CHARACTERISTICS OF RELATIVE COEFFICIENTS OF VISCOSITY OF BLOOD FLOWING IN A NARROW CHANNEL

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In the present paper, a two-layered blood flow model through a narrow vessel has been considered with a peripheral layer of Bingham fluid and the core region a of Casson type. The relative coefficients of viscosity for peripheral and core layer have been determined and their variations are shown graphically for different values of maximum hematocrit, shape parameter etc.

1. Introduction

In large vessels, blood is treated as a Newtonian fluid and it is assumed that the rheological properties of blood do not influence the nature of its flow characteristics **[11, 12]**. But near the quasi-steady conditions and in the vicinity of changes of the cross-sectional area of a vessel, where the flow rate is comparatively low, some kind of exceptional cases are observed. It is noticed that as the diameter of the vessel reduces, the rheological properties of blood become more and more important than the shear thinning. In the narrow vessels, the flow becomes more complicated by phase separation. Various authors **[3, 6, 7]** discussed some properties of blood flow in a narrow vessel such as the blunting of the velocity profile near the axis of the vessel, the occurrence of plasma layer, the Fahraeus-Lindqvist effect and its inversion, the sigma effect etc.

Blood consists of suspension of cells in an aqueous solution, called plasma, which is composed of 90% water and 7% protein. There are about 5×10^9 cells in a milliliter of healthy human blood of which 95% are red cells or erythrocytes. The main function of the erythrocytes is to transport oxygen from the lungs to all cells of the body and to remove carbon-dioxide formed by metabolic processes in the body to the lungs. About 45% of the volume in an average human is occupied by erythrocytes, the fraction being known as the hematocrit. It is normally 38 to 52 % for males and 37 to 47 % for females. The white cells or leucocytes constitute 1% of the blood playing the role in the resistance of the body to infection and the platelets form 5% of the total, performing a function related to blood clotting.

It is observed that the hematocrit value influences the relative coefficient of viscosity of blood. Also during the flow in a narrow vessel the aforementioned cells tend to migrate along the center of the vessel giving rise to form a relatively cell-free slower moving plasma layer. Photographic records of blood flow in narrow vessels of animals give the evidence of the existence of a peripheral plasma layer near the wall of a narrow vessel **[1]** , which leads to consider a two-layered blood flow model in case of narrow vessels . Many authors **[8, 9, 14, 15, 16]** studied the Fahraeus-Lindqvist phenomena. Bugliarello et al. **[2]** made experiments in vitro in glass capillaries with diameters ranging from 40-83 μ taking blood samples with acid-citrate-dextrose and varying hematocrits were used there. Chaturani and Biswas **[4, 5]** made their investigations on two-layered and three-layered blood flow model considering blood as a polar fluid. Majumdar et al. **[10]** considered a two-layered blood flow model taking the peripheral and core layer as power law fluids and studied the relative consistency coefficient of blood. Sanyal and Sarkar **[13]** also considered a two-layered blood flow model in which the peripheral plasma layer has been taken as a Newtonian fluid and the core layer as a Casson type non-Newtonian fluid . They characterized the blood flow by the relative coefficient of viscosity.

In this paper we consider a two-layered blood flow model , assuming the peripheral plasma layer consists of Bingham plastic while the core layer as Casson type fluid . As the plasma layer is not completely cell free and its nature is not purely Newtonian, the peripheral plasma layer is taken to be Bingham plastic. The analytical expressions for the relative coefficients of viscosity in both the peripheral and core layers are

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obtained and their nature are shown graphically for different values of maximum hematocrit, shape parameter etc.

2. Mathematical Formulation

We consider a symmetric, laminar, two-dimensional flow of blood in a sufficiently narrow, rigid cylindrical blood vessel . As the peripheral plasma layer consists of Bingham fluid, the shear stress (τ_p) and shear rate (e_p) are related by

$$\tau_p = \mu_p e_p + \tau_{0p} \; ; \qquad \tau_p \geq \tau_{0p}$$

$$\text{and} \qquad e_p = 0 \; ; \qquad \tau_p < \tau_{0p} . \qquad (1)$$

On the other hand, the core region being filled with Casson type fluid , the shear stress (τ_c) and shear rate (e_c) are related by

$$\tau_c^{1/2} = \mu_c^{1/2} e_c^{1/2} + \tau_{0c}^{1/2} \; ; \qquad \tau_c \geq \tau_{0c}$$

$$\text{and} \qquad e_c = 0 \; ; \qquad \tau_c < \tau_{0c} . \qquad (2)$$

In the above, τ_{0p}, τ_{0c} and μ_p, μ_c are yield stresses and coefficients of viscosities of the two regions .

Let us assume that the thickness of the peripheral plasma layer be δ, the radius of the vessel be R, radius of the plug region be r_p and P=$\left(-\frac{dp}{dz}\right)$ is the constant pressure gradient. The geometry of the vessel is shown in fig.-1.

Considering the forces on the core region, we get

$$P \times \pi (R-\delta)^2 = \tau_{0c} \times 2\pi (R-\delta)$$

$$\text{or,} \quad \tau_{0c} = \frac{1}{2} P (R-\delta) . \qquad (3)$$

Also for the force in the plug region, we get

$$P \times \pi r_p^2 = \tau_{0p} \times 2\pi r_p$$

$$\text{or,} \quad \tau_{0p} = \frac{1}{2} P r_p . \qquad (4)$$

The required equations of motion for the peripheral plasma layer and the core region are given respectively by

$$\mu_p \left(-\frac{dv_{zp}}{d\eta}\right) = \frac{PR^2}{2} \{\eta - (1-\delta/R)\} \quad , \qquad 1-\delta/R \leq \eta \leq 1 \qquad (5)$$

and

$$\frac{d}{d\eta}\left\{\eta \mu_c \left(-\frac{dv_{zc}}{d\eta}\right)\right\} = \frac{PR^2}{2}\left\{\left(\eta^{1/2} - c_p^{1/2}\right) + \eta^{1/2}\left(\eta^{1/2} - c_p^{1/2}\right)\right\} \quad , \qquad c_p \leq \eta \leq 1-\delta/R . \quad (6)$$

where $\eta = \frac{r}{R}$, $c_p = \frac{r_p}{R}$, v_{zp} is the axial velocity component in the peripheral region and v_{zc} is the axial velocity component in the core region .

Let us express μ_p and μ_c as

$$\mu_p = \frac{\mu_w}{\left\{1 - \beta h_m \left(1-\eta^n\right)\right\}} \quad , \qquad 1-\delta/R \leq \eta \leq 1, \qquad (7)$$

$$\mu_c = \frac{\mu_s}{\left\{1 - \beta h_m \left(\frac{\eta}{1-\delta/R}\right)^n\right\}} \quad , \qquad c_p \leq \eta \leq 1-\delta/R , \qquad (8)$$

where μ_w and μ_s are the coefficients of viscosity at the wall and at the surface of separation of the two-layers respectively , h_m is the maximum hematocrit at the centre of the vessel , β is a constant and n is the shape parameter [15] .

The boundary conditions are given by

$$\begin{aligned}
&\text{(i)} \quad v_{zp}=0 \quad \text{at } \eta=1,\\
&\text{(ii)} \quad \frac{dv_{zc}}{d\eta}=0 \quad \text{at } 0\le\eta\le c_p,\\
&\text{(iii)} \quad v_{zp}=v_{zc} \quad \text{at } \eta=1-\delta/R,\\
&\text{(iv)} \quad \mu_p=\mu_c \quad \text{at } \eta=1-\delta/R.
\end{aligned} \tag{9}$$

For simplicity we take $L=\beta h_m$ and $a=(1-\delta/R)^{-1}$.

3. Solutions

Using the boundary condition (i) in equation (5) we get on integration

$$v_{zp}=\frac{PR^2}{2\mu_w}\left[\frac{(1-L)}{2}(1-\eta^2)-\frac{(1-L)}{a}(1-\eta)+\frac{L}{n+2}(1-\eta^{n+2})-\frac{L}{a(n+1)}(1-\eta^{n+1})\right]. \tag{10}$$

Again using the boundary condition (ii) in equation (6) we get by solving

$$v_{zc}=-\frac{PR^2}{2\mu_s}\left[(1-L)\left(\frac{\eta^2}{2}+\eta c_p-\frac{4\eta^{3/2}c_p^{1/2}}{3}\right)+La^n\left(\frac{\eta^{n+2}}{n+2}+\frac{c_p\eta^{n+1}}{n+1}-\frac{4\eta^{3/2+n}c_p^{1/2}}{3+2n}\right)\right]+A_0 \tag{11}$$

where A_0 is the constant of integration .

Applying the boundary condition (iv) in equations (7) and (8) we obtain the relation between μ_w and μ_s as

$$\mu_s=\frac{\mu_w}{\left[1+L(a^{-n}-1)\right]}. \tag{12}$$

Now applying the boundary condition (iii) in equation (11) we obtain

$$\begin{aligned}
A_0=&\frac{PR^2}{2\mu_w}\left[\frac{(1-L)}{2}(1-\frac{1}{a^2})-\frac{(1-L)}{a}\left(1-\frac{1}{a}\right)+\frac{L}{n+2}\left(1-\frac{1}{a^{n+2}}\right)-\frac{L}{a(n+1)}\left(1-\frac{1}{a^{n+1}}\right)\right]\\
&+\frac{PR^2}{2\mu_w}\left[(1-L)\left(\frac{1}{2a^2}+\frac{c_p}{a}-\frac{4c_p^{1/2}}{3a^{3/2}}\right)+L\left\{\frac{1}{(n+2)a^2}+\frac{c_p}{(n+1)a}-\frac{4c_p^{1/2}}{(3+2n)a^{3/2}}\right\}\right].
\end{aligned} \tag{13}$$

Putting the values of μ_s from equation (12) and A_0 from equation (13) in the equation (11) we get

$$v_{zc}=-\frac{PR^2}{2\mu_w}\left[\left(1+L(a^{-n}-1)\right)\left\{(1-L)\left(\frac{\eta^2}{2}+c_p\eta-\frac{4\eta^{3/2}c_p^{1/2}}{3}\right)+La^n\left(\frac{\eta^{n+2}}{n+2}+\frac{c_p\eta^{n+1}}{n+1}-\frac{4\eta^{3/2+n}c_p^{1/2}}{3+2n}\right)\right\}-F_0\right], \tag{14}$$

where

$$F_0=\left[\frac{(1-L)}{2}\left(1-\frac{1}{a^2}\right)-\frac{(1-L)}{a}\left(1-\frac{1}{a}\right)+\frac{L}{n+2}\left(1-\frac{1}{a^{n+2}}\right)-\frac{L}{a(n+1)}\left(1-\frac{1}{a^{n+1}}\right)\right]$$

$$+\left(1+L\left(a^{-n}-1\right)\right)\left[(1-L)\left(\frac{1}{2a^2}+\frac{c_p}{a}-\frac{4c_p^{1/2}}{3a^{3/2}}\right)+L\left(\frac{1}{(n+2)a^2}+\frac{c_p}{(n+1)a}-\frac{4c_p^{1/2}}{(3+2n)a^{3/2}}\right)\right]. \quad (15)$$

The velocity at the plug region is given by

$$v_p=\left(v_{zc}\right)_{\eta=c_p}$$

$$=-\frac{PR^2}{2\mu_w}\left[\left(1+L\left(a^{-n}-1\right)\right)\left\{(1-L)\frac{c_p^2}{6}+\frac{La^n c_p^{n+2}}{(n+1)(n+2)(2n+3)}\right\}-\mathrm{F}_0\right]. \quad (16)$$

Let us denote the fluxes in the peripheral plasma layer by Q_p , plug region (sub core region) by Q_{sc} and non-plug core region by Q_c , so that the flux on the core layer is Q_c+Q_{sc} .

Now we have

$$Q_p=\int_{R-\delta}^{R} v_{zp}2\pi r dr$$

$$=2\pi R^2\int_{1-\frac{\delta}{R}}^{1} v_{zp}\eta d\eta$$

$$=\frac{\pi PR^2}{\mu_w}\Bigg[\frac{(1-L)}{2}\left\{\frac{1}{4}-\left(\frac{1}{2a^2}-\frac{1}{4a^4}\right)\right\}-\frac{(1-L)}{a}\left\{\frac{1}{6}-\left(\frac{1}{2a^2}-\frac{1}{3a^3}\right)\right\}$$
$$+\frac{L}{n+2}\left\{\frac{n+2}{2(n+4)}-\left(\frac{1}{2a^2}-\frac{1}{(n+4)a^{n+4}}\right)\right\}$$
$$-\frac{L}{a(n+1)}\left\{\frac{n+1}{2(n+3)}-\left(\frac{1}{2a^2}-\frac{1}{(n+3)a^{n+3}}\right)\right\}\Bigg]$$

$$=\frac{\pi PR^4}{\mu_w}Q_p^0, \quad (17)$$

where $$Q_{\mathrm{p}}^0=\frac{(1-L)}{2}\left\{\frac{1}{4}-\left(\frac{1}{2a^2}-\frac{1}{4a^4}\right)\right\}-\frac{(1-L)}{a}\left\{\frac{1}{6}-\left(\frac{1}{2a^2}-\frac{1}{3a^3}\right)\right\}$$
$$+\frac{L}{n+2}\left\{\frac{n+2}{2(n+4)}-\left(\frac{1}{2a^2}-\frac{1}{(n+4)a^{n+4}}\right)\right\}$$
$$-\frac{L}{a(n+1)}\left\{\frac{n+1}{2(n+3)}-\left(\frac{1}{2a^2}-\frac{1}{(n+3)a^{n+3}}\right)\right\}. \quad (18)$$

Again

$$Q_c=\int_{r_p}^{R-\delta} v_{zc}2\pi r dr$$

$$=2\pi R^2\int_{c_p}^{1-\frac{\delta}{R}} v_{zc}\eta d\eta$$

$$=-\frac{\pi PR^4}{\mu_w}\Bigg[\left(1-L\left(a^{-n}-1\right)\right)\Big\{(1-L)\left(\frac{1}{8}\left(\frac{1}{a^4}-c_p^4\right)+\frac{c_p}{3}\left(\frac{1}{a^3}-c_p^3\right)-\frac{8c_p^{1/2}}{21}\left(\frac{1}{a^{7/2}}-c_p^{7/2}\right)\right)$$
$$+La^n\left(\frac{1}{(n+2)(n+4)}\left(\frac{1}{a^{n+4}}-c_p^{n+4}\right)+\frac{c_p}{(n+1)(n+3)}\left(\frac{1}{a^{n+3}}-c_p^{n+3}\right)-\frac{8c_p^{1/2}}{(2n+3)(2n+7)}\left(\frac{1}{a^{n+7/2}}-c_p^{7/2}\right)\right)\Big\}$$
$$-\frac{\mathrm{F}_0}{2}\left(\frac{1}{a^2}-c_p^2\right)\Bigg]$$

$$=\frac{\pi PR^4}{\mu_w}Q_c^0, \tag{19}$$

where $Q_c^0=\left(1-L\left(a^{-n}-1\right)\right)\Big\{(1-L)\left(\frac{1}{8}\left(\frac{1}{a^4}-c_p^4\right)+\frac{c_p}{3}\left(\frac{1}{a^3}-c_p^3\right)-\frac{8c_p^{1/2}}{21}\left(\frac{1}{a^{7/2}}-c_p^{7/2}\right)\right)$

$$+La^n\left(\frac{1}{(n+2)(n+4)}\left(\frac{1}{a^{n+4}}-c_p^{n+4}\right)+\frac{c_p}{(n+1)(n+3)}\left(\frac{1}{a^{n+3}}-c_p^{n+3}\right)-\frac{8c_p^{1/2}}{(2n+3)(2n+7)}\left(\frac{1}{a^{n+7/2}}-c_p^{7/2}\right)\right)\Big\}$$

$$-\frac{F_0}{2}\left(\frac{1}{a^2}-c_p^2\right) \tag{20}$$

and

$$Q_{sc}=\pi r_p^2\left(v_{zc}\right)_{\eta=c_p}$$
$$=\pi R^2c_p^2v_p$$
$$=-\frac{\pi PR^4}{2\mu_w}\left[\left(1+L\left(a^{-n}-1\right)\right)\left\{(1-L)\frac{c_p^4}{6}+\frac{La^nc_p^{n+4}}{(n+1)(n+2)(2n+3)}\right\}-F_0c_p^2\right]$$
$$=-\frac{\pi PR^4}{2\mu_w}Q_{sc}^0 \tag{21}$$

with $Q_{sc}^0=\left(1+L\left(a^{-n}-1\right)\right)\left\{(1-L)\frac{c_p^4}{6}+\frac{La^nc_p^{n+4}}{(n+1)(n+2)(2n+3)}\right\}-F_0c_p^2.$ (22)

Thus the total flux is given by

$$Q_T=Q_p+Q_c+Q_{sc}$$
$$=\frac{\pi PR^4}{\mu_w}Q_p^0-\frac{\pi PR^4}{\mu_w}Q_c^0-\frac{\pi PR^4}{2\mu_w}Q_{sc}^0. \tag{23}$$

If the tube were filled by a Newtonian fluid with coefficient of viscosity μ_N, then the flux is

$$Q_N=\frac{\pi PR^4}{8\mu_N}. \tag{24}$$

Taking the fluxes Q_T and Q_N to be the same we can easily determine the peripheral relative coefficient of viscosity μ_{pr} from the equations (23) and (24) as

$$\mu_{pr}=\frac{\mu_w}{\mu_N}$$
$$=8Q_p^0-8Q_c^0-4Q_{sc}^0 \quad , \quad 1-\delta/R\le\eta\le 1. \tag{25}$$

Again the core relative coefficient of viscosity is given by

$$\mu_{cr}=\frac{\mu_s}{\mu_N}$$
$$=\frac{8Q_p^0-8Q_c^0-4Q_{sc}^0}{1+L\left(a^{-n}-1\right)} \quad , \quad c_p\le\eta\le 1-\delta/R. \tag{26}$$

If we take the shape parameter n to be zero, it is observed that both peripheral and core relative coefficients become constants and are equal, i.e., $\mu_{pr}=\mu_{cr}$. In this case we denote both μ_{pr} and μ_{cr} as μ_r and is termed as relative coefficient of viscosity.

4. Numerical results and discussions

From figures 2-6 we observe that the peripheral and core relative coefficients of viscosity decrease as we increase the maximum hematocrit. With the increase of shape parameter n both of the relative coefficients of viscosity decrease.

For a fixed shape parameter n and maximum hematocrit h_m , both of the relative coefficients of viscosity decrease up to a certain level and then they increase .

5. Conclusions

In the case of anemia, multiple myeloma, rheumatoid arthritis, the value of hematocrit decreases whereas in case of dehydration, erythrocytosis, polycythemia vera the value of hematocrit increases. Clinical results show that the blood pressure increases due to the increase in hematocrit. So in various cases where hematocrit values deviate from normal level we can determine the corresponding coefficients of viscosity using the aforementioned mathematical expressions, which may be helpful for medical researches also .

References

1. E. L. BLOCH : A quantitative study of the hemodynamics in the living microvascular system, *Amer. J. Anatomy*. **110** (1962) 125 – 153.
2. G. BUGLIARELLO et al. : The profile velocity and other characteristics of blood flow in a non-uniform shear fluid, Proc. Four. Int. Congr. On Rheology, A.L.Copley [Ed.], Interscience, N.Y., (1965), 351 – 370.
3. G. BUGLIARELLO and J. SEVILLA : Velocity distribution and other characteristics of steady and pulsatile blood flow in fine glass tubes, *Biorheology*. **7** (1970) 85 – 107.
4. P. CHATURANI and D. BISWAS : Three layered Couette flow of polar fluid with non-zero particle spin boundary condition at the interfaces with application to blood flow, *Biorheology*. **20** (1983) 734 - 744.
5. P. CHATURANI and D. BISWAS : A Comparative study of two-layered Poiseuille flow of a polar fluid under different boundary conditions with application to blood flow, Proc. of the Ninth International Congress of Mathematical Biology, 1983, at Paris, France, *Biorheology*, Revue de Bio-Mathematique(Bio-Mathematics). **101** (1988) 47 – 56.
6. L. DINTENFASS : An inversion of the Fahreaeus-Lindqvist phenomenon in blood flow through capillaries of diminishing radius, *Nature*. **215** (1967) 1099 – 1100.
7. R. FAHREAEUS and T. LINDQVIST : The viscosity of blood in narrow capillary tubes, *Amer. J. Physiol.*. **96** (1931) 562 – 568.
8. R. H. HAYNES and A. C. BURTON : Role of non-Newtonian behavior of blood in hemodynamics , *Amer. J. Physiol.*. **197** (1959) 943.
9. S. N. MAJHI and L. USHA : A mathematical note on the Fahreaeus-Lindqvist effect in power-law fluid, *Bull. Math. Biol.*. **47** (1985) 765 – 769.
10. H. P. MAJUMDAR et al. : On the consistency coefficient of a power law flow of blood through the narrow vessel, Engng., *Trans. Polish Academy of Sciences*, Institute of Fundamental Technological Research. **43** (1995) 373 – 383.
11. D. QUEMADA : Blood rheology and its implication in flow of blood, in arteries and arterial blood flow – C.M.Rodkiewicz [Ed.], Springer Verlag, Wien, (1983), 1 - 27.
12. C. M. RODKIEWICZ : Flow in large arteries , in arteries and arterial blood flow - C.M.Rodkiewicz [Ed.], Springer Verlag, Wien, (1983), 327 - 411.
13. D. C. SANYAL and B. SARKAR : Some anomalous features of two-layered blood flow through narrow vessel, Jour. of Indian Acad. Math.. **28** (2006) 455 – 474.

14. J. B. SUKLA et al. : Effects of peripheral layer viscosity on blood flow through the artery with mild stenosis, *Bull. Math. Biol.*. **42** (1980) 797 – 805.

15. P. N. TANDON and K. KUSHWAHA : A study of Fahreaeus-Lindqvist effect in oscillatory capillary blood-blood flows. *Proc. Nat. Acad. Sci. India*. **62(A)** (1992) 797 – 805.

16. R. L. WHITMORE : A theory of blood flow in small vessels, *J. Appl. Physiol.*. **22** (1967) 767 – 771.

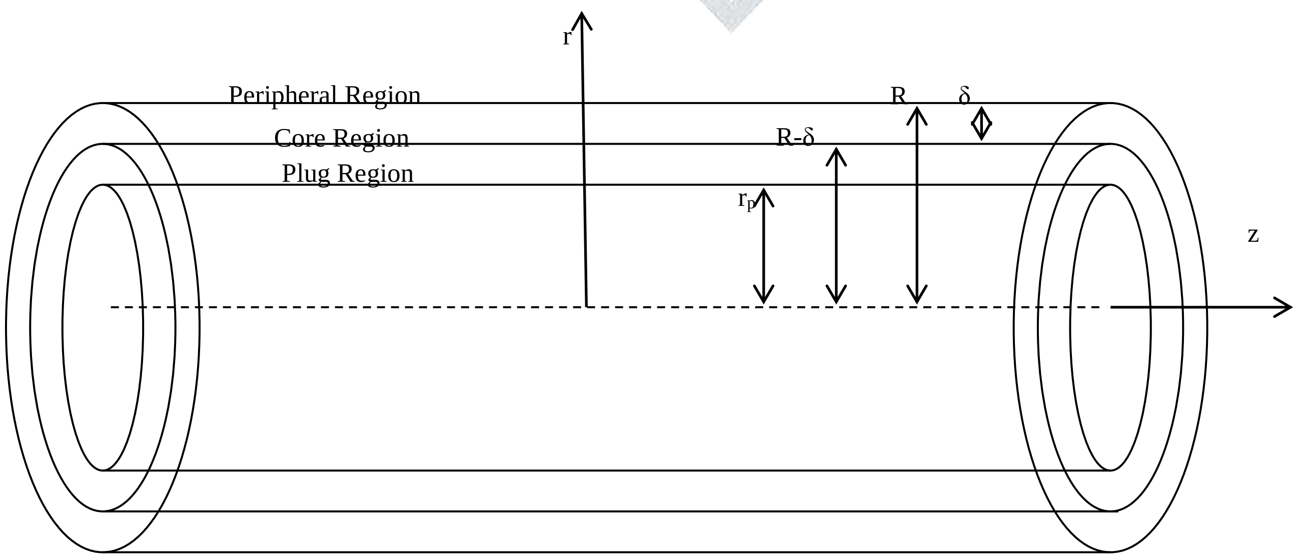

Fig – 1. Geometry of a narrow vessel with two layered flow

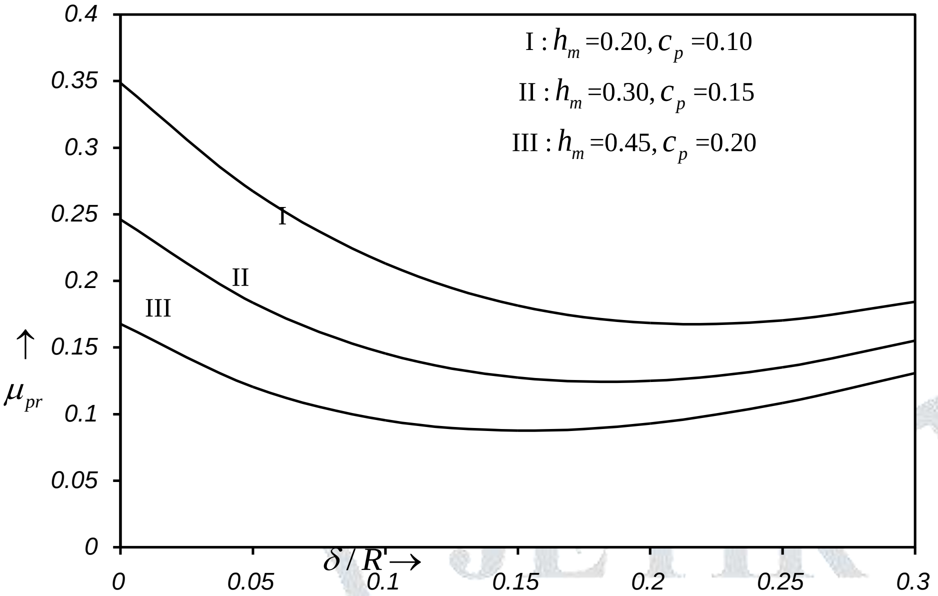

Fig.–2. Variations of peripheral relative coefficient of viscosity for different values of h_m and c_p taking n=2.

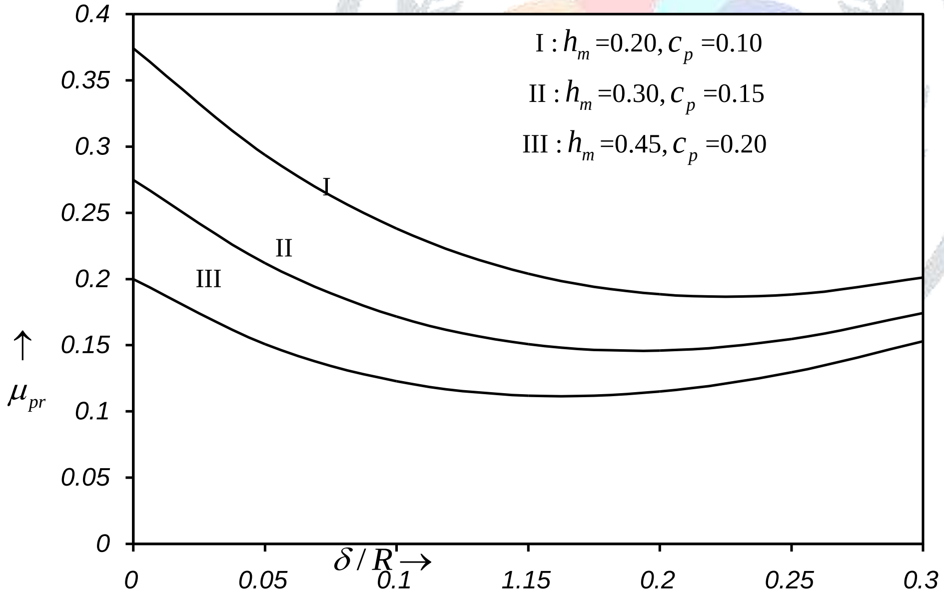

Fig.–3. Variations of peripheral relative coefficient of viscosity for different values of h_m and c_p taking n=1.

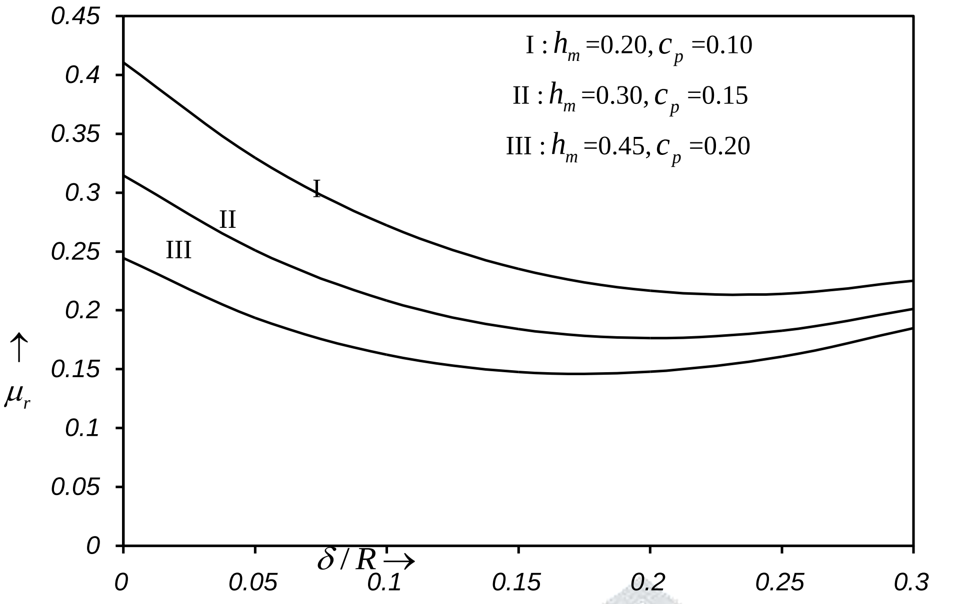

Fig.–4. Variations of relative coefficient of viscosity for different values of h_m and c_p taking n=0.

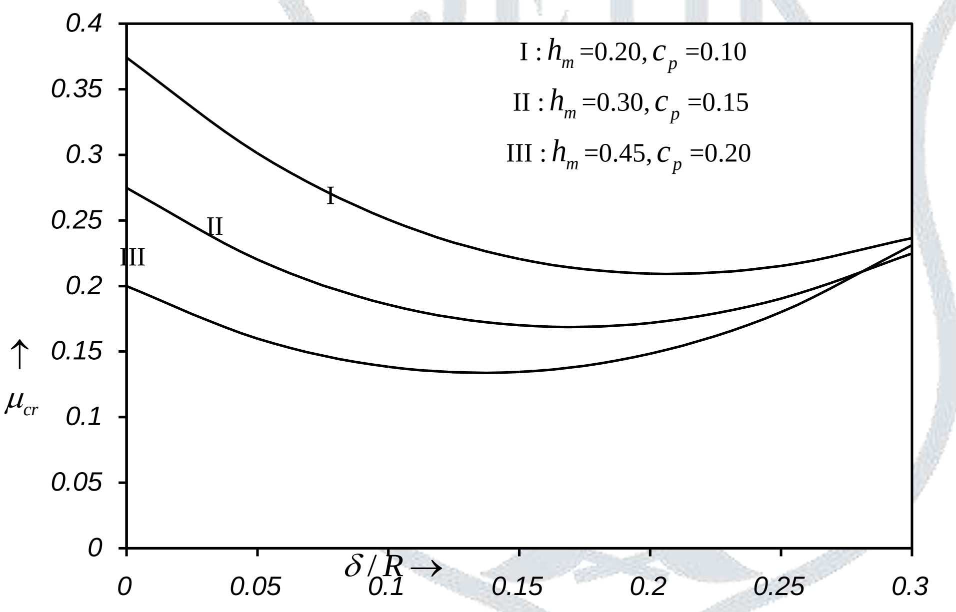

Fig.–5. Variations of core relative coefficient of viscosity for different values of h_m and c_p taking n=1.

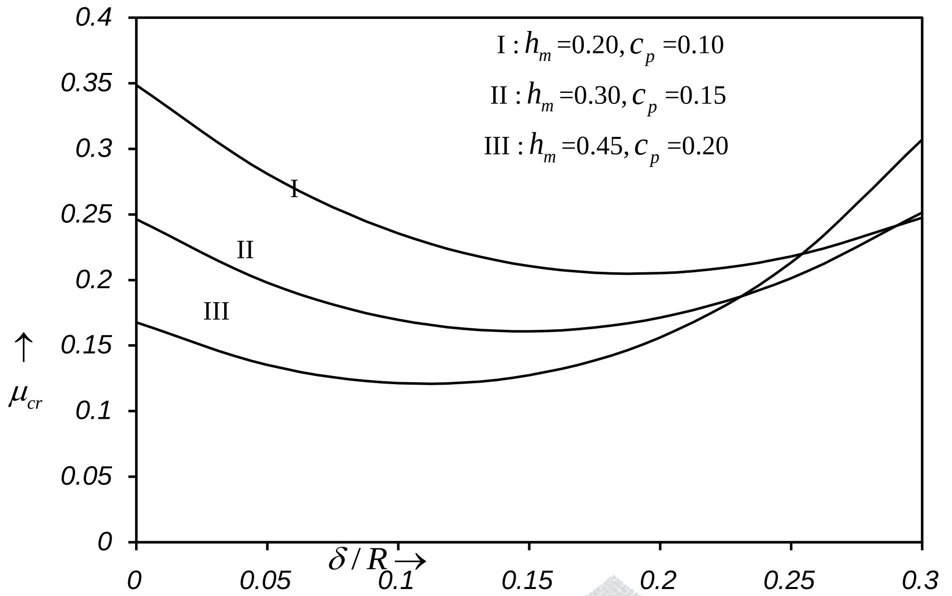

Fig.–6. Variations of core relative coefficient of viscosity for different values of h_m and c_p taking n=2.