# GENERALISATION OF MATRICES GIVING ABSOLUTELY PERMANENT SERIES -TO - SEQUENCE AND SERIES -TO - SERIES TRANSFORMATION

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***Abstract:*** We obtain the theorems of matrices absolutely permanent series – to – sequence and series – to – series transformation. In this paper we prove the product matrix H = GB exists and is a $\gamma_A$ – matrix for every $\gamma_A$ matrix G iff B is an $\alpha_A$ – matrix, the product GB of a $\gamma_A$ – matrix G and an $\alpha_A$ – matrix B exists and is a $\gamma_A$- matrix and the product of two – matrices are an -matrix.

***Index Terms*: matrix method for summability, sequence - to-sequence transformation etc.**

## 1 INTRODUCTION

Matrices giving absolutely permanent series - to-sequence and series - to - series transformation. The subject of infinite matrices; being a recent one is abounding in good research problems. A very important application of matrices, namely to the theory of summability of divergent sequence and the series was initiated by Toeplitz [9] in 1911. Since, then, it has attached almost all researches in the field of summability methods.

1.1 Although the concept of "absolute summability" was introduced as early as in 1911 by Fekete [10] in case of Cesar o[12] methods, and the same for Riesz and Abel[11] methods was defined by Obrechkoff [13] and Whittaker[14] in 1928 and 1932 respectively, for matrix transformation in general thesis was considered in 1937 by Mears. Ordinary summability by the general matrix transformation has been investigated in considerable detailed thus haseached a stage where a sufficiently unified theory can be presented. But absolute summability on the other hands is still in its infancy.

It was proved by Bosanquet [1, 2,5and 6], in 1931 that the necessary and sufficient condition that an RF- transformation.

$$\gamma_n=\sum_{k=1}^{\infty} g_{nk} u_k, \ (n > k) \tag{1.1.1}$$

should tend to finite limits whenever $\sum u_k = S$ is that G = $(g_{nk})$ should be a $\gamma$ matrices.
In 1949, Vermes [4, 7, 8] considered the RR- transformation

$$\gamma_n = \sum_{k=1}^{\infty} b_{nk} u_k \tag{1.1.2}$$

And proved that the transformation iff the matrix G = $(g_{nk})$ defined as

$$g_{nk} = b_{1k} + b_{2k} + \ldots + b_{nk} \tag{1.1.3}$$

(n, k$\geq 1$) is a $\gamma$ – matrix such a matrix
B = $(b_{nk})$ is called an $\alpha$- matrix

In the same paper vermes has prove the following theorems involving product of and matrices [3]. Sufficient of the conditions was first proved for a lower semi - matrices by H. Bohr in 1909. Carmichael and perron proved the sufficiency of the conditions for a general matrices. It was Hahn who proved the necessity of the condition and also gave a complete proof.

**Theorem A**

The product GB of a $\gamma$ – matrix G and $\alpha$- matrix B exits and is a $\gamma$ – matrix.

**Theorem B**

A necessary and sufficient for the matrix product H = GB to exits and be a $\gamma$ – matrix for every $\gamma$ matrix G is that b should be an $\alpha$- matrix..

**Theorem C**

The product of two $\alpha$- matrices is an $\alpha$- matrix. Knoop and Lorentz proved the RR- transformation (1.1.2) is absolutely permanent iff the transformation matrix B = $(b_{nk})$ is and $\alpha_A -$ matrix.

## 2 PRELIMINARIES

**Theorem 2.1**

If $g_{nk} = b_{1k}+b_{2k} + \cdots + b_{nk}$, (2.1.1)
(n,k$\geq 1$) then G = ($g_{nk}$) is $a\ \gamma_A$– matrix iff B = ($b_{nk}$) is an $\gamma_A$ – matrix.

**Theorem2.2**

The product GB of a $\gamma_A$ – matrix G and $\alpha_A$ $matrix$ B- exits and is a $\gamma_A$ – matrix.

**Theorem 2.3**

The product BG and $\alpha_A$ matrix $B\ and$ $a$ $\gamma_A$ – matrix G may not exits.

**Theorem2.4**

The product of $\gamma_A$ – matrix G and as $\alpha_A$ matrix B is not commutative.

**Theorem2.5**

The product matrix H = GB exists and is a $\gamma_A$ – matrix for every $\gamma_A$ – matrix G iff B is an $\alpha_A$ -matrix GB is an $\alpha_A$ – matrix.

**Theorem 2.6**

The product of two $\alpha_A$ – matrices is a $\alpha_A$ – matrix.

**Theorem 2.7**

The product of two $\gamma_A$ - matrices may not be a $\gamma_A$ - matrix.

## 3 Proof of the theorems are shown as follows

### 3.1 Proof of theorem 2.1

In (2.1.1) if G is a $\gamma_A$ – matrix then form $\underset{n\to\infty}{limit} J_{nk} = 1$ ∀ k=1,2,…

$\lim_{n\to\infty} ( \sum_{i=1}^{\infty} b_{ik}) = \lim_{n\to\infty} g_{nk} = 1$ (3.1.1)

Also, $|b_{nk}| = |g_{nk} - g_{n-1,k}| \leq |g_{nk}| + |g_{n-1,k}|$

$\leq k_{n-1}(G) + k_n(G)$ (3.1.2)

Since, each $|g_{nk}| < k_n(G)$, independent of k

$\sum_{n=1}^{\infty} | b_{nk}| = | g_{1k}| + \sum_{n=2}^{\infty} | g_{nk} - g_{n-1,k}| < M(G)$ (3.1.3)

By condition $\sum_{n=2}^{\infty} | J_{nk} - J_{n-1,k}|$<M(G)

Where, M(G) is absolutely constant independent of k. Conditions (3.1.1) – (3.1.3) are precisely the condition for the matrix B to be an $\alpha_A$ matrix. Conversely, if the matrix B = ($b_{nk}$) is an $\alpha_A$ matrix. Then in the definition,

$b_{nk} = g_{nk} - g_{n-1,k}$, (n >1 , k $\geq 1$)

$b_{1k} = g_{1k}$ (k$\geq 1$) (3.1.4)

We have to prove that G = $(g_{nk})$ is a $\gamma_A$ – matrix. It is easy to see from (3.1.3)

That, if $\sum_{n=1}^{\infty} | b_{nk}| < M(B)$, then

$\sum_{n=2}^{\infty} | g_{nk} - g_{n-1,k}| < M(G)$ independent of k (3.1.5)

From$| b_{nk}| < k_n(B)$, it follows that

$|g_{nk}| = |b_{1k} + b_{2k} + \cdots + b_{nk}| \leq |b_{1k}| + |b_{2k}| + \cdots + |b_{nk}|$

$< k_n(G)$ Independent of k (3.1.6)

Lastly by definition $\sum_{i=1}^{n} b_{ik} = b_{1k}+\sum_{i=2}^{n} b_{2k} = g_{1k} + \sum_{i=2}^{n}(g_{ik} - g_{i-1,k})$

$= g_{nk}$

Therefore,

$\underset{n\to\infty}{limit} g_{nk} = \sum_{i=1}^{\infty} b_{ik} = 1$, (3.1.7)

Thus the conditions (3.1.5) – (3.1.7) show that the matrix G = $( g_{nk})$ in (3.1.4) is a $\gamma_A$ – matrix. Hence, the theorem is completely established.

**3.2 Proof of theorem 2.2**

Here G = $(g_{nk})$ is a $\gamma_A$ – matrix. Consequently if the Series $\sum q_j$ is convergent then the G- transform of $\sum a_j$ namely.

$\sum_{j=1}^{\infty} g_{nj} a_j$ exits for all n, and is a sequence of bounded variation.

Also,

$\lim_{n\to\infty}(\sum_{j=1}^{\infty} g_{nj} a_j) = \sum_{j=1}^{\infty} a_j$ choose $a_j = b_{jk}$, where $b_{jk}$ are the elements of the $\alpha_A$ – matrix B.

Now, [By The RR- transformation $\gamma_n = \sum_{n=1}^{\infty} b_{nk} u_k$ is absolutely permanent iff the conditions $\sum_{n=1}^{\infty} |b_{nk}| < M(B)$, $|b_{nk}| < k_n(B)$,]

$\sum_{j=1}^{\infty} a_j = \sum_{j=1}^{\infty} b_{jk} = 1$, k = 1, 2…

Also, for an $\alpha_A$ – matrixB,

$\sum_{j=1}^{\infty} |b_{jk}| < M(B)$ independent of k. Therefore, if H = GB, then

$|b_{nk}| = |\sum_{j=1}^{\infty} g_{nj} b_{jk}| < |g_{nj}||b_{jk}| < k_n(H)$, independent of k. (3.2.1)

Therefore the product matrix H = $(h_{nk})$ exits for all and k as well as

$\lim_{n\to\infty}(h_{nk}) = 1$ (3.2.2)

Now,

$\sum_{n=2}^{\infty} |h_{nk} - h_{n-1,k}| = \sum_{n=2}^{\infty} |\sum_{j=1}^{\infty} (g_{nj} - g_{n-1,j}).b_{jk}| < \sum_{n=2}^{\infty} \sum_{j=1}^{\infty} |g_{nj} - g_{n-1,j}||b_{jk}|$

$= \sum_{j=1}^{\infty} |b_{jk}| \sum_{n=2}^{\infty} |g_{nj} - g_{n-1,j}|$

$< M(G)\ \ M(B)$ (3.2.3)

$< M(H)$, independent of k. The conditions (3.2.1 – 3.2.3) show that H = GB is a $\gamma_A$ -matrix.

**3.3 Proof of theorem 2.3**

We consider the following example,

Let $b_{nk} = 1$ for n=1, k = 1, 2,... (3.3.1)

= 0 for n > 1

And $g_{nk} = 1\ \forall$ n, k ≥1 (3.3.2)

These matrices B,G defined in (3.3.1) and (3.3.2) are $\alpha_A$ – and $\gamma_A$ - matrices respectively. Their product,

$(GB)_{nk} = (G)_{nk}$ exits and is the $\gamma_A$ - matrix G of (3.3.2) but the product

$(BG)_{nk} = \sum_{j=1}^{\infty} (b_{nj} g_{jk})$

$= \sum_{j=\_}^{\infty} (1 + 1 + \cdots)_{nk}$ doesn't exits. Hence, the required result follows

**3.4 Proof of the theorem 2. 4 and 2.5**

Theorem 2.4 and sufficiency of the condition in theorem2.5 follows by combining the theorem 2.2 and 2.3.

To prove the necessity part of theorem2.5, we consider a $\gamma_A$ - matrix G,

defined as $g_{nk} = 1$ for $k \le n$ (3.4.1)

$= 0$ for $k > n$

Then the product matrix H = (GB) is

$h_{nx} = \sum_{j=1}^{\infty} g_{nj} b_{jk}$ (3.4.2)

$= \sum_{j=1}^{n} b_{jk}$

By theorem2.1, the matrix

H = $(h_{nk})$ in (3.4.2) is a $\gamma_A$ - matrix, only if B is an $\alpha_A$ - matrix.

**3.5 Proof of the theorem 2.6**

Let A and B be any two $\alpha_A$ – matrices. We define a matrix

G = $(g_{nk})$ as

$g_{nk} = a_{1k} + a_{2k} + \cdots + a_{nk}$ , $(n \ge 1, k \ge 1)$ (3.5.1)

Then by theorem2.1, $(g_{nk})$ is a $\gamma_A$ - matrix and by theorem 2.2,

The product

$(H)_{nk} = (GB)_{nk}$ is a $\gamma_A$ - matrix. If we further define

$C_{nk} = h_{nk} - h_{n-1,k}$, $(n > 1, k \ge 1)$ , (3.5.2)

$C_{1k} = h_{1k}$

Then C = $(c_{nk})$ is an $\alpha_A$- matrix and

$c_{nk} = \sum_{j=1}^{\infty} g_{nj}\, b_{jk} - \sum_{j=1}^{\infty} g_{n-1,j}$

$= \sum_{j=1}^{\infty}( g_{nj} - g_{n-1,j})$

$= \sum_{j=1}^{\infty} a_{nj} b_{jk}$

$\therefore (c)_{nk} = (AB)_{nk}$

By definition (3.5.1) of the matrix G. Hence the proof is complete.

**3.6 Proof of the Theorem 2.7**

An example is sufficient for the part of the theorem. Consider the $\gamma_A$ - matrix

G = $(g_{nk})$ defined in (3.4.1) another $\gamma_A$ - matrix H = $(h_{nk})$,

$h_{nk} = 1 \;\forall$ n, k$\geq 1$ (3.6.1)

The product matrix F = GH of the two $\gamma_A$ - matrices is given by

$f_{nk} = \sum_{j=1}^{\infty} g_{nj}\, h_{jk}$

$= \sum_{j=1}^{n} g_{aj} b_{jk}$

= n

$\therefore \lim_{n\to\infty}( f_{nk}) = \infty \neq 1 \;\forall$ k

And each fixed n, this shows that F is not a $\gamma_A$ - matrix.

**4 Conclusion**

In this paper we prove the product matrix H = GB exists and is a $\gamma_A$ – matrix for every $\gamma_A$ matrix G iff B is an $\alpha_A$ – matrix, the product GB of a $\gamma_A$ – matrix G and an $\alpha_A$ – matrix B exists and is a $\gamma_A$- matrix and the product of two $\alpha_A$ – matrices is an $\alpha_A$-matrix.

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