Calculation of Some Performance Measures in M/M/1 Queuing Models with Finite Capacity Using the Method of Order Statistics

Bidyut Das
Assistant Professor, Department of Statistics,
Nowgong College, India
bidyutindrani@gmail.com

Abstract:

This present paper deals with some performance measures in single server with finite capacity Markovian queuing system using the moments of order statistics. The average value and the variance of the minimum number of customer in the system as well as the average value and the variance of the maximum number of customer in the system are obtained.

Keywords: Average, Markovian, Moments, Order statistics, Queuing model.

I. INTRODUCTION

Queuing system has wide applications in modeling and analysis in the field of computer science and communication systems, and various other engineering and management systems. In managing various queuing system, the study of queuing system often concerned with the busy period and waiting time in the queue. Neuts (1973) was first formulated the virtual waiting time distribution for single server queue in discrete time cases. Serfozo (1987) studied the asymptotic behaviour of the maximum value of the recurrent and transient birth-death processes over large time intervals in M/M/c queues. Kinateder and Lee (2000) used Laplace transformation to study the length of a busy period of an M/M/1 queue with constrained workload. Draief and Mairess (2005) analysed the service times of customers depending on their position in a busy period in an M/M/1 queue. They also provided a law of the service of a customer by considering a family of polynomial generating series associated with Dyck paths of length of 2n at the start, in the middle and at the end of the busy period. Guillemin and Pinchon (1998) studied details about using a busy period in an M/M/1 queue. Takagi and Tarabia (2009) deal with more general model M/M/1/L, where L is the capacity of the system and provided an explicit probability density function of the length of a busy period beginning with 'i' consumers. Hanbali and Boxma (2010) during the busy period studied the transient behaviour of a state-dependent M/M/1/L queue. Gross and Harris (2003) derived the steady-state virtual waiting time distribution for an M/M/c queuing model. Berger and Whitt (1995) obtained different approximation and simulation techniques for various queuing processes. They also derived expressions for waiting time, virtual waiting time, and the queue length. Tarabia (2002) computed moments of the busy period for the single-server model by using a recursive procedure. By investigating the extreme values of the maximum queue length he also proved the limit theorems for the virtual waiting time and waiting time for various queue models. Artalejo et al (2007) derived an efficient algorithm for computing the distribution for the maximum number of customers in orbit and in the system during a busy period for the M/M/c retrial queue. The key idea of their algorithm is to diminish the computation of the distribution of the maximum customer number in orbit by computing certain absorption probabilities. Our motivation is to derive some corresponding measures of performance of an M/M/1 queuing model. The average value and the variance of the minimum number of customer in the system as well as the average value and the variance of the maximum number of customer in the system are discussed.

Let us divide the number of arrival of customers into j intervals, and let Y_i be the number of customers in each interval. The corresponding order statistics is defined by $Y_{i:j}$. Three special cases are introduced: (a) $i = j$ defines the maximum number of customers presented in the system, (b) $i = 1$ defines the minimum number of customers in the system, and (c) $i = j = 1$ defines the regular performance measures. So, our interest is to compute $\mu_{1:j} = E(Y^{min})$, $\sigma^2_{1:j} = Var(Y^{min})$, $\mu_{j:j} = E(Y^{max})$, $\sigma^2_{j:j} = Var(Y^{max})$ where $Y^{min} = Min\{Y_i\}$ and $Y^{max} = Max\{Y_i\}$ for $1 \leq i \leq k$.

The present paper is organised as given below.

In section 2, the model is described and some order statistics related theorems are given. In section 3, the proposed works on measures of performance are derived and in section 4, conclusions are drawn.

II. DESCRIPTION OF THE MODEL

Consider a queuing model with single server whose inter-arrival time follows exponential distribution with rates λ and service time follows exponential distribution with μ where capacity of the system is limited, say k.

The physical interpretation of this model may be either

(i) that there is only one server with capacity of k units. or

(ii) that the arriving customers will go for their service elsewhere permanently, if the waiting line is too long $(\leq k)$.

Considering above model, in the steady state, when $t \to \infty, P_n(t) \to P_n$ *and hence* $P_n'(t) \to 0$ then we get

$$p_n = \frac{1-\rho}{1-\rho^{k+1}}\rho^n, n = 0, 1, 2, \ldots, k. \text{ where } \rho = \frac{\lambda}{\mu} \quad (2.1)$$

Saxena and Surendran (1990).

Let the number of customers be N in the system. Then cumulative distribution function (cdf) of N is defined as

$$F(y) = \Pr\{N \leq y\} = \sum_{n=0}^{y} p_n \quad (2.2)$$

In the steady state for M/M/1: (k/FCFS), (2.2) can be written as

$$F(y) = \sum_{n=0}^{y} \frac{1-\rho}{1-\rho^{k+1}}\rho^n = \frac{1-\rho^{y+1}}{1-\rho^{k+1}} \quad (2.3)$$

Now we state two theorems which are useful in our derivation. The theorem 2.1 by Barakat and Abdelkader (2004) deals with the s[th] moments of i[th] order statistics, $Y_{i:j}$ in a sample of size j in a continuous case and the theorem 2.2 by Arnold et al (1992) gives the expressions for the first two moments of the i[th] order statistics, $Y_{i:j}$ in a sample of size j in discrete case.

Theorem 2.1. Let $Y_i, 1 \leq i \leq j$ be a non-negative random variable with distribution function $F_i(y)$. Then in a sample of size j, the s[th] moment of the i[th] order statistics is given by

$$\mu_{i:j}^{(s)} = s\int_0^{\infty} y^{s-1}\left(1 - F_{i:j}(y)\right)dy \quad (2.4)$$

In the case of independently and identically distributed (iid) random variable, when i=j and i=1we get respectively

$$\mu_{j:j}^{(s)} = s\int_0^{\infty} y^{s-1}(1 - [F(y)]^j)dy \quad (2.5)$$

$$\mu_{1:j}^{(s)} = s\int_0^{\infty} y^{s-1}\left(1 - F(y)\right)^j dy \quad (2.6)$$

where $F_{j:j}(y) = [F(y)]^j$ and $F_{1:j}(y) = 1 - \left(1 - F(y)\right)^j$

Theorem 2.2. Let S be a subset of non-negative integers. Then

$$E\left(Y_{i:j}\right) = \mu_{i:j} = \sum_{y=0}^{\infty}\left(1 - F_{i:j}(y)\right) \quad (2.7)$$

$$E\left(Y_{i:j}^{2}\right) = \mu_{i:j}^{(2)} = 2\sum_{y=0}^{\infty} y\left(1 - F_{i:j}(y)\right) + \mu_{i:j} \qquad (2.8)$$

if moments are exist.

Hence variance is given by

$$\sigma_{i:j}^{2} = \mu_{i:j}^{(2)} - \mu_{i:j}^{2} \qquad (2.9)$$

In the case of iid random variables, the expected value and variance of maximum of order statistics are given by

$$\mu_{j:j} = \sum_{y=0}^{\infty}(1 - (F(y))^{j}) \qquad (2.10)$$

$$\mu_{j:j}^{(2)} = 2\sum_{y=0}^{\infty} y(1 - (F(y))^{j}) + \mu_{j:j} \qquad (2.11)$$

$$\sigma_{j:j}^{2} = \mu_{j:j}^{(2)} - \mu_{j:j}^{2} \qquad (2.12)$$

Similarly, the expected value and variance of minimum of order statistics are given by

$$\mu_{1:j} = \sum_{y=0}^{\infty}(1 - F(y))^{j} \qquad (2.13)$$

$$\mu_{1:j}^{(2)} = 2\sum_{y=0}^{\infty} y(1 - F(y))^{j} + \mu_{1:j} \qquad (2.14)$$

$$\sigma_{1:j}^{2} = \mu_{1:j}^{(2)} - \mu_{1:j}^{2} \qquad (2.15)$$

III. MEASURES OF PERFORMANCES

This section of the paper deals with proposed measures of performance which are very useful in managing queuing system. The mean value and the variance of the minimum number of customer in the system as well as the mean value and the variance of the maximum number of customer in the system are obtained. The following lemma helps for the computation of these measures.

Lemma 3.1. Let Y_i be iid random variables, the cdf for the minimum $F_{1:j}(y)$ and the maximum $F_{j:j}(y)$ are given by

$$F_{1:j}(y) = 1 - \left(1 - \frac{1-\rho^{y+1}}{1-\rho^{k+1}}\right)^{j} \qquad (3.1)$$

$$F_{j:j}(y) = \left(\frac{1-\rho^{y+1}}{1-\rho^{k+1}}\right)^{j} \qquad (3.2)$$

3.1 The expected value and the variance of the minimum number of customer in the system.

The expected value of minimum number of customer in the system is given by

$$\mu_{1:j} = \sum_{y=0}^{\infty}\left(1 - F(y)\right)^{j}$$

$$= \sum_{y=0}^{\infty}\left(1 - \frac{1-\rho^{y+1}}{1-\rho^{k+1}}\right)^{j}$$

$$= \frac{1}{\left(1-\rho^{k+1}\right)^{j}}\sum_{y=0}^{\infty}(\rho^{y+1} - \rho^{k+1})^{j}$$

$$= \frac{\rho^{j}}{\left(1-\rho^{k+1}\right)^{j}}\sum_{y=0}^{\infty}(\rho^{y} - \rho^{k})^{j}$$

$$= \frac{\rho^{j}}{\left(1-\rho^{k+1}\right)^{j}}\sum_{y=0}^{\infty}\sum_{l=0}^{j}(-1)^{l}\binom{j}{l}\rho^{kj}\rho^{y(j-l)}$$

$$= \frac{\rho^j}{(1-\rho^{k+1})^j} \sum_{l=0}^{j} (-1)^l \binom{j}{l} \rho^{kj} \sum_{y=0}^{\infty} \rho^{y(j-l)}$$

$$= \frac{1}{(1-\rho^{k+1})^j} \sum_{l=0}^{j} (-1)^l \binom{j}{l} \frac{\rho^{(k+1)j}}{1-\rho^{(j-l)}}$$

The second order moment is given by

$$\mu_{1:j}{}^{(2)} = 2 \sum_{y=0}^{\infty} y(1-F(y))^j + \mu_{1:j}$$

$$= 2 \sum_{y=0}^{\infty} y \left(1 - \frac{1-\rho^{y+1}}{1-\rho^{k+1}}\right)^j + \mu_{1:j}$$

$$= \frac{2\rho^j}{(1-\rho^{k+1})^j} \sum_{l=0}^{j} (-1)^l \binom{j}{l} \rho^{kj} \sum_{y=0}^{\infty} y\rho^{y(j-l)} + \mu_{1:j}$$

$$= \frac{2}{(1-\rho^{k+1})^j} \sum_{l=0}^{j} (-1)^l \binom{j}{l} \rho^{(k+1)j} \frac{\rho^{(j-l)}}{(1-\rho^{(j-l)})^2} + \frac{1}{(1-\rho^{k+1})^j} \sum_{l=0}^{j} (-1)^l \binom{j}{l} \frac{\rho^{(k+1)j}}{1-\rho^{(j-l)}}$$

$$= \frac{1}{(1-\rho^{k+1})^j} \sum_{l=0}^{j} (-1)^l \binom{j}{l} \left[\frac{2\rho^{(kj+2j-l)}}{(1-\rho^{(j-l)})^2} + \frac{\rho^{(k+1)j}}{1-\rho^{(j-l)}}\right]$$

$$= \frac{1}{(1-\rho^{k+1})^j} \sum_{l=0}^{j} (-1)^l \binom{j}{l} \rho^{(k+1)j} \left[\frac{1+\rho^{(j-l}}{(1-\rho^{(j-l)})^2}\right]$$

The variance of minimum number of customer in the system is given by

$$\sigma_{1:j}^2 = \mu_{1:j}{}^{(2)} - \left(\mu_{1:j}\right)^2$$

$$= \frac{1}{(1-\rho^{k+1})^j} \sum_{l=0}^{j} (-1)^l \binom{j}{l} \left[\frac{\rho^{(kj+2j-l)} + \rho^{(k+1)j}}{(1-\rho^{(j-l)})^2}\right] - \left(\frac{1}{(1-\rho^{k+1})^j} \sum_{l=0}^{j} (-1)^l \binom{j}{l} \frac{\rho^{(k+1)j}}{1-\rho^{(j-l)}}\right)^2$$

3.2 The expected value and the variance of the maximum number of customer in the system.

The expected value of maximum number of customer in the system is given by

$$\mu_{j:j} = \sum_{y=0}^{\infty} (1-(F(y))^j)$$

$$= \sum_{y=0}^{\infty} \left(1 - \left(\frac{1-\rho^{y+1}}{1-\rho^{k+1}}\right)^j\right)$$

$$= \sum_{y=0}^{\infty} \left(1 - \left(1 - \frac{\rho^{y+1}-\rho^{k+1}}{1-\rho^{k+1}}\right)^j\right)$$

$$= \sum_{y=0}^{\infty} \left(1 - \sum_{l=0}^{j} (-1)^l \binom{j}{l} \left(\frac{\rho^{y+1}-\rho^{k+1}}{1-\rho^{k+1}}\right)^l\right)$$

$$= \sum_{y=0}^{\infty} \sum_{l=1}^{j} (-1)^{l+1} \binom{j}{l} \left(\frac{\rho^{y+1}-\rho^{k+1}}{1-\rho^{k+1}}\right)^l$$

$$= \sum_{y=0}^{\infty} \sum_{l=1}^{j} (-1)^{l+1} \binom{j}{l} \frac{\rho^l}{(1-\rho^{k+1})^l} (\rho^y - \rho^k)^l$$

$$= \sum_{y=0}^{\infty} \sum_{l=1}^{j} (-1)^{l+1} \binom{j}{l} \frac{\rho^l}{(1-\rho^{k+1})^l} \sum_{m=0}^{l} (-1)^m \binom{l}{m} (\rho^k)^m (\rho^y)^{l-m}$$

$$= \sum_{l=1}^{j} (-1)^{l+1} \binom{j}{l} \frac{\rho^l}{(1-\rho^{k+1})^l} \sum_{m=0}^{l} (-1)^m \binom{l}{m} (\rho^k)^m \sum_{y=0}^{\infty} \rho^{y(l-m)}$$

$$= \sum_{l=1}^{j} (-1)^{l+1} \binom{j}{l} \frac{\rho^l}{(1-\rho^{k+1})^l} \sum_{m=0}^{l} (-1)^m \binom{l}{m} \rho^{km} \frac{1}{1-\rho^{l-m}}$$

The second order moment is given by

$$\mu_{j:j}^{(2)} = 2\sum_{y=0}^{\infty} y\left(1-(F(y))^j\right) + \mu_{j:j}$$

$$= 2\sum_{y=0}^{\infty} y\left(1-\left(\frac{1-\rho^{y+1}}{1-\rho^{k+1}}\right)^j\right) + \mu_{j:j}$$

$$= 2\sum_{y=0}^{\infty} y\left(1-\left(1-\frac{\rho^{y+1}-\rho^{k+1}}{1-\rho^{k+1}}\right)^j\right) + \mu_{j:j}$$

$$= 2\sum_{y=0}^{\infty} y\left(1-\sum_{l=0}^{j}(-1)^l\binom{j}{l}\left(\frac{\rho^{y+1}-\rho^{k+1}}{1-\rho^{k+1}}\right)^l\right) + \mu_{j:j}$$

$$= 2\sum_{y=0}^{\infty} y\sum_{l=1}^{j}(-1)^{l+1}\binom{j}{l}\left(\frac{\rho^{y+1}-\rho^{k+1}}{1-\rho^{k+1}}\right)^l + \mu_{j:j}$$

$$= 2\sum_{y=0}^{\infty} y\sum_{l=1}^{j}(-1)^{l+1}\binom{j}{l}\frac{\rho^l}{(1-\rho^{k+1})^l}(\rho^y-\rho^k)^l + \mu_{j:j}$$

$$= 2\sum_{y=0}^{\infty} y\sum_{l=1}^{j}(-1)^{l+1}\binom{j}{l}\frac{\rho^l}{(1-\rho^{k+1})^l}\sum_{m=0}^{l}(-1)^m\binom{l}{m}(\rho^k)^m(\rho^y)^{l-m} + \mu_{j:j}$$

$$= 2\sum_{l=1}^{j}(-1)^{l+1}\binom{j}{l}\frac{\rho^l}{(1-\rho^{k+1})^l}\sum_{m=0}^{l}(-1)^m\binom{l}{m}(\rho^k)^m\sum_{y=0}^{\infty} y\rho^{y(l-m)} + \mu_{j:j}$$

$$= 2\sum_{l=1}^{j}(-1)^{l+1}\binom{j}{l}\frac{\rho^l}{(1-\rho^{k+1})^l}\sum_{m=0}^{l}(-1)^m\binom{l}{m}\rho^{km}\frac{\rho^{l-m}}{(1-\rho^{l-m})^2}$$

$$+\sum_{l=1}^{j}(-1)^{l+1}\binom{j}{l}\frac{\rho^l}{(1-\rho^{k+1})^l}\sum_{m=0}^{l}(-1)^m\binom{l}{m}\rho^{km}\frac{1}{1-\rho^{l-m}}$$

$$= \sum_{l=1}^{j}(-1)^{l+1}\binom{j}{l}\frac{\rho^l}{(1-\rho^{k+1})^l}\sum_{m=0}^{l}(-1)^m\binom{l}{m}\rho^{km}\left[\frac{1+\rho^{l-m}}{(1-\rho^{l-m})^2}\right]$$

The variance of maximum number of customer in the system is given by

$$\sigma_{j:j}^2 = \mu_{j:j}^{(2)} - \left(\mu_{j:j}\right)^2$$

$$= \sum_{l=1}^{j}(-1)^{l+1}\binom{j}{l}\frac{\rho^l}{(1-\rho^{k+1})^l}\sum_{m=0}^{l}(-1)^m\binom{l}{m}\rho^{km}\left[\frac{1+\rho^{l-m}}{(1-\rho^{l-m})^2}\right]$$

$$-\left(\sum_{l=1}^{j}(-1)^{l+1}\binom{j}{l}\frac{\rho^l}{(1-\rho^{k+1})^l}\sum_{m=0}^{l}(-1)^m\binom{l}{m}\rho^{km}\frac{1}{1-\rho^{l-m}}\right)^2$$

IV. CONCLUSIONS

In this paper, we have considered the Markovian queuing model with a single-server and finite capacity system. The paper presents some complementary measures of performance which are depending on the methods of order statistics. The average value and the variance of the minimum number of customers in the system as well as the average value and the variance of the maximum number of customers in the system are derived.

ACKNOWLEDGEMENT

The authors would like to thank the anonymous referees for their detailed, careful, and exhaustive comments.

REFERENCES

[1] M. F. Neuts, “The single server queue in discrete time-numerical ananlysis,” Naval Research Logistic Quart., vol. 20, no. 1, pp. 297–304, 1973.
[2] R. F. Serfozo, “Extreme values of birth and death processes and queues,” Stochastic Processes and Their Applications, vol. 27, no. 2, pp. 291–306, 1987.
[3] K. K. J. Kinateder and E. Y. Lee, “A new approach to the busy period of the M/M/1 queue,” Queueing Systems, vol. 35, no. 1–4, pp. 105–115, 2000.
[4] M. K. Draief and J. Mairesse, “Services within a busy period of an M/M/1 queue Dyck paths,” Queueing Systems, vol. 49, no. 1, pp. 73–84, 2005.
[5] F. Guillemin and D. Pinchon, “On the area swept under the occupation process of an M/M/1 queue in a busy period,” Queueing Systems, vol. 29, no. 2–4, pp. 383–398, 1998.
[6] H. Takagi and A. M. K. Tarabia, “Explicit probability density function for the length of a busy period in an M/M/1/K queue,” in Advances in Queueing Theory and Network Applications, chapter 12, pp. 213–226, Springer, New York, NY, USA, 2009.
[7] A. Al Hanbali and O. Boxma, “Busy period analysis of the state dependent M/M/1/K queue,” Operations Research Letters, vol. 38, no. 1, pp. 1–6, 2010.
[8] D. Gross and C. Harris, Fundamentals of Queuing Theory, Wiley Series in Probability, 3rd edition, 2003.
[9] A. W. Berger and W. Whitt, “Maximum values in queueing processes,” Probability in the Engineering and Informational Sciences, vol. 9, no. 3, pp. 375–409, 1995.
[10] A. M. K. Tarabia, “A new formula for the transient behaviour of a non-empty M/M/1/∞ queue,” Applied Mathematics and Computation, vol. 132, no. 1, pp. 1–10, 2002.
[11] J. R. Artalejo, A. Economou, and M. J. Lopez-Herrero, “Algorithmic analysis of the maximum queue length in a busy period for the M/M/c retrial queue,” INFORMS Journal on Computing, vol. 19, no. 1, pp. 121–126, 2007.12 Journal of Applied Mathematics .
[12] H. M. Barakat and Y. H. Abdelkader, “Computing the moments of order statistics from nonidentical random variables,” Statistical Methods & Applications, vol. 13, no. 1, pp. 15–26, 2004.
[13] B. C. Arnold, N. Balakrishnan, and H. N. Nagaraja, A First Course in Order Statistics, Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, John Wiley & Sons, New York, NY, USA, 1992.