THE DUST PARTICLES INFLUENCING THE AIR FLOW IN TRACHEA IN ACCOUNT WITH TIME DEPENDENT PRESSURE GRADIENT

Dr. Subhrojyoti Debnath.
Department of Mathematics
Brahmananda Keshab Chandra College,
111/2 B. T. Road, Kolkata – 700108.

Abstract. The unsteady flow of a dusty incompressible Newtonian fluid with time dependent pressure gradient through the trachea of human respiratory system has been investigated. The effects of two parameters viz. f, the mass concentration and τ_r, the relaxation time on fine and coarse dust particles are studied. The analytical expressions for velocity, flow rate and shear stress are obtained for clean air, air with fine dust particles and air with coarse dust particles respectively and their natures are shown graphically for different radial coordinate and different time due to dust parameters f and τ_r to elucidate the problem.

AMS Mathematics subject classification : 76Z05

1. Introduction.

During inhalation various types of liquid and solid particles are transported with air through trachea and eventually lead to lungs. These types of particles usually generate from different kinds of natural and industrial sources including condensation, smokes, soils and sands, volcanic activities, pollens and micro flora, known as aerosols. Dusts are small particles of substances in a solid state, most of which are microscopic, i.e., not visible to the naked eye. These dust particles circulate in the air and often come into contact with human organs, which lead to number of health hazards. Various types of dusts containing the compounds of sulphar, iron, nitrogen, lead, carbon etc. which usually generates from mines and industries cause serious health hazards, e.g.

(i) allergic reactions and patches in the skin;

(ii) irritation on eyes which may lead to partial loss of vision;

(iii) poisoning of blood and harmful reactions on the excretory system leading to glomerulo nephritis;

(iv) various types of lung diseases such as pneumoconiosis, bronchitis which may lead to bronchial asthma, lung cancer etc.

So it is of great interest to the researchers to trace out the characteristics of various types of dusty fluids within the human organs. Saffman (1961) studied the stability of laminar flow of a dusty gas by assuming that the dust particles are uniform in size and shape and uniformly distributed but the bulk concentration of the dust is very small. The fluid embedded with particles have been studied by Michael and Norey (1966), Healey and Young (1972). Liu (1966, 1967) and Michael and Miller (1966) studied the flow produced by the motion of an infinite plate in a dusty gas occupying the semi-infinite space above it. The concerning health effects of particulate matter was analyzed by Lodge et al. (1981). S. Rao (1969) studied the unsteady flow of a dusty fluid through a uniform pipe under the influence of experimental pressure gradient with respect to time. Singh (1976) and Gupta and Gupta (1976) discussed the flow of a dusty fluid through a channel with arbitrary time varying pressure gradient. Ratchagar and Chitra (2007) considered the motion of dusty air within the trachea taking different sizes of the dust particles. The particles with size in the range between 0.1 μm and 2 μm are termed as fine particles whereas the particles with size above 2 μm are termed as coarse particles.

In this paper we analyze the unsteady laminar flow of dusty air within the trachea taking different sizes of dust particles and assuming the pressure gradient to be time dependent. Since the velocity of air in the lung airways are always small as compared with the speed of sound, the compressibility effects can be neglected (1985) and so, here, we consider the air to be incompressible fluid. In this case the dust is represented by the number density N of small dust particles whose volume concentration is small but has appreciable mass concentration. We also distinguish between fine and coarse dust particles as in the work of Ratchagar and Chitra (2007). When the dusts are fine, the relaxation time τ_r decreases whereas if the dusts are coarse then the relaxation time τ_r increases in a manner proportional to the surface area of the particles. For these two extreme values of relatively small and large relaxation time τ_r, it is possible to simplify the equation of motion and we shall examine these two limiting cases. The analytical expressions for flow velocity, flow rate and wall shear stress are derived. Also some particular cases when the pressure gradient is an exponentially decreasing function of time, periodic function of time etc. are discussed. The effects of relaxation time τ_r, mass concentration f of the dust particles on flow velocity, flow rate and wall shear stress are shown through the graphs.

2. Mathematical Formulation of the Problem. Let us consider the flow of clean air and dusty air through the symmetric form of a circular tube of trachea. The equations of motion for dusty air are

$$\frac{\partial u_a}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu_a\left\{\frac{\partial^2 u_a}{\partial r^2} + \frac{1}{r}\frac{\partial u_a}{\partial r}\right\} + \frac{KN_0}{\rho}\left(u_d - u_a\right), \tag{2.1}$$

$$m\frac{\partial u_d}{\partial t} = K\left(u_a - u_d\right), \tag{2.2}$$

where $u_a(r,z,t)$ is the velocity of particles of clean air, $u_d(r,z,t)$ is the velocity of dust particles in the direction of the axis of the tube, which is taken as the z axis.

The initial and boundary conditions are given by

$$(i)\quad \left.\begin{array}{ll} u_a = u_0 & \text{at } t = 0 \\ u_a = u_0 & \text{at } t = 0 \end{array}\right\} \quad \text{(initial conditions)},$$

$$(ii)\quad \left.\begin{array}{ll} \dfrac{\partial u_a}{\partial r} = 0 & \text{at } r = 0 \\ \dfrac{\partial u_d}{\partial r} = 0 & \text{at } r = 0 \\ u_a = 0 & \text{at } r = R_0 \\ u_d = 0 & \text{at } r = R_0 \end{array}\right\} \quad \text{(boundary conditions)}. \tag{2.3}$$

Here ρ is the density of the clean air, p is the pressure, ν_a is the kinematic coefficient of viscosity, m is the mass of the dust particle, K is the Stokes' drag coefficient (for spherical particles of radius r, K=$6\pi r\mu$), N_0 is the number density of the dust particles and t is the time.

For simplicity, let us introduce non-dimensional quantities by putting

$$r' = \frac{r}{R_0}\ ,\ z' = \frac{z}{L}\ ,\ p' = \frac{pL^2}{\rho {\nu_a}^2}\ ,\ t' = \frac{t\nu_a}{L^2}\ ,\ u_a' = \frac{u_a L}{\nu_a}\ ,\ u_d' = \frac{u_d L}{\nu_a}\ ,\ \tau_r = \frac{\nu_a m}{KL^2}\ ,$$

$$\beta = \frac{KN_0 L^2}{\rho \nu_a}\ ,\ \beta\tau_r = \frac{mN_0}{\rho} = f\ ,$$

where τ_r is the relaxation time, f is the mass concentration of the dust particles, R_0 is the radius of the trachea and L is the length of the trachea.

Using the above non-dimensional quantities in equations (2.1) to (2.3) we get (dropping the primes)

$$\frac{\partial u_a}{\partial t} = -\frac{\partial P}{\partial z} + \frac{L^2}{{R_0}^2}\left\{\frac{\partial^2 u_a}{\partial r^2} + \frac{1}{r}\frac{\partial u_a}{\partial r}\right\} + \beta\left(u_d - u_a\right), \tag{2.4}$$

$$\tau_r \frac{\partial u_d}{\partial t} = \left(u_a - u_d\right), \tag{2.5}$$

(i) $$\left.\begin{matrix} u_a = u_0(r) \\ u_d = u_0(r) \end{matrix}\right\} \quad \text{at } t = 0; \tag{2.6}$$

(ii) $$\left.\begin{matrix} \dfrac{\partial u_a}{\partial r} = 0 \\ \dfrac{\partial u_d}{\partial r} = 0 \end{matrix}\right\} \quad \text{at } r = 0; \tag{2.7}$$

(iii) $$\left.\begin{matrix} u_a = 0 \\ u_d = 0 \end{matrix}\right\} \quad \text{at } r = 1. \tag{2.8}$$

Here the time varying pressure gradient is taken in the form

$$-\frac{\partial p}{\partial z} = p_0 + p_1(t)\,, \tag{2.9}$$

where p_0 is constant and $p_1(t)$ is a function of time.

3. Solutions of the Problem. To find the solutions, we decompose the velocity into a steady and an unsteady part as

$$\left.\begin{matrix} u_a(r,t) = u_{as}(r) + u_{at}(r,t) \\ u_d(r,t) = u_{ds}(r) + u_{dt}(r,t) \end{matrix}\right\}, \tag{3.1}$$

where u_{as} and u_{at} are the steady and unsteady parts of clean air and u_{ds} and u_{dt} are the steady and unsteady parts of dusty air respectively.

Using (3.1) in equations (2.4) and (2.5) and then separating steady part we get

$$\frac{d^2 u_{as}}{dr^2} + \frac{1}{r}\frac{du_{as}}{dr} + p_0 = 0\,. \tag{3.2}$$

The corresponding boundary conditions are

$$\left.\begin{aligned} \frac{du_{as}}{dr} &= 0 \qquad \text{at } r = 0, \\ u_{as} &= 0 \qquad \text{at } r = 1 \end{aligned}\right\}. \tag{3.3}$$

The solution of (3.2) subject to the boundary conditions (3.3) is

$$u_{as}(r) = \frac{p_0}{4}(1 - r^2). \tag{3.4}$$

Again using (3.1) in (2.4) and (2.5) and then considering unsteady parts we get

$$\frac{\partial u_{at}}{\partial t} = p_1(t) + \frac{\partial^2 u_{at}}{\partial r^2} + \frac{1}{r}\frac{\partial u_{at}}{\partial r} + \beta\left(u_{dt} - u_{at}\right), \tag{3.5}$$

$$\tau_r \frac{\partial u_{dt}}{\partial t} = u_{at} - u_{dt}. \tag{3.6}$$

The corresponding initial and boundary conditions are

$$\left.\begin{aligned} u_{at} = u_{dt} = \frac{p_0}{4}\left(1 - r^2\right) &\qquad \text{at } t = 0, \\ \frac{\partial u_{at}}{\partial r} = \frac{\partial u_{dt}}{\partial r} = 0 &\qquad \text{at } r = 0, \\ u_{at} = u_{dt} = 0 &\qquad \text{at } r = 1 \end{aligned}\right\}. \tag{3.7}$$

Applying Laplace transformations we may rewrite the boundary conditions of (3.7) as

$$\left.\begin{aligned} \frac{\partial \mathrm{U}}{\partial \mathrm{r}} = \frac{\partial V}{\partial r} = 0 &\qquad \text{at } r = 0, \\ \mathrm{U(r,s)=V(r,s)=0} &\qquad \text{at } r = 1 \end{aligned}\right\}, \tag{3.8}$$

where U and V are the Laplace transforms of u_{at} and u_{dt} respectively and s is the Laplace transform parameter.

Again from (3.5) and (3.6) and using (3.7) we get

$$\frac{d^2U}{dr^2} + \frac{1}{r}\frac{dU}{dr} - \beta U + \beta V = sU - \frac{p_0}{4}(1 - r^2) - \overline{p_1}(s) \tag{3.9}$$

and

$$(\tau_r s + 1)V - \frac{\tau_r p_0}{4}(1 - r^2) = U. \tag{3.10}$$

Eliminating V between (3.9) and (3.10) we get

$$\frac{d^2U}{dr^2} + \frac{1}{r}\frac{dU}{dr} - \chi U = -\overline{p_1}(s) - \frac{A}{4}(1 - r^2) \tag{3.11}$$

where $\chi = \dfrac{\tau_r s^2 + (1 + f)s}{\tau_r s + 1}$ and $A = \dfrac{(\tau_r s + 1 + f)p_0}{\tau_r s + 1}$.

Also applying the finite Hankel transform on (3.11) and by considering (3.8) we get

$$\overline{U}(\xi_n, s) = \overline{p_1}(s)\frac{J_0(\xi_n)}{\xi_n}\frac{1}{\xi_n^{\;2}+\chi} + A\frac{J_1(\xi_n)}{\xi_n^{\;3}}\frac{1}{\xi_n^{\;2}+\chi}, \tag{3.12}$$

where $\overline{\mathrm{U}}(\xi_n, s)$ is the Hankel transform of U(r,s) with $rJ_0(\xi_n r)$ as the kernel of the transform and ξ_n are the roots of the equation $J_0(\xi) = 0$.

Taking inverse Hankel transform of (3.12) we get

$$U(r,s) = 2\sum_{n=1}^{\infty}\overline{p_1}(s)\frac{1}{\xi_n^{\;2}+\chi}\frac{J_0(\xi_n r)}{\xi_n J_1(\xi_n)} + 2\sum_{n=1}^{\infty}A\frac{1}{\xi_n^{\;2}+\chi}\frac{J_0(\xi_n r)}{\xi_n^{\;3}J_1(\xi_n)}. \tag{3.13}$$

For clean air $\tau_r = 0$ and f =0 which give A= p_0 and $\chi = s$.

Substituting the values of A and χ in (3.13) we get

$$U(r,s) = 2\sum_{n=1}^{\infty}\overline{p_1}(s)\frac{1}{\xi_n^{\;2}+s}\frac{J_0(\xi_n r)}{\xi_n J_1(\xi_n)} + 2\sum_{n=1}^{\infty}p_0\frac{1}{\xi_n^{\;2}+s}\frac{J_0(\xi_n r)}{\xi_n^{\;3}J_1(\xi_n)}. \tag{3.14}$$

Taking inverse Laplace transform we get

$$u_{at}(r,t) = 2\sum_{n=1}^{\infty}\left[\int_0^t p_1(t-\eta)e^{-\xi_n^{\;2}\eta}d\eta\right]\frac{J_0(\xi_n r)}{\xi_n J_1(\xi_n)} + 2\sum_{n=1}^{\infty}p_0 e^{-\xi_n^{\;2}t}\frac{J_0(\xi_n r)}{\xi_n^{\;3}J_1(\xi_n)}\ . \tag{3.15}$$

Hence, the expression for the velocity of the clean air is obtained from (3.4) and (3.15) as

$$u_{at}(r,t) = \frac{p_0}{4}(1-r^2) + 2\sum_{n=1}^{\infty}\left[\int_0^t p_1(t-\eta)e^{-\xi_n^{\;2}\eta}d\eta\right]\frac{J_0(\xi_n r)}{\xi_n J_1(\xi_n)} + 2\sum_{n=1}^{\infty}p_0 e^{-\xi_n^{\;2}t}\frac{J_0(\xi_n r)}{\xi_n^{\;3}J_1(\xi_n)}. \tag{3.16}$$

Now we consider the effects of fine and coarse dust particles which enter into the model through the parameters f and τ_r. If the dusts are fine then the relaxation time τ_r decreases whereas when the dusts are coarse, the relaxation time τ_r increases in a manner proportional to the surface area of the particles. For these two extreme values of τ_r relatively small or large it is possible to simplify the equations of motion (2.4) and (2.5) and examine these two limiting cases.

Considering $\overline{\overline{L}}$ as length scale and $\overline{\overline{U}}$ as velocity scale we observe that the left hand side of (2.5) is of order $\frac{\overline{\overline{L}}}{\overline{\overline{U}}}$ and right hand side is of order $\frac{u_a - u_d}{\tau_r}$.

Case(i): Fine dust particles. For fine dust particles $\tau_r << \frac{\overline{\overline{L}}}{\overline{\overline{U}}}$ and $u_d = u_a$ for disturbances with length scale $\overline{\overline{L}}$ or larger. Thus from (2.5) we get

$$u_a - u_d \cong \tau_r\frac{\partial u_a}{\partial t}. \tag{3.17}$$

Substituting (3.17) in (2.4) we get

$$(1+f)\frac{\partial u_a}{\partial t} = p_0 + p_1(t) + \left\{\frac{\partial^2 u_a}{\partial r^2} + \frac{1}{r}\frac{\partial u_a}{\partial r}\right\}. \tag{3.18}$$

Taking u_f as the velocity of fine dust particles we obtain

$$(1+f)\frac{\partial u_f}{\partial t} = p_0 + p_1(t) + \left\{\frac{\partial^2 u_f}{\partial r^2} + \frac{1}{r}\frac{\partial u_f}{\partial r}\right\}. \tag{3.19}$$

The corresponding initial and boundary conditions are

$$\left.\begin{aligned} u_f &= u_f(r) && \text{at } t = 0, \\ \frac{\partial u_f}{\partial r} &= 0 && \text{at } r = 0, \\ u_f &= 0 && \text{at } r = 1 \end{aligned}\right\}. \tag{3.20}$$

As above we decompose $u_f(r,t)$ into a steady and an unsteady parts as

$$u_f(r,t) = u_{fs}(r) + u_{ft}(r,t) \tag{3.21}$$

so that from (3.19) we get

$$\frac{d^2 u_{fs}}{dr^2} + \frac{1}{r}\frac{du_{fs}}{dr} + p_0 = 0 \tag{3.22}$$

as the steady state equation. The corresponding boundary conditions as obtained from (3.20) are

$$\left.\begin{aligned} \frac{du_{fs}}{dr} &= 0 && \text{at } r = 0, \\ u_{fs} &= 0 && \text{at } r = 1 \end{aligned}\right\}. \tag{3.23}$$

Using (3.23) we obtain the solution of (3.22) as

$$u_{fs}(r) = \frac{p_0}{4}(1 - r^2). \tag{3.24}$$

Similarly we obtain the unsteady state equation as

$$(1+f)\frac{\partial u_{ft}}{\partial t} = \frac{\partial^2 u_{ft}}{\partial r^2} + \frac{1}{r}\frac{\partial u_{ft}}{\partial r} + p_1(t). \tag{3.25}$$

The initial and boundary conditions (3.20) by employing (3.24) become

$$\left.\begin{aligned} u_{fs}(r) = u_{ft}(r) &= \frac{p_0}{4}(1 - r^2) && \text{at } t = 0, \\ \frac{\partial u_{ft}}{\partial r} &= 0 && \text{at } r = 0, \\ u_{ft}(r,t) &= 0 && \text{at } r = 1 \end{aligned}\right\}. \tag{3.26}$$

Then applying Laplace transform we may rewrite the boundary conditions of (3.26) as

$$\left.\begin{aligned} \frac{\partial U_{ft}}{\partial r} &= 0 && \text{at } r = 0, \\ U_{ft}(r,s) &= 0 && \text{at } r = 1 \end{aligned}\right\} \tag{3.27}$$

where U_{ft} is the Laplace transform of u_{ft} and s is the Laplace transform parameter.

Also the equation (3.25) transforms to

$$\frac{d^2U_{ft}}{dr^2}+\frac{1}{r}\frac{dU_{ft}}{dr}-s(1+f)U_{ft}=-\frac{(1+f)p_0}{4}(1-r^2)-\overline{p_1}(s). \tag{3.28}$$

Applying the finite Hankel transform on (3.28) and then using (3.27) we get

$$\overline{U}_{ft}=\frac{\overline{p_1}(s)J_1(\xi_n)}{(1+f)\xi_n}\frac{1}{s+\frac{\xi_n^{\,2}}{1+f}}+\frac{p_0J_1(\xi_n)}{\xi_n^{\,3}}\frac{1}{s+\frac{\xi_n^{\,2}}{1+f}}. \tag{3.29}$$

where $\overline{U}_{ft}(\xi_n,s)$ is the Hankel transform of U_{ft} (r,s) with $rJ_0(\xi_n r)$ as the kernel of the transform and ξ_n are the roots of the equation $J_0(\xi)=0$.

Inverse Hankel transform of (3.29) gives

$$\overline{U}_{ft}(r)=2\sum_{n=1}^{\infty}\frac{\overline{p_1}(s)}{(1+f)}\frac{1}{s+\frac{\xi_n^{\,2}}{1+f}}\frac{J_0(\xi_n r)}{\xi_n J_1(\xi_n)}+2\sum_{n=1}^{\infty}p_0\frac{1}{s+\frac{\xi_n^{\,2}}{1+f}}\frac{J_0(\xi_n r)}{\xi_n^{\,3}J_1(\xi_n)}. \tag{3.30}$$

Taking inverse Laplace transform of (3.30) we get

$$u_{ft}(r,t)=2\sum_{n=1}^{\infty}\left[\int_0^t p_1(t-\eta)e^{-\frac{\xi_n^{\,2}}{1+f}\eta}d\eta\right]\frac{J_0(\xi_n r)}{\xi_n J_1(\xi_n)}+2\sum_{n=1}^{\infty}p_0e^{-\frac{\xi_n^{\,2}}{1+f}t}\frac{J_0(\xi_n r)}{\xi_n^{\,3}J_1(\xi_n)}. \tag{3.31}$$

Thus the required velocity for fine dust is obtained from (3.24) and (3.31) as

$$u_{ft}(r,t)=\frac{p_0}{4}(1-r^2)+2\sum_{n=1}^{\infty}\left[\int_0^t p_1(t-\eta)e^{-\frac{\xi_n^{\,2}}{1+f}\eta}d\eta\right]\frac{J_0(\xi_n r)}{\xi_n J_1(\xi_n)}+2\sum_{n=1}^{\infty}p_0e^{-\frac{\xi_n^{\,2}}{1+f}t}\frac{J_0(\xi_n r)}{\xi_n^{\,3}J_1(\xi_n)}. \tag{3.32}$$

Case(ii): Coarse dust particles. For coarse dust particles, $\tau_r \geq \frac{\overline{\overline{L}}}{\overline{\overline{U}}}$ which arises due to sufficiently coarse dust particles or sufficiently fast flow. In this case, for disturbances of time scale $\frac{\overline{\overline{L}}}{\overline{\overline{U}}}$ or smaller, the dust perturbation velocity u_d is negligible. Thus equation (2.4) reduces to

$$\frac{\partial u_a}{\partial t}=p_0+p_1(t)+\frac{\partial^2 u_a}{\partial r^2}+\frac{1}{r}\frac{\partial u_a}{\partial r}\beta u_a. \tag{3.33}$$

Taking u_c as the velocity of the coarse dust particle we rewrite (3.33) as

$$\frac{\partial u_c}{\partial t}=p_0+p_1(t)+\frac{\partial^2 u_c}{\partial r^2}+\frac{1}{r}\frac{\partial u_c}{\partial r}\beta u_c. \tag{3.34}$$

The corresponding initial and boundary conditions are

$$\left.\begin{array}{ll} u_c = u_c(r) & \text{at } t = 0, \\ \dfrac{\partial u_c}{\partial r} = 0 & \text{at } r = 0, \\ u_c = 0 & \text{at } r = 1 \end{array}\right\}. \tag{3.35}$$

As above we decompose u_c in a steady and an unsteady part as

$$u_c(r,t) = u_{cs}(r) + u_{ct}(r,t) . \tag{3.36}$$

From (3.34) we obtain the steady state equation as

$$\frac{d^2 u_{cs}}{dr^2} + \frac{1}{r}\frac{du_{cs}}{dr} - \beta u_{cs} + p_0 = 0 \tag{3.37}$$

and the corresponding boundary conditions are

$$\left.\begin{array}{ll} \dfrac{du_{cs}}{dr} = 0 & \text{at } r = 0, \\ u_{cs} = 0 & \text{at } r = 1 \end{array}\right\}. \tag{3.38}$$

Applying finite Hankel transform on (3.37) and then using (3.38) we obtain

$$\overline{U}_{cs} = \frac{p_0 J_1(\xi_n)}{\xi_n}\frac{1}{\xi_n{}^2 + \beta} . \tag{3.39}$$

where $\overline{U}_{cs}(\xi_n, s)$ is the Hankel transform of U_{cs} (r,s) with $rJ_0(\xi_n r)$ as the kernel of the transform and ξ_n are the roots of the equation $J_0(\xi) = 0$.

Taking inverse Hankel transform of (3.39) we get

$$u_{cs}(r) = 2\sum_{n=1}^{\infty} p_0 \frac{J_0(\xi_n r)}{(\xi_n{}^2 + \beta)\xi_n J_1(\xi_n)} . \tag{3.40}$$

Finally from (3.34) we get the equation for unsteady state as

$$\frac{\partial u_{ct}}{\partial t} = \frac{\partial^2 u_{ct}}{\partial r^2} + \frac{1}{r}\frac{\partial u_{ct}}{\partial r} - \beta u_{ct} + p_1(t) . \tag{3.41}$$

and the corresponding initial and boundary conditions are

$$\left.\begin{array}{ll} u_{ct} = u_{cs} = 2\displaystyle\sum_{n=1}^{\infty} p_0 \frac{J_0(\xi_n r)}{(\xi_n{}^2 + \beta)\xi_n J_1(\xi_n)} & \text{at } t = 0, \\ \dfrac{\partial u_{ct}}{\partial r} = 0 & \text{at } r = 0, \\ u_{ct} = 0 & \text{at } r = 1 \end{array}\right\}. \tag{3.42}$$

Applying Laplace transform we rewrite the boundary conditions of (3.42) as

$$\left.\begin{array}{ll} \dfrac{\partial U_{ct}}{\partial r} = 0 & \text{at } r = 0, \\ U_{ct} = 0 & \text{at } r = 1 \end{array}\right\} \tag{3.43}$$

where U_{ct} is the Laplace transform of u_{ct} and s is the Laplace transform parameter.

Thus from (3.41) and (3.42) we get

$$\frac{d^2U_{ct}}{dr^2}+\frac{1}{r}\frac{dU_{ct}}{dr}-(\beta+s)U_{ct}=-\overline{p_1}(s)-2\sum_{n=1}^{\infty}p_0\frac{J_0(\xi_n r)}{(\xi_n{}^2+\beta)\xi_n J_1(\xi_n)}. \qquad (3.44)$$

Applying finite Hankel transform we solve (3.44) as

$$\overline{U}_{ct}=\frac{\overline{p_1}(s)J_1(\xi_n)}{\xi_n(\xi_n{}^2+\beta+s)}+\frac{p_0}{(\xi_n{}^2+\beta)}\frac{J_1(\xi_n)}{\xi_n(\xi_n{}^2+\beta+s)} \qquad (3.45)$$

where $\overline{U}_{ct}(\xi_n,s)$ is the Hankel transform of U_{ct} (r,s) with $rJ_0(\xi_n r)$ as the kernel of the transform and ξ_n being the roots of the equation $J_0(\xi)=0$. Inverse Hankel transform of (3.45) gives

$$U_{ct}=2\sum_{n=1}^{\infty}\overline{p_1}(s)\frac{J_0(\xi_n r)}{(\xi_n{}^2+\beta+s)\xi_n J_1(\xi_n)}+2\sum_{n=1}^{\infty}\frac{p_0}{\left(\xi_n{}^2+\beta\right)}\frac{1}{\xi_n{}^2+\beta+s}\frac{J_0(\xi_n r)}{\xi_n J_1(\xi_n)} \qquad (3.46)$$

and inverse Laplace transform of (3.46) gives

$$u_{ct}(r,t)=2\sum_{n=1}^{\infty}\left[\int_0^t p_1(t-\eta)e^{-(\xi_n{}^2+\beta)\eta}d\eta\right]\frac{J_0(\xi_n r)}{\xi_n J_1(\xi_n)}+2\sum_{n=1}^{\infty}p_0 e^{-(\xi_n{}^2+\beta)t}\frac{J_0(\xi_n r)}{(\xi_n{}^2+\beta)\xi_n J_1(\xi_n)}. \qquad (3.47)$$

Thus the required velocity for coarse dust is obtained from (3.40) and (3.47) as

$$u_{ct}(r,t)=2\sum_{n=1}^{\infty}\left[\int_0^t p_1(t-\eta)e^{-(\xi_n{}^2+\beta)\eta}d\eta\right]\frac{J_0(\xi_n r)}{\xi_n J_1(\xi_n)}+2\sum_{n=1}^{\infty}p_0\frac{J_0(\xi_n r)}{(\xi_n{}^2+\beta)\xi_n J_1(\xi_n)}\left[1+e^{-(\xi_n{}^2+\beta)t}\right]. \qquad (3.48)$$

Flow rate

The flow rate for clean air is given by

$$Q_a(t)=\frac{2\pi R_0{}^2\nu_a}{L}\int_0^1 u_a r\,dr$$

$$=\frac{2\pi R_0{}^2\nu_a}{L}\left[\frac{p_0}{16}+2\sum_{n=1}^{\infty}\left\{\int_0^t p_1(t-\eta)e^{-(\xi_n{}^2+\beta)\eta}d\eta\right\}\frac{1}{\xi_n{}^2}+2\sum_{n=1}^{\infty}p_0\frac{e^{-\xi_n{}^2 t}}{\xi_n{}^4}\right]. \qquad (3.49)$$

The flow rate for fine dusty air is given by

$$Q_f(t)=\frac{2\pi R_0{}^2\nu_a}{L}\int_0^1 u_f r\,dr$$

$$=\frac{2\pi R_0{}^2\nu_a}{L}\left[\frac{p_0}{16}+2\sum_{n=1}^{\infty}\left\{\int_0^t p_1(t-\eta)e^{-\frac{\xi_n{}^2}{1+f}\eta}d\eta\right\}\frac{1}{\xi_n{}^2}+2\sum_{n=1}^{\infty}p_0\frac{e^{-\frac{\xi_n{}^2}{1+f}t}}{\xi_n{}^4}\right]. \qquad (3.50)$$

and that for coarse dusty air is given by

$$Q_c(t)=\frac{2\pi R_0^{\,2}\nu_a}{L}\int_0^1 u_c r\,dr$$

$$=\frac{2\pi R_0^{\,2}\nu_a}{L}\left[2\sum_{n=1}^{\infty}\left\{\int_0^t p_1(t-\eta)e^{-(\xi_n^{\,2}+\beta)\eta}d\eta\right\}\frac{1}{\xi_n^{\,2}}+2\sum_{n=1}^{\infty}p_0\frac{1+e^{-(\xi_n^{\,2}+\beta)t}}{(\xi_n^{\,2}+\beta)\xi_n^{\,2}}\right]. \qquad (3.51)$$

Wall shear stress

The wall shear stress for clean air is

$$\tau_a(t)=\left[-\mu_a\rho\nu_0^{\,2}\frac{\partial u_a}{\partial r}\right]_{r=1}$$

$$=\mu_a\rho\nu_0^{\,2}\left[\frac{p_0}{2}+2\sum_{n=1}^{\infty}\left\{\int_0^t p_1(t-\eta)e^{-\xi_n^{\,2}\eta}d\eta\right\}+2\sum_{n=1}^{\infty}p_0\frac{e^{-\xi_n^{\,2}t}}{\xi_n^{\,2}}\right]. \qquad (3.52)$$

The wall shear stress for fine dusty air is

$$\tau_f(t)=\left[-\mu_a\rho\nu_0^{\,2}\frac{\partial u_f}{\partial r}\right]_{r=1}$$

$$=\mu_a\rho\nu_0^{\,2}\left[\frac{p_0}{2}+2\sum_{n=1}^{\infty}\left\{\int_0^t p_1(t-\eta)e^{-\frac{\xi_n^{\,2}}{1+f}\eta}d\eta\right\}+2\sum_{n=1}^{\infty}p_0\frac{e^{-\frac{\xi_n^{\,2}}{1+f}t}}{\xi_n^{\,2}}\right] \qquad (3.53)$$

whereas that for coarse dusty air is

$$\tau_c(t)=\left[-\mu_a\rho\nu_0^{\,2}\frac{\partial u_c}{\partial r}\right]_{r=1}$$

$$=\mu_a\rho\nu_0^{\,2}\left[2\sum_{n=1}^{\infty}\left\{\int_0^t p_1(t-\eta)e^{-(\xi_n^{\,2}+\beta)\eta}d\eta\right\}+2\sum_{n=1}^{\infty}p_0\frac{1+e^{-(\xi_n^{\,2}+\beta)t}}{\xi_n^{\,2}+\beta}\right]. \qquad (3.54)$$

Particular cases :

Case (i). *When the pressure gradient is exponentially decreasing function of time*

Here we consider $p_1(t)=Ce^{-\lambda t}$. Then from (3.16)

$$u_a(r,t)=\frac{p_0}{4}(1-r^2)+2\sum_{n=1}^{\infty}\frac{Ce^{-\lambda t}}{\xi_n^{\,2}-\lambda}\left[1-e^{-\left(\xi_n^{\,2}-\lambda\right)t}\right]\frac{J_0(\xi_n r)}{\xi_n J_1(\xi_n)}+2\sum_{n=1}^{\infty}p_0e^{-\xi_n^{\,2}t}\frac{J_0(\xi_n r)}{\xi_n^{\,3}J_1(\xi_n)}. \qquad (3.55)$$

Also from (3.32), we have

$$u_f(r,t) = \frac{p_0}{4}(1-r^2) + 2\sum_{n=1}^{\infty} \frac{Ce^{-\lambda t}}{\frac{\xi_n^{\,2}}{1+f} - \lambda}\left[1 - e^{-\left(\frac{\xi_n^{\,2}}{1+f} - \lambda\right)t}\right]\frac{J_0(\xi_n r)}{\xi_n J_1(\xi_n)} + 2\sum_{n=1}^{\infty} p_0 e^{-\frac{\xi_n^{\,2}}{1+f}t}\frac{J_0(\xi_n r)}{\xi_n^{\,3} J_1(\xi_n)} \qquad (3.56)$$

and from (3.48)

$$u_c(r,t) = 2\sum_{n=1}^{\infty} \frac{Ce^{-\lambda t}}{\xi_n^{\,2} + \beta - \lambda}\left[1 - e^{-\left(\xi_n^{\,2} + \beta - \lambda\right)t}\right]\frac{J_0(\xi_n r)}{\xi_n J_1(\xi_n)} + 2\sum_{n=1}^{\infty} p_0 \frac{J_0(\xi_n r)}{(\xi_n^{\,2} + \beta)\xi_n J_1(\xi_n)}\left[1 + e^{-\left(\xi_n^{\,2} + \beta\right)t}\right]. \qquad (3.57)$$

Flow rate

The flow rate for clean air from (3.49) is given by

$$Q_a(t) = \frac{2\pi R_0^{\,2} v_a}{L}\left[\frac{p_0}{16} + 2\sum_{n=1}^{\infty} \frac{Ce^{-\lambda t}}{\xi_n^{\,2} - \lambda}\left\{1 - e^{-(\xi_n^{\,2} - \lambda)t}\right\}\frac{1}{\xi_n^{\,2}} + 2\sum_{n=1}^{\infty} p_0 \frac{e^{-\xi_n^{\,2} t}}{\xi_n^{\,4}}\right]. \qquad (3.58)$$

and that for fine dusty air from (3.50) is

$$Q_f(t) = \frac{2\pi R_0^{\,2} v_a}{L}\left[\frac{p_0}{16} + 2\sum_{n=1}^{\infty} \frac{Ce^{-\lambda t}}{\frac{\xi_n^{\,2}}{1+f} - \lambda}\left\{1 - e^{-(\frac{\xi_n^{\,2}}{1+f} - \lambda)t}\right\}\frac{1}{\xi_n^{\,2}} + 2\sum_{n=1}^{\infty} p_0 \frac{e^{-\frac{\xi_n^{\,2}}{1+f}t}}{\xi_n^{\,4}}\right]. \qquad (3.59)$$

Finally, the flow rate for coarse dusty air from (3.51) is obtained as

$$Q_c(t) = \frac{2\pi R_0^{\,2} v_a}{L}\left[2\sum_{n=1}^{\infty} \frac{Ce^{-\lambda t}}{\xi_n^{\,2} + \beta - \lambda}\left\{1 - e^{-(\xi_n^{\,2} + \beta - \lambda)t}\right\}\frac{1}{\xi_n^{\,2}} + 2\sum_{n=1}^{\infty} p_0 \frac{1 + e^{-(\xi_n^{\,2} + \beta)t}}{(\xi_n^{\,2} + \beta)\xi_n^{\,2}}\right]. \qquad (3.60)$$

Wall shear stress

The wall shear stress for clean air from (3.52) is

$$\tau_a(t) = \mu_a \rho v_0^{\,2}\left[\frac{p_0}{2} + 2\sum_{n=1}^{\infty} \frac{Ce^{-\lambda t}}{\xi_n^{\,2} - \lambda}\left\{1 - e^{-(\xi_n^{\,2} - \lambda)t}\right\} + 2\sum_{n=1}^{\infty} p_0 \frac{e^{-\xi_n^{\,2} t}}{\xi_n^{\,2}}\right]. \qquad (3.61)$$

The wall shear stress for fine dusty air from (3.53) is

$$\tau_f(t) = \mu_a \rho v_0^{\,2}\left[\frac{p_0}{2} + 2\sum_{n=1}^{\infty} \frac{Ce^{-\lambda t}}{\frac{\xi_n^{\,2}}{1+f} - \lambda}\left\{1 - e^{-(\frac{\xi_n^{\,2}}{1+f} - \lambda)t}\right\} + 2\sum_{n=1}^{\infty} p_0 \frac{e^{-\frac{\xi_n^{\,2}}{1+f}t}}{\xi_n^{\,2}}\right]. \qquad (3.62)$$

The wall shear stress for coarse dusty air from (3.54) is

$$\tau_c(t) = \mu_a \rho v_0^{\,2}\left[2\sum_{n=1}^{\infty} \frac{Ce^{-\lambda t}}{\xi_n^{\,2} + \beta - \lambda}\left\{1 - e^{-(\xi_n^{\,2} + \beta - \lambda)t}\right\} + 2\sum_{n=1}^{\infty} p_0 \frac{1 + e^{-(\xi_n^{\,2} + \beta)t}}{\left(\xi_n^{\,2} + \beta\right)}\right]. \qquad (3.63)$$

Case (ii). *When the pressure gradient is constant*

Here we put λ =0 in case (i) and then we get similar results as in Ratchagar and Chitra (2007).

In a similar manner we may consider the following cases :

Case (iii). *When the pressure gradient is periodic function of time*

i.e., $p_1(t) = C\sin\omega t$.

Case (iv). *When* $p_1(t) = Cte^{-\lambda t}$.

4. Numerical Results and Discussions.

From the above mathematical analysis it is clear that the velocity, flow rate and wall shear stress depend upon the dust parameters f and τ_r . Here the effects of these parameters on velocity, flow rate and wall shear stress are shown through figures 1 - 4 .

In fig. 1, the changes in velocity for clean air, fine dusty air and coarse dusty air with radial coordinate are shown. It is observed in fig. 1 that increase in time t decreases velocities. Also the velocity for clean air attains its maximum, whereas the velocity for coarse dusty air attains its minimum at the same values of the parameters, i.e.,

$$u_a(r,t) > u_f(r,t) > u_c(r,t) \quad \text{for } 0 < r < 1.$$

In fig. 2, the changes in velocity for clean air, fine dusty air and coarse dusty air with time are shown. Here we observe that

$$u_a(r,t) > u_f(r,t) > u_c(r,t) \quad \text{for } 0 < t < 1.$$

In fact, in the case of clean air there is no resistance due to the dust particles, whereas due to the presence of fine dust particles the velocity of the fine dusty air is lesser than the clean air. In the case of coarse dust particles the resistance is high and for this reason, the velocity of coarse dusty air is minimum.

In fig. 3, the variations in flow rate with time is shown. The resistance of dust particles reduces the flow rate of air. Hence the flow rate of clean air is maximum and coarse dusty air is minimum, i.e.,

$$Q_a(t) > Q_f(t) > Q_c(t) \quad \text{for } 0 < t < 1.$$

In fig. 4, the changes in wall shear stress with time is shown. Due to the resistance of dust particles, the wall shear stress of coarse dusty air becomes maximum and the wall shear stress of clean air becomes minimum, i.e.,

$$\tau_c(t) > \tau_f(t) > \tau_a(t) \quad \text{for } 0 < t < 1.$$

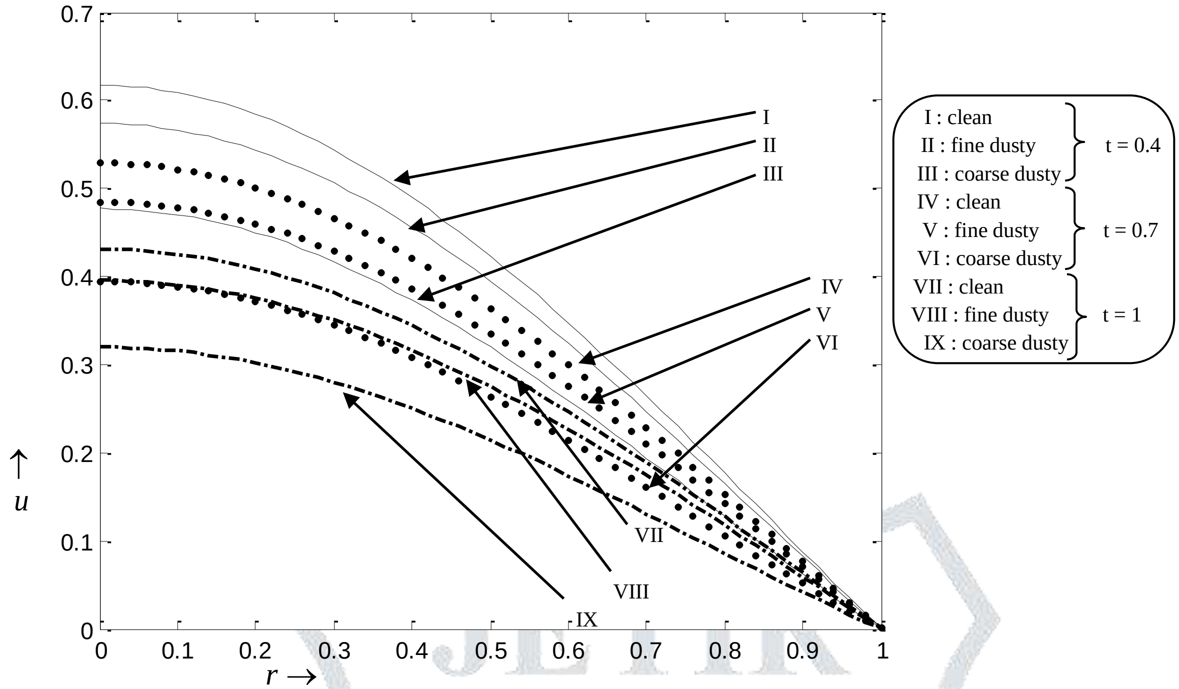

Fig. 1 : Radial coordinate r versus axial velocity u taking different values of time t

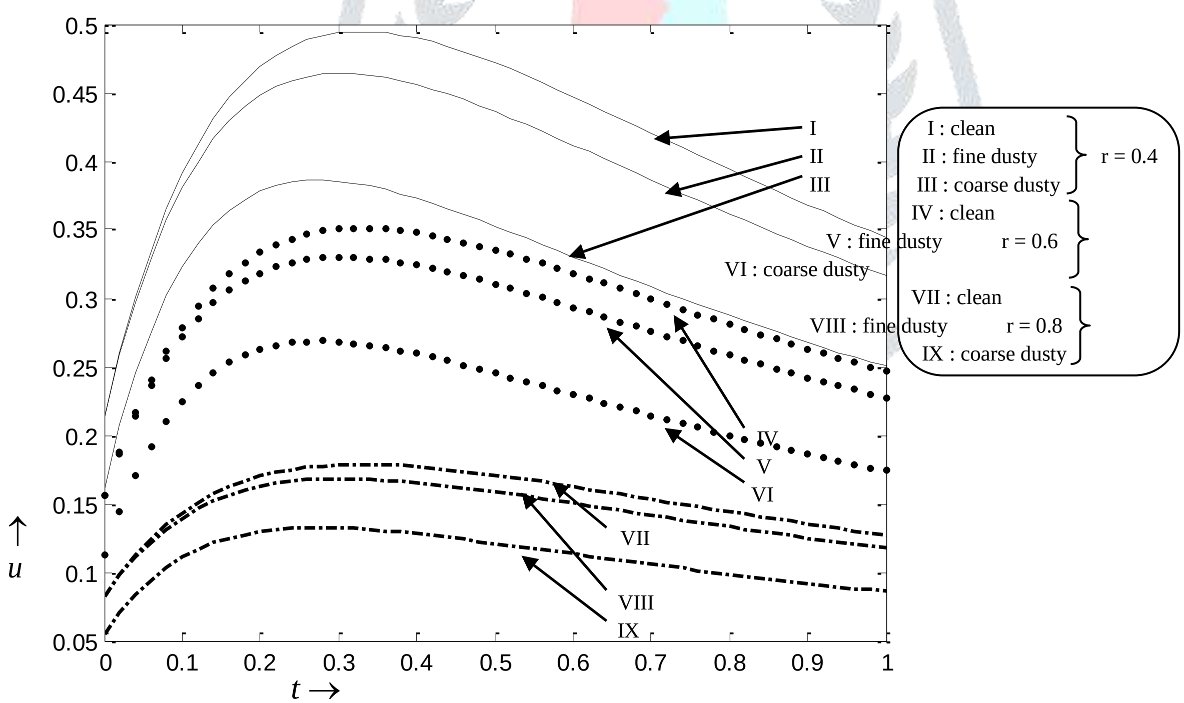

Fig. 2 : Time t versus axial velocity u taking different values of radial coordinate r

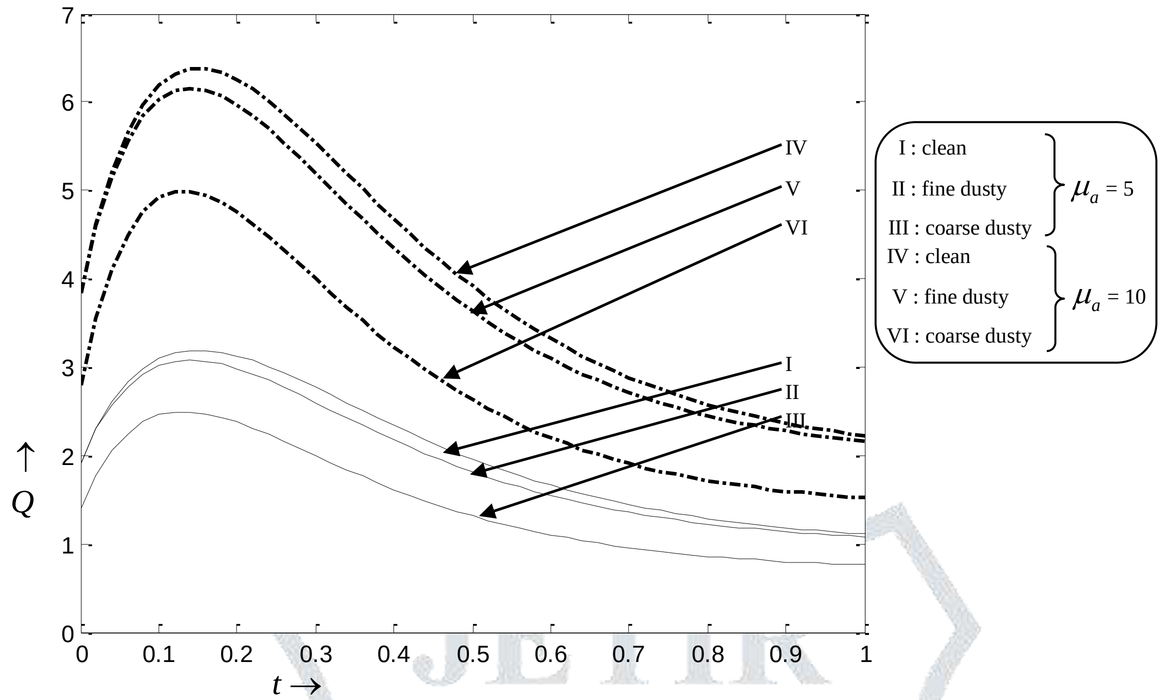

Fig. 3 : Time t versus flow rate Q taking different values of viscosity

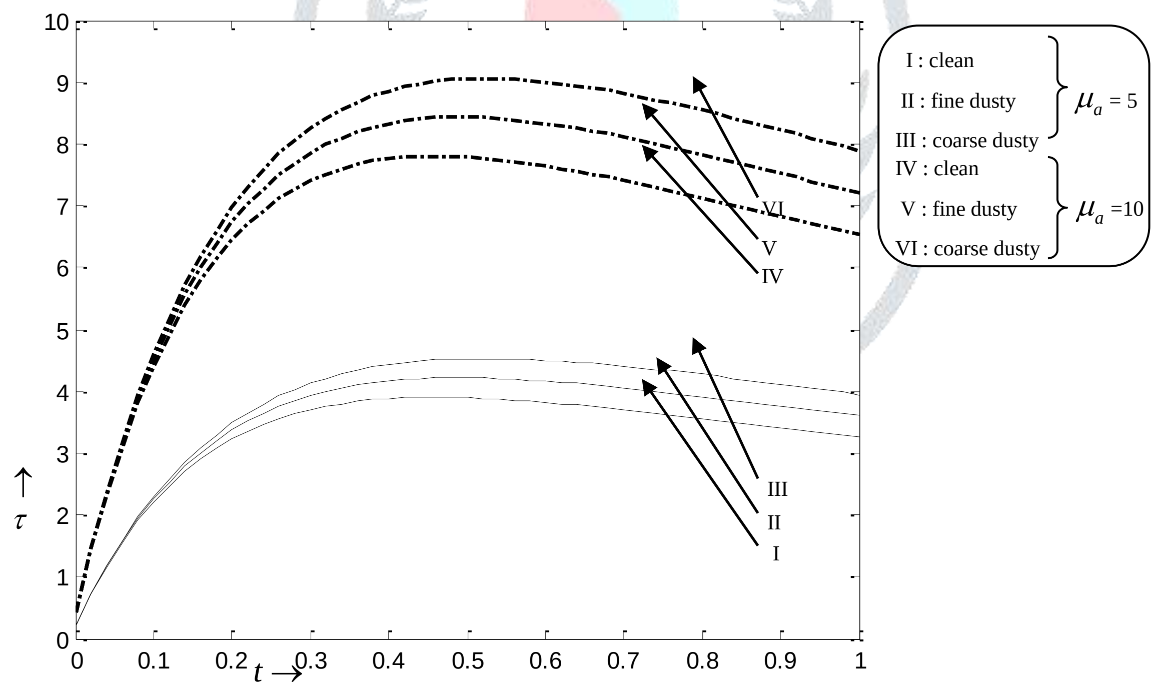

Fig. 4 : Time t versus wall shear stress τ taking different values of viscosity

5. Conclusions. In industrial areas various types of metallic dusts are found in the surrounding air accompanied with several kinds of harmful compounds. Also in the mines dusts of several kinds of ores are found in the air. These dusts usually enters into the human organs during inhalation. Thus it is quite necessary to trace out the flow of dusty air in trachea to reveal the flow velocity, flow rate and wall shear stress of the dusty air. It is expected that the dust particles resist the velocity of air within trachea, so it is quite natural that the velocity of clean air become more than that of the fine dusty air, which is again more than that of the coarse dusty air. The flow rate is obviously maximum for clean air and minimum for coarse dusty air. As the resistance of dust particles increases the wall shear stress, the wall shear stress of coarse dusty air is maximum whereas that is minimum for clean air. Our present discussion also gives the identical results.

References

Gupta, R. K. and Gupta, S. C. (1976) : "Flow of dusty gas through a channel with arbitrary time varying pressure gradient", *ZAMP*, **27**, pp. 119.

Healey, J. V. and Young, H. T. (1972) : "The Stokes' problem for a suspension of particles", *Astronautica Acta*, **17**, pp. 851.

Kapur, J. N. (1985) : "Mathematical models in biology and medicine", *Affiliated East-West Press Pvt. Ltd.*, pp. 408.

Liu, T. J. (1966) : "Flow induced by an oscillating infinite plate in a dusty gas", *Phys. Fluid*, **9**, pp. 1716.

Liu, T. J. (1967) : "Flow indused by the impulsive motion of an infinite plate in a dusty gas", *Astronautica Acta*, **13**, pp. 369.

Lodge, J. P., Waggonerc, A. P., Klodt, D. T., Crain, C. N. (1981) : "Non-health effects of airborne particulate matter", *Atmospheric Environment*, **15**, pp. 431-482.

Michael, T. T. and Norey, P. W. (1966) : "Plane parallel flow of a dusty gas between rotating cylinders", *Q. J. Mech. Math.*, **21**, pp. 375.

Michael, D. H. and Miller, D. A. (1966) : "Plane parallel flow of a dusty gas", *Mathematica*, **13**, pp. 97-109.

Ratchagar, N. P. and Chitra, M. (2007) : "Effects of fine and coarse dust particles on the transport of air in the trachea", *J. of Indian Acad. Math.*, **29(2)**, pp. 551-570.

Saffman, P. G. (1961) : "On the stability of laminar flow of a dusty gas", *J. Fluid Mech.*, **13(1)**, pp. 120-128.

Sambasiva Rao, P. (1969) : "Unsteady flow of dusty viscous liquid through a circular cylinder", *Def. Sc. J.*, **19**, pp. 135.

Singh, K. K. (1976) : "Unsteady flow of a conducting dusty fluid through a rectangular channel with time dependent pressure gradient", *Ind. J. of Pure and Applied Mathematics*, **8**, pp. 1124-1131.