map is proved by Nieto and Lopez [9].
Definition 4.4
Let (X, d) be a metric space and T be a self map on X then for x ∈ X we define 𝕆(x; T) of T at x by
𝕆(x; T) = {x, Tx, T^2x, ---, T^nx, ---}
Definition 4.5
The map T is called T-orbitally continuous on X if for any sequence $\{x_n\} \subseteq$ 𝕆(x; T) such that $x_n \to x^* \Rightarrow Tx_n \to Tx^*$ for each x ∈ X.
Definition 4.6
Let (X, d) be a metric space and T be a self map on X and metric space X is said to be T-orbitally complete if every Cauchy sequence $\{x_n\} \subseteq$ 𝕆(x; T) converges to x^*.
Definition 4.7
A self map T defined on X is said to be partially continuous at a point y ∈ E if for each $\epsilon > 0$ there exist a $\delta > 0$ such that $\|Tx - Ty\| < \epsilon$ whenever x is comparable to y with $\|x - y\| < \delta$. T is said to be partially continuous on X if it is partially continuous at every point of X.
Clearly if T is partially continuous on X, then T is continuous on every chain C contained in X.
Definition 4.8
A map Φ : [0, ∞) → [0, ∞) is said to be a dominating function if it is an upper semi-continuous and monotonic increasing function with Φ(0) = 0.
Theorem 4.9
Let (X, ≼) be a POSET with a metric d, f is self map and T: X → X be a monotonic increasing map there exists a dominating function Φ such that d (Tfx, Tfy) ≤ Φ (d (fx, fy)) for all comparable elements x, y ∈ X, where Φ(p) < p, p > 0. Suppose that either X is T-orbitally complete and T-orbitally continuous or T is partially T-orbitally continuous and X is regular and every sequence $\{x_n\}$ in X whose consecutive terms are comparable having a monotone subsequence. Furthermore if there is an element $x_0 \in$ X satisfying $x_0 \preccurlyeq$ Tx_0 or $x_0 \succcurlyeq$ Tx_0, then T has a unique fixed point $x^*$and the sequence $\{T^n x_0\}$ of iterations converges to x^* if every pair of elements in X has a lower and an upper bound.
Proof: The inequality d (Tfx, Tfy) ≤ Φ (d (fx, fy)) gives us the map T is partially T-orbitally continuous on X. If we consider y = Tfx in the inequality and it reduces to d (Tfx, T^2fx) ≤ Φ (d (fx, Tfx)) and by Banach fixed point theorem in POSET metric space T has a fixed point x^* and the sequence $\{T^n f x_0\}$ of successive iterations converges to x^*, the uniqueness follows from Nieto and Lopez theorem 4.3[9].

5. Applications of Hybrid Fixed Point Theorem in POSET related metric spaces to Differential and Integral equations.

Let I be a closed and bounded Interval such that I = $[t_0, t_0 + r] \subset \mathbb{R}$ for some a_0, $r \in \mathbb{R}$ and r > 0.
Consider the initial value problem of first order ordinary nonlinear hybrid differential equation

$$y'(t) = f(t; y(\omega(t)), \ t \in I, \text{ and } y(t_0) = y_0 \in \mathbb{R}. \quad \text{and} \quad \omega \in \mathbb{R} \qquad \text{--- (5.1)}$$

where f : I × ℝ → ℝ is continuous function.
The solution of (5.1) is a function y(t) ∈ C(I, ℝ) that satisfies d (Tx, Ty) ≤ λ d (x, y), where C(I, ℝ) is the class of continuous real-valued functions defined on I, clearly C(I, ℝ) is a Banach space with respect to the supremum norm and POSET with respect to the partially order relation on I. Also the partially ordered Banach space C(I, ℝ) is regular.
Definition 5.1
A function $\varphi \in$ C(I, ℝ) is said to be a lower solution of hybrid differential equation (5.1) and d (Tx, Ty) ≤ λ d (x, y) if it satisfies

$$\varphi'(t) \le f(t; \varphi(\omega(t)), \varphi(t_0) = \varphi_0 \ \text{ for all } \ t \in I \qquad \text{--- (5.2)}$$

Suppose the following conditions holds:
(I) There exists constants $\alpha, \beta > 0$ with $\alpha \ge \beta$ such that $\frac{-\beta\,(x-y)}{1+(x-y)} \le [f(t, x) + \alpha x] - [f(t, y) + \alpha y] \le 0$, for all t ∈ I and x, y ∈ ℝ, x ≥ y .
(II) The hybrid differential equation (5.1) and d (Tx, Ty) ≤ λ d (x, y) has a lower solution $\varphi \in$ C(I, ℝ)
Consider the initial value problem of the hybrid differential equation

$$y'(t) + \alpha\, y(\omega(t)) = f^*(t; y(\omega(t)), y(t_0) = y_0, \text{ for all } t \in I, \qquad \text{--- (5.3)}$$

where f*, f : I × ℝ → ℝ and f*(t; y($\omega(t)$) = f(t; y($\omega(t)$)) + α y($\omega(t)$)
Theorem 5.2
A function $\varphi \in$ C(I, ℝ) is a solution of the hybrid differential equation (5.3) if and only if it is a solution of the nonlinear integral equation,

$$y(\omega(t)) = c\, e^{-\lambda t} + e^{-\lambda t} \int_{t_0}^{t} e^{\lambda \xi} f^*(\xi; y(\omega(\xi))\, d\xi \qquad \text{--- (5.4)}$$

for all t ∈ I where c ∈ ℝ defined as c = $y_0 e^{t_0}$.

Theorem 5.3

Suppose that conditions (I) and (II) holds. Then the hybrid differential equation (5.3) d (Tx, Ty) ≤ λ d (x, y) has a unique solution x^* defined on I and the sequence $\{x_n\}$ of successive approximations given by

$$y_n(\omega(t)) = \mathrm{c}\, e^{-\lambda t} + e^{-\lambda t}\int_{t_0}^{t} e^{\lambda\xi}\, \mathrm{f}^*(\xi;\, y_n(\omega(\xi))\, \mathrm{d}\xi \qquad \text{--- (5.5)}$$

where $y_0 = \varphi$, converges to x^*.

Proof: Let $\mathcal{L}$ be the operator defined on $\mathrm{C}(\mathrm{I}, \mathbb{R})$ by

$$\mathcal{L}\mathrm{y}(\omega(t)) = \mathrm{c}\, e^{-\lambda t} + e^{-\lambda t}\int_{t_0}^{t} e^{\lambda\xi}\, \mathrm{f}^*(\xi;\, \mathrm{y}(\omega(\xi))\, \mathrm{d}\xi, \qquad \text{--- (5.6)}$$

for all t ∈ I.

By the continuity of integral equation the operator $\mathcal{L}$ defines the self map on $\mathrm{C}(\mathrm{I}, \mathbb{R})$ the hybrid differential equation (5.1) is equivalent operator equation $\mathcal{L}$ y($\omega(t)$) = y($\omega(t)$), for all t ∈ I. --- (5.7)

The operator $\mathcal{L}$ satisfies Theorem 4.9 and we get $\|\mathcal{L}x - \mathcal{L}y\| < \varphi(\|x-y\|)$, for all x, y ∈ $\mathrm{C}(\mathrm{I}, \mathbb{R})$ with x ≥ y.

Hence $\mathcal{L}$ satisfies the contraction principle given in Theorem 4.9 on $\mathrm{C}(\mathrm{I}, \mathbb{R})$ hence operator $\mathcal{L}$ is a partially T-orbitally continuous on $\mathrm{C}(\mathrm{I}, \mathbb{R})$. Also y($\omega(t)$) satisfies operator inequality y($\omega(t)$) ≤ $\mathcal{L}$y($\omega(t)$) and condition (II) then hybrid differential equation has a lower solution y($\omega(t)$). Then

$$\mathrm{y}'(\omega(t)) \le \mathrm{f}(\mathrm{t};\, \mathrm{y}(\omega(t)), \quad \mathrm{y}(\omega(t_0)) \le y_0 = \varphi \qquad \text{--- (5.8)}$$

combining λ y($\omega(t)$) on both sides of the above inequality (5.8) we get,

$$\mathrm{y}'(\omega(t)) + \lambda\, \mathrm{y}(\omega(t)) \le \mathrm{f}(\mathrm{t};\, \mathrm{y}(\omega(t)) + \lambda\, \mathrm{y}(\omega(t)) \qquad \text{--- (5.9)}$$

for all t ∈ I. Multiplying the above inequality(5.9) by $e^{\lambda t}$ we get

$$\left(e^{\lambda t}\mathrm{y}(\omega(t))\right)' \le e^{\lambda t}\, \mathrm{f}^*(\,\mathrm{t};\, \mathrm{y}(\omega(t)) \qquad \text{--- (5.10)}$$

Integrating both sides of inequality (5.10) from t_0 to t we get

$$\mathrm{y}(\omega(t)) = \mathrm{c}\, e^{-\lambda t} + e^{-\lambda t}\int_{t_0}^{t} e^{\lambda\xi}\, \mathrm{f}^*(\xi;\, \mathrm{y}(\omega(\xi))\, \mathrm{d}\xi$$

for all t ∈ I.

Hence the operator equation $\mathcal{L}$ y($\omega(t)$) = y($\omega(t)$) has a solution corresponding to the integral equation and the hybrid differential equation has a solution x^*defined on $\mathrm{C}(\mathrm{I}, \mathbb{R})$. Consequently the sequence $\{y_n\}$ of successive approximations given in (5.5) converges to x^*.

This proves the theorem.

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