SOME PROPERTIES OF $\breve{\theta}$-$\mathcal{J}$-CLOSED SETS WITH RESPECT TO AN IDEAL TOPOLOGICAL SPACES

M. VIJAYASANKARI[1], G. RAMKUMAR[2] AND NIVETHA MARTIN[3]

[1]Assistant Professor, Department of Mathematics, Hajee Karutha Rowther Howdia College, Uthamapalayam, Theni, Tamil Nadu, India.

[2]Assistant Professor, Department of Mathematics, Arul Anandar College, Karumathur, Madurai, Tamil Nadu, India.

[3]Assistant Professor, Department of Mathematics, Arul Anandar College, Karumathur, Madurai, Tamil Nadu, India.

Abstract: In this paper, we introduce the properties of $\breve{\theta}$-$\mathcal{J}$-cld sets and $\breve{\theta}$-$\mathcal{J}_\alpha$-cld sets in ideal topological spaces.We discuss about the $\breve{\theta}$-$\mathcal{J}$-closure, ${}_g T\breve{\theta}$-$\mathcal{J}$-spaces and α $T\breve{\theta}$-$\mathcal{J}$-spaces.

2010 Mathematics Subject Classification: 54A05

Keywords: $\breve{\theta}$-$\mathcal{J}$-open, $\breve{\theta}$-$\mathcal{J}$-interior, $\breve{\theta}$-$\mathcal{J}$-closure, ${}_g T\breve{\theta}$-$\mathcal{J}$-spaces, α $T\breve{\theta}$-$\mathcal{J}$-spaces

1. INTRODUCTION

In 1973, M. H. Stone [14], introduced the applications of the theory of Boolean rings to general topology. In 2000, P. Sundaram and M. Rajamani [15], introduced some decompositions of regular generalized continuous maps in topological spaces.

In 2009, V. Renukadevi [13], introduced the note on IR-closed and AIR-sets. A. Acikgoz and et al. [1], introduced the on α-I-continuous and α-I-open functions.In 1966,K. Kuratowski [6], introduced topology. S. Jafari and N. Rajesh [7], introduced the generalized closed sets with respect to an ideal. N. Levine [8], introduced the generalized closed sets in topology.

In this paper, we introduce the properties of $\breve{\theta}$-$\mathcal{J}$-cld sets and $\breve{\theta}$-$\mathcal{J}_\alpha$-cld sets in ideal topological spaces. We discuss about the $\breve{\theta}$-$\mathcal{J}$-closure, ${}_g T\breve{\theta}$-$\mathcal{J}$-spaces and α $T\breve{\theta}$-$\mathcal{J}$-spaces.

2.PRELIMINARIES

An ideal I on a topological space (briefly, TPS) (X, τ) is a nonempty collection of subsets of X which satisfies

(1) $A \in I$ and $B \subseteq A \Rightarrow B \in I$ and

(2) $A \in I$ and $B \in I \Rightarrow A \cup B \in I$.

Given a topological space (X, τ) with an ideal I on X if $\wp(X)$ is the set of all subsets of X, a set operator $(\bullet)^*: \wp(X) \rightarrow \wp(X)$, called a local function [5] of A with respect to τ and I is defined as follows: for $A \subseteq X$, $A^*(I, \tau) = \{x \in X : U \cap A \notin I$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau : x \in U\}$. A Kuratowski closure operator

$cl^\star(\bullet)$ for a topology $\tau^\star(I, \tau)$, called the $\star$-topology and finer than τ, is defined by $cl^\star(A) = A \cup A^\star(I, \tau)$ [5]. We will simply write $A^\star$ for $A^\star(I, \tau)$ and $\tau^\star$ for $\tau^\star(I, \tau)$. If I is an ideal on X, then (X, τ, I) is called an ideal topological space(briefly, ITPS). A subset A of an ideal topological space (X, τ, I) is $\star$-closed (briefly, $\star$-cld) [5] if $A^\star \subseteq A$. The interior of a subset A in (X, $\tau^\star(I)$) is denoted by $int^\star(A)$.

Definition 2.1 A subset K of a TPS X is called:
(i) semi-open set [3] if K$\subseteq$ cl(int(K));
(ii) α-open set [9] if K$\subseteq$ int(cl(int(K)));
(iii) β-open set (Semi-pre-open) [3] if K$\subseteq$ cl(int(cl(K)));
(iv) regular open set [14] if A = int(cl(A))

The complements of the above-mentioned open sets are called their respective closed sets.

Definition 2.2 A subset K of a TPS X is called

(i) g -closed set (briefly, g -cld) [9] if cl(K) $\subseteq$ V whenever K $\subseteq$ V and V is open.
(ii) αgs -closedset (briefly,αgs -cld)[9] if αcl(K) $\subseteq$V whenever K$\subseteq$V and V is semi-open.

(iii) semi-generalized closed (briefly, sg-cld)[3] if scl(K) $\subseteq$V whenever K$\subseteq$V and V is semi-open.

(iv) ψ -closed (briefly, ψ -cld) [12] if scl(K) $\subseteq$V whenever K$\subseteq$V and V is sg-open.

(v) generalized semi-closed (briefly, gs-cld)[10] if scl(K) $\subseteq$V

whenever K$\subseteq$V and V is open.

(vi) α -generalized closed (briefly, α g-cld)[9] if α cl(K) $\subseteq$V whenever K$\subseteq$V and V is open.

(vii) generalized semi-pre-closed(briefly, gsp-cld) [10] if spcl(K) $\subseteq$V whenever K$\subseteq$V and V is open .

The complements of the above-mentioned closed sets are called their respective open sets.

Definition 2.3

A space (X, τ) is called

(i) $T_{1/2}$-space [8] if every g-closed set is closed.

(ii) T_b-space [2] if every gs-closed set is closed.

(iii) αT_b-space [9] if every α g-closed set is closed.

(iv) T_ω -space [17] if every ω -closed set is closed.

(v) T_p*-space [16] if every g*p-closed set is closed.

(vi) $*_sT_p$-space [16] if every gsp-closed set is g*p-closed.

(vii) αT_d-space [9] if every α g-closed set is g-closed.

(viii) α -space [11] if every α -closed set is closed.

Definition 2.4 [3]

The intersection of all sg-open subsets of X containing K is called the sg-kernel of K and denoted by sg-ker(K).

Definition 2.5 [9]

A subset K of X is called locally closed (briefly, lc) if K = U ∩ F, where U is open and F is closed in X.

Definition 2.6 A subset K of a ITPS X is called I_g -closed (briefly, I_g -cld) set [4] if K* ⊆ V whenever K ⊆ V and V is open.

The complements of the above-mentioned closed set are called their respective open set.

3. PROPERTIES OF $\breve{\theta}$-$\mathcal{J}$-CLOSED SETS

Definition 3.1

A subset K of X is called

(i) $\breve{\theta}$-$\mathcal{J}$-closed (briefly, $\breve{\theta}$-$\mathcal{J}$-cld) if K* ⊆ V whenever K ⊆ V and V is sg-open.

The complement of $\breve{\theta}$-$\mathcal{J}$-cld is called $\breve{\theta}$-$\mathcal{J}$-open.

The family of all $\breve{\theta}$-$\mathcal{J}$-cld in X is denoted by $\breve{\theta}$-$\mathcal{J}$C(X).

(ii) $\breve{\theta}$-$\mathcal{J}_\alpha$-closed (briefly, $\breve{\theta}$-$\mathcal{J}_\alpha$-cld) if α cl(K*) ⊆ V whenever K ⊆ V and V is sg-open.

The complement of $\breve{\theta}$-$\mathcal{J}_\alpha$-cld is called $\breve{\theta}$-$\mathcal{J}_\alpha$-open.

(i) I-ĝ-closed (briefly, I-ω-cld) if K* ⊆ V whenever K ⊆ V and V is semi-open.

The complements of the above-mentioned closed set are called their respective open set.

Lemma 3.2

A subset K of X is $\breve{\theta}$-$\mathcal{J}$-cld if and only if K*⊆ sg-ker(K).

Proof

Suppose that K is $\breve{\theta}$-$\mathcal{J}$-cld. Then K*⊆V whenever K⊆V and V is sg-open. Let x ∈K*. If x ∉ sg-ker(K), then there is a sg-open V containing K such that x ∉V. Since V is a sg-open containing K, we have x ∉K* and this is a contradiction.

Conversely, let K*⊆ sg-ker(K). If V is any sg-open containing K, then K*⊆ sg-ker(K) ⊆V. Therefore, K is $\breve{\theta}$-$\mathcal{J}$-cld.

Theorem 3.3

If a subset K of X is $\breve{\theta}$-$\mathcal{J}$-cld then X_1 ∩K*⊆K, where X_1 = {x ∈ X : {x} is nowhere dense}.

Proof

Suppose that K is $\breve{\theta}$-$\mathcal{J}$-cld. Let x ∈ X_1 ∩K*. Then x ∈ X_1 and x ∈K*. Since x ∈ X_1, int(({x})*) = ϕ. Therefore, {x} is semi-cld, since int(({x})*) ⊆ {x}. Since every semi-cld is sg-cld, we have {x} is sg-cld. If x ∉K and if V = X \ {x}, then V is a sg-open containing K and so K*⊆V, a contradiction.

Lemma 3.4

If G is a ⋆-cld of X. Then the following properties hold:

(i) If K is semi-cld in X, then K∩G is semi-cld in X.

(ii) If K is sg-cld in X, then K∩G is sg-cld in X.

Corollary 3.5

If K is a $\breve{\theta}$-$\mathcal{I}$-cld and G is a ⋆-cld, then K∩ F is a $\breve{\theta}$-$\mathcal{I}$-cld .

Proof

Let V be a sg-open of X such that K∩G⊆V. By Lemma 3.4, it is show that K⊆V∪ (X \ G) and V∪ (X \ G) is sg-open in X. Since K is $\breve{\theta}$-$\mathcal{I}$-cld in X, we have K*⊆V∪ (X \ G) and so (K∩G)*⊆K*∩G* = K*∩G⊆ (V∪ (X \ G)) ∩G = V∩G⊆V. Therefore, K∩G is $\breve{\theta}$-$\mathcal{I}$-cld in X.

Proposition 3.6

If K and L are $\breve{\theta}$-$\mathcal{I}$-cld in X, then K∪L is $\breve{\theta}$-$\mathcal{I}$-cld in X.

Proof

If K∪L⊆H and H is sg-open, then K⊆H and L⊆H. Since K and L are $\breve{\theta}$-$\mathcal{I}$-cld, H⊇K* and H⊇L* and hence H⊇K*∪L* = (K∪L)*. Thus K∪L is $\breve{\theta}$-$\mathcal{I}$-cld in X.

Proposition 3.7

If a set K is$\breve{\theta}$-$\mathcal{I}$-cld in X, then K*–K contains no nonempty ⋆-cld set in X.

Proof

Suppose that K is $\breve{\theta}$-$\mathcal{I}$-cld. Let G be a ⋆-cld subset of K*–K. Then K⊆G^c. But K is $\breve{\theta}$-$\mathcal{I}$-cld, therefore K*⊆G^c. Consequently, G⊆ $(K^*)^c$. We already have G⊆K*. Thus G⊆K*∩ $(K^*)^c$ and G is empty.

The converse of Proposition 3.7 need not be true as seen from the following example.

Example 3.8

Let X = {a, b, c} with τ = {ϕ, {a}, X} and I={ϕ}. Then $\breve{\theta}$-$\mathcal{I}$-C(X) = {ϕ, {b, c}, X}. If K = {b}, then K*–K = {c} does not contain any nonempty ⋆-cld . But K is not $\breve{\theta}$-$\mathcal{I}$-cld in X.

Proposition 3.9

If K is $\breve{\theta}$-$\mathcal{I}$-cld in X and K⊆L⊆K*, then L is $\breve{\theta}$-$\mathcal{I}$-cld in X.

Proof

Since L⊆K*, we have L*⊆K*. Then, L*–L⊆K*–K. Since K*–K has no nonempty sg-cld subsets, neither does L*–L. We haveL is$\breve{\theta}$-$\mathcal{I}$-cld.

Proposition 3.10

Let K⊆Z⊆X and suppose that K is $\breve{\theta}$-$\mathcal{I}$-cld in X. Then K is $\breve{\theta}$-$\mathcal{I}$-cld relative to Z.

Proof

Let K⊆Z∩H, where H is sg-open in X. Then K⊆H and hence K*⊆H. This implies that Z∩K*⊆Z∩H. Thus K is $\breve{\theta}$-$\mathcal{J}$-cld relative to Z.

Proposition 3.11

If K is a sg-open and$\breve{\theta}$-$\mathcal{J}$-cld in X, then K is ⋆-cld in X.

Proof

Since K is sg-open and $\breve{\theta}$-$\mathcal{J}$-cld, K*⊆K and hence K is ⋆-cld in X.

Theorem 3.12

Let X be extremally disconnected and K a semi-open subset of X. Then K is $\breve{\theta}$-$\mathcal{J}$-cld if and only if it is sg-cld.

Proof

It follows from the fact that if X is extremally disconnected and K is a semi-open subset of X, then scl(K) = K*.

Theorem 3.13

Let K be a locally closed set of X. Then K is ⋆-cld if and only if K is $\breve{\theta}$-$\mathcal{J}$-cld.

Proof

(i) ⇒ (ii). It is fact that every ⋆-cld is $\breve{\theta}$-$\mathcal{J}$-cld.

(ii) ⇒ (i). We haveK∪ (X –K*) is open in X, since K is locally closed. Now K∪ (X –K*) is sg-open of X such that K⊆K∪ (X –K*). Since K is $\breve{\theta}$-$\mathcal{J}$-cld, then K*⊆K∪ (X –K*). Thus, we have K*⊆K and hence K is a ⋆-cld.

Proposition 3.14

For each x ∈ X, either {x} is sg-cld or $\{x\}^c$ is $\breve{\theta}$-$\mathcal{J}$-cld in X.

Proof

Suppose that {x} is not sg-cld in X. Then $\{x\}^c$ is not sg-open and the only sg-open containing $\{x\}^c$ is the space X itself. Therefore $(\{x\}^c)^*$⊆ X and so $\{x\}^c$ is $\breve{\theta}$-$\mathcal{J}$-cld in X.

Theorem 3.15

Let K be a $\breve{\theta}$-$\mathcal{J}$-cld set of a ideal topological space X. Then,

(i) sint(K) is $\breve{\theta}$-$\mathcal{J}$-cld.

(ii) If K is regular open, then pint(K) and scl(K) are also $\breve{\theta}$-$\mathcal{J}$-cld.

(iii) If K is regular closed, then pcl(K) is also $\breve{\theta}$-$\mathcal{J}$-cld.

Proof

(i) Since (int(K))* is a ⋆-cld set in X, by Corollary 3.5, sint(K) = K$\cap$ (int(K))* is $\breve{\theta}$-$\mathcal{J}$-cld in X.

(ii) Since K is regular open in X, K = int(K*). Then scl(K) = K$\cup$ int(k*) = K. Thus, scl(K) is $\breve{\theta}$-$\mathcal{J}$-cld in X. Since pint(K) = K$\cap$ int(K*) = K, pint(K) is $\breve{\theta}$-$\mathcal{J}$-cld.

(iii) Since K is regular closed in X, K = (int(K))*. Then pcl(K) = K$\cup$ (int(K))* = K. Thus, pcl(K) is $\breve{\theta}$-$\mathcal{J}$-cld in X.

The converses of the statements in the Theorem 3.15 are not true as we can see in the following examples.

Example 3.16

Let X = {p, q, r} with τ = {ϕ, {r}, {q, r}, X} and I = {ϕ}. Then $\breve{\theta}$-$\mathcal{J}$-C(X) = {ϕ, {p}, {p, q}, X}. Then the set K = {q} is not a $\breve{\theta}$-$\mathcal{J}$-cld set. However sint(K) = ϕ is a $\breve{\theta}$-$\mathcal{J}$-cld.

Example 3.17

Let X = {p, q, r} with τ = {ϕ, {p}, {q}, {p, q}, {p, r}, X} and I = {ϕ}. Then $\breve{\theta}$-$\mathcal{J}$-C(X) = {ϕ, {q}, {r}, {p, r}, {q, r}, X}. Then the set K = {r} is not regular open. However K is $\breve{\theta}$-$\mathcal{J}$-cld and scl(K) = {r} is a $\breve{\theta}$-$\mathcal{J}$-cld and pint(K) = ϕ is also $\breve{\theta}$-$\mathcal{J}$-cld.

Example 3.18

Let X = {p, q, r} with τ = {ϕ, {p, q}, X} and I = {ϕ}. Then $\breve{\theta}$-$\mathcal{J}$-C(X) = {ϕ, {r}, {p, r}, {q, r}, X}. Then the set K = {r} is not regular closed. However K is a $\breve{\theta}$-$\mathcal{J}$-cld and pcl(K) = {r} is $\breve{\theta}$-$\mathcal{J}$-cld.

4. $\breve{\theta}$-$\mathcal{J}$-CLOSURE AND $\breve{\theta}$-$\mathcal{J}$-OPEN SETS

Definition 4.1

For every set K$\subseteq$ X, we define the $\breve{\theta}$-$\mathcal{J}$-closure of K to be the intersection of all $\breve{\theta}$-$\mathcal{J}$-cld sets containing K.

i.e., $\breve{\theta}$-$\mathcal{J}$-cl(K) = $\cap$ {G : K$\subseteq$G$\in\breve{\theta}$-$\mathcal{J}$-C(X)}.

Lemma 4.2

For any K$\subseteq$ X, K$\subseteq\breve{\theta}$-$\mathcal{J}$-cl(K) $\subseteq$K*.

Proof

It follows from Proposition 3.6.

Remark 4.3

Both containment relations in Lemma 4.2 may be proper as seen from the following example.

Example 4.4

Let X = {p, q, r} and τ = {ϕ, {p, q}, X} and I = {ϕ}. Let K = {p}. Then $\breve{\theta}$-$\mathcal{I}$-cl(K) = {p, r} and so K$\subset\breve{\theta}$-$\mathcal{I}$-cl(K) $\subset$K*.

Lemma 4.5

For any K$\subseteq$ X, I-ω-cl(K) $\subseteq\breve{\theta}$-$\mathcal{I}$-cl(K), where I-ω-cl(K) is given by I-ω-cl(K) = $\cap$ {G : K$\subseteq$G$\in$I-ω-C(X)}.

Proof

It follows from Proposition 3.6.

Remark 4.6

Containment relation in the above Lemma 4.5 may be proper as seen from the following example.

Example 4.7

Let X = {p, q, r, s} with τ = {ϕ, {p}, {q, r}, {p, q, r}, X} and I = {ϕ}. Then $\breve{\theta}$-$\mathcal{I}$-C(X) = {ϕ, {s}, {p, s}, {q, r, s}, X} and I-ω-C(X) = {ϕ, {s}, {p, s}, {q, s}, {r, s}, {p, q, s}, {p, r, s}, {q, r, s}, X}. Let K = {q, s}. Then $\breve{\theta}$-$\mathcal{I}$-cl(K) = {q, r, s} and I-ω-cl(K) = {q, s}. So, I-ω-cl(K) $\subset\breve{\theta}$-$\mathcal{I}$-cl(K).

The following two Propositions are easy consequences from definitions.

Proposition 4.8

For any K$\subseteq$ X, the following holds:

(i) $\breve{\theta}$-$\mathcal{I}$-cl(K) is the smallest $\breve{\theta}$-$\mathcal{I}$-cld set containing K.

(ii) K is$\breve{\theta}$-$\mathcal{I}$-cld if and only if $\breve{\theta}$-$\mathcal{I}$-cl(K) = K.

Proposition 4.9

For any two subsets K and L of X, the following holds:

(i) If K$\subseteq$L, then $\breve{\theta}$-$\mathcal{I}$-cl(K) $\subseteq\breve{\theta}$-$\mathcal{I}$-cl(L).

(ii) $\breve{\theta}$-$\mathcal{I}$-cl(K$\cap$L) $\subseteq\breve{\theta}$-$\mathcal{I}$-cl(K) $\cap\breve{\theta}$-$\mathcal{I}$-cl(L).

Definition 4.10

A subset K of X is called $\breve{\theta}$-$\mathcal{I}$-open in X if K^c is $\breve{\theta}$-$\mathcal{I}$-cld in X.

The collection of all $\breve{\theta}$-$\mathcal{I}$-open sets in X is denoted by $\breve{\theta}$-$\mathcal{I}$-O(X).

Proposition 4.11

If K and L are $\breve{\theta}$-$\mathcal{I}$-open sets in X, then K$\cap$L is $\breve{\theta}$-$\mathcal{I}$-open in X.

We introduce the following definition.

Definition 4.12

For any K$\subseteq$ X, $\breve{\theta}$-$\mathcal{J}$-int(K) is defined as the union of all $\breve{\theta}$-$\mathcal{J}$-open sets contained in K. i.e., $\breve{\theta}$-$\mathcal{J}$-int(K) = $\cup$ {H : H$\subseteq$K and H is $\breve{\theta}$-$\mathcal{J}$-open}.

Lemma 4.13

For any K$\subseteq$ X, int(K) $\subseteq\breve{\theta}$-$\mathcal{J}$-int(K) $\subseteq$K.

Proof

It follows from Definition3.1 (i).

The following two Propositions are easy consequences from definitions.

Proposition 4.14

For any K$\subseteq$ X, the following holds:

(i) $\breve{\theta}$-$\mathcal{J}$-int(K) is the largest $\breve{\theta}$-$\mathcal{J}$-open set contained in K.

(ii) K is $\breve{\theta}$-$\mathcal{J}$-open if and only if $\breve{\theta}$-$\mathcal{J}$-int(K) = K.

Proposition 4.15

For any subsets K and L of X, the following holds:

(i) $\breve{\theta}$-$\mathcal{J}$-int(K$\cap$L) $\subseteq\breve{\theta}$-$\mathcal{J}$-int(K) $\cap\breve{\theta}$-$\mathcal{J}$-int(L).

(ii) $\breve{\theta}$-$\mathcal{J}$-int(K$\cup$L) $\supseteq\breve{\theta}$-$\mathcal{J}$-int(K) $\cup\breve{\theta}$-$\mathcal{J}$-int(L).

(iii) If K$\subseteq$L, then $\breve{\theta}$-$\mathcal{J}$-int(K) $\subseteq\breve{\theta}$-$\mathcal{J}$-int(L).

(iv) $\breve{\theta}$-$\mathcal{J}$-int(X) = X and $\breve{\theta}$-$\mathcal{J}$-int(ϕ) = ϕ.

Theorem 4.16

Let K be any subset of X. Then

(i) ($\breve{\theta}$-$\mathcal{J}$-int(K))^c = $\breve{\theta}$-$\mathcal{J}$-cl(K^c).

(ii) $\breve{\theta}$-$\mathcal{J}$-int(K) = ($\breve{\theta}$-$\mathcal{J}$-cl(K^c))^c.

(iii) $\breve{\theta}$-$\mathcal{J}$-cl(K) = ($\breve{\theta}$-$\mathcal{J}$-int(K^c))^c.

Proof

(i) Let x $\in$ ($\breve{\theta}$-$\mathcal{J}$-int(K))^c. Then x $\notin\breve{\theta}$-$\mathcal{J}$-int(K). That is, every $\breve{\theta}$-$\mathcal{J}$-open set V containing x is such that V$\not\subseteq$K. That is, every $\breve{\theta}$-$\mathcal{J}$-open set V containing x is such that V$\cap$K^c$\neq\phi$. We have, x $\in\breve{\theta}$-$\mathcal{J}$-cl(K^c) and therefore ($\breve{\theta}$-$\mathcal{J}$-int(K))^c$\subseteq\breve{\theta}$-$\mathcal{J}$-cl(K^c).

Conversely, let x $\in\breve{\theta}$-$\mathcal{J}$-cl(K^c). Then by Lemma 4.5., every $\breve{\theta}$-$\mathcal{J}$-open set V containing x is such that V$\cap$K^c$\neq\phi$. That is, every $\breve{\theta}$-$\mathcal{J}$-open set V containing x is such that V$\not\subseteq$K. This implies by Definition 4.12, x $\notin\breve{\theta}$-$\mathcal{J}$-int(K). That is, x $\in$ ($\breve{\theta}$-$\mathcal{J}$-int(K))^c and so $\breve{\theta}$-$\mathcal{J}$-cl(K^c) $\subseteq$ ($\breve{\theta}$-$\mathcal{J}$-int(K))^c. Thus ($\breve{\theta}$-$\mathcal{J}$-int(K))^c = $\breve{\theta}$-$\mathcal{J}$-cl(K^c).

(ii) Follows by taking complements in (i).

(iii) Follows by replacing K by K^c in (i).

5. SOME APPLICATIONS OF $\breve{\theta}$-$\mathcal{I}$-CLOSED SETS

As applications of $\breve{\theta}$-$\mathcal{I}$-cld sets, we introduce the notions called T$\breve{\theta}$-$\mathcal{I}$-spaces, $_g$T$\breve{\theta}$-$\mathcal{I}$-spaces and α T$\breve{\theta}$-$\mathcal{I}$-spaces and obtain their properties and characterizations.

Definition 5.1

A space X is called a T$\breve{\theta}$-$\mathcal{I}$-space if every $\breve{\theta}$-$\mathcal{I}$-cld set in it is ⋆-cld.

Example 5.2

Let X = {p, q, r} with τ = {ϕ, {q}, X} and I ={ϕ}. Then $\breve{\theta}$-$\mathcal{I}$-C(X) = {ϕ, {p, r}, X}. Thus X is a T$\breve{\theta}$-$\mathcal{I}$-space.

Example 5.3

Let X = {p, q, r} with τ = {ϕ, {p, r}, X} and I = {ϕ}. Then $\breve{\theta}$-$\mathcal{I}$-C(X) = {ϕ, {q}, {p, q}, {q, r}, X}. Thus X is not a T$\breve{\theta}$-$\mathcal{I}$-space.

Proposition 5.4

Every $T_{1/2}$-space is T$\breve{\theta}$-$\mathcal{I}$-space but not conversely.

Proof

Follows from Definition 2.3 (i).

The converse of Proposition 5.4 need not be true as seen from the following example.

Example 5.5

In the Example 5.2. Then we have G C(X) = {ϕ, {p}, {r}, {p, q}, {p, r}, {q, r}, X}. Thus X is not a $T_{1/2}$-space.

Proposition 5.6

Every Tω-space is T$\breve{\theta}$-$\mathcal{I}$-space but not conversely.

Proof

Follows from Definition2.3(iv).

The converse of Proposition 5.6 need not be true as seen from the following example.

Example 5.7

Let X = {p, q, r} with τ = {ϕ, {q}, {p, r}, X} and I = {ϕ}. Then ωC(X) = P(X) and$\breve{\theta}$-$\mathcal{I}$-C(X) = {ϕ, {q}, {p, r}, X}. Thus X is T$\breve{\theta}$-$\mathcal{I}$-space but not a Tω-space.

Proposition 5.8

Every α T_b-space is T$\breve{\theta}$-$\mathcal{I}$-space but not conversely.

Proof

Follows from Definiton 2.3(iii).

The converse of Proposition 5.8 need not be true as seen from the following example.

Example 5.9

In the Example 5.2. Then we have αG C(X) = {ϕ, {p}, {r}, {p, q}, {p, r}, {q, r}, X}. Thus X is T$\breve{\theta}$-$\mathcal{J}$-space but X is not a α T_b-space.

Proposition 5.10

Every T_b-space is T$\breve{\theta}$-$\mathcal{J}$-space but not conversely.

Proof

Follows from Definition 2.3(ii).

The converse of Proposition 5.10 need not be true as seen from the following example.

Example 5.11

In the Example 5.2. Then we have GS C(X) = {ϕ, {p}, {r}, {p, q}, {p, r}, {q, r}, X}. Thus X is not a T_b-space.

Remark 5.12

We conclude from the next two examples that T$\breve{\theta}$-$\mathcal{J}$-spaces and α -spaces are independent.

Example 5.13

In the Example 5.2. Then we have α C(X) = {ϕ, {p}, {r}, {p, r}, X}. Thus X is a T$\breve{\theta}$-$\mathcal{J}$-space but not an α -space.

Example 5.14

In the Example 5.3. Then we have α C(X) = {ϕ, {q}, X}. Thus X is an α -space but not a T$\breve{\theta}$-$\mathcal{J}$-space.

Theorem 5.15

For a space X, the following properties are equivalent:

(i) X is a T$\breve{\theta}$-$\mathcal{J}$-space.

(ii) Every singleton subset of X is either sg-cld or open.

Proof

(i) ⇒ (ii). Assume that for some x ∈ X, the set {x} is not a sg-cld in X. Then the only sg-open set containing $\{x\}^c$ is X and so $\{x\}^c$ is $\breve{\theta}$-$\mathcal{J}$-cld in X. By assumption $\{x\}^c$ is⋆-cld in X or equivalently {x} is open.

(ii) $\Rightarrow$ (i). Let K be a $\breve{\theta}$-$\mathcal{I}$-cld subset of X and let $x \in K^*$. By assumption {x} is either sg-cld or open.

Case (a) Suppose that {x} is sg-cld. If $x \notin K$, then K^*-K contains a nonempty sg-cld set {x}, which is a contradiction. Therefore $x \in K$.

Case (b) Suppose that {x} is open. Since $x \in K^*$, $\{x\} \cap K \neq \phi$ and so $x \in K$. Thus in both case, $x \in K$ and therefore $K^* \subseteq K$ or equivalently K is a $\star$-cld set of X.

Theorem 5.16

For a space X, the following properties hold:

(i) If X is sg-T_1, then it is $T\breve{\theta}$-$\mathcal{I}$.

(ii) If X is $T\breve{\theta}$-$\mathcal{I}$, then it is sg-T_0 .

Proof

(i) The proof is obvious from Proposition 3.11.

(ii) Let x and y be two distinct elements of X. Since the space X is $T\breve{\theta}$-$\mathcal{I}$, we have that {x} is sg-cld or open. Suppose that {x} is open. Then the singleton {x} is a sg-open set such that $x \in \{x\}$and $y \notin \{x\}$. Also, if {x} is sg-cld, then $X \setminus \{x\}$ is sg-open such that $y \in X \setminus \{x\}$ and $x \notin X \setminus \{x\}$. Thus, in the above two cases, there exists a sg-open set U of X such that $x \in U$ and $y \notin U$ or $x \notin U$ and $y \in U$. Thus, the space X is sg-T_0 .

Theorem 5.17

For a sg-R_0ideal topological space X, the following properties are equivalent:

(i) X is sg-T_0.

(ii) X is$T\breve{\theta}$-$\mathcal{I}$.

(iii) X is sg-T_1.

Proof

It suffices to prove only (i) $\Rightarrow$ (iii). Let $x \neq y$ and since X is sg-T_0, we may assume that $x \in U \subseteq X \setminus \{y\}$ for some sg-open set U. Then $x \in X \setminus$ sg-cl({y}) and $X \setminus$ sg-cl({y}) is sg-open. Since X is sg-R_0, we have sg-cl({x}) $\subseteq X \setminus$ sg-cl({y}) $\subseteq X \setminus \{y\}$ and hence $y \notin$ sg-cl({x}). There exists sg-open set V such that $y \in V \subseteq X \setminus \{x\}$ and X is sg-T_1.

Definition 5.18

A space X is called a $_gT\breve{\theta}$-$\mathcal{I}$-space if every g-cld set in it is $\breve{\theta}$-$\mathcal{I}$-cld.

Example 5.19

In the Example 5.3. Then X is a $_gT\breve{\theta}$-$\mathcal{I}$-space and the space X in the Example 5.2 is not a $_gT\breve{\theta}$-$\mathcal{I}$-space.

Proposition 5.20

Every $T_{1/2}$-space is ${}_gT\breve{\theta}$-$\mathcal{J}$-space but not conversely.

Proof

Follows from Definition 2.3 (i).

The converse of Proposition 5.20 need not be true as seen from the following example.

Example 5.21

In the Example 5.3. Then X is a ${}_gT\breve{\theta}$-$\mathcal{J}$-space but not a $T_{1/2}$-space.

Remark 5.22

$T\breve{\theta}$-$\mathcal{J}$-spaces and ${}_gT\breve{\theta}$-$\mathcal{J}$-spaces are independent.

Example 5.23

In the Example 5.3 is a ${}_gT\breve{\theta}$-$\mathcal{J}$-space but not a $T\breve{\theta}$-$\mathcal{J}$-space and the space X in the Example 5.2 is a $T\breve{\theta}$-$\mathcal{J}$-space but not a ${}_gT\breve{\theta}$-$\mathcal{J}$-space.

Theorem 5.24

If X is a ${}_gT\breve{\theta}$-$\mathcal{J}$-space, then every singleton subset of X is either g-cld or $\breve{\theta}$-$\mathcal{J}$-open.

Proof

Assume that for some $x \in X$, the set $\{x\}$ is not a g-cld in X. Then $\{x\}$ is not a $\star$-cld set, since every $\star$-cld set is a g-cld set. So $\{x\}^c$ is not open and the only open set containing $\{x\}^c$ is X itself. Therefore $\{x\}^c$ is trivially a g-cld set and by assumption, $\{x\}^c$ is a $\breve{\theta}$-$\mathcal{J}$-cld set or equivalently, $\{x\}$ is $\breve{\theta}$-$\mathcal{J}$-open.

The converse of Theorem 5.24 need not be true as seen from the following example.

Example 5.25

In the Example 5.2. Then the sets $\{p\}$ and $\{r\}$ are g-cld in X and the set $\{q\}$ is $\breve{\theta}$-$\mathcal{J}$-open. But the space X is not a ${}_gT\breve{\theta}$-$\mathcal{J}$-space.

Theorem 5.26

A space X is $T_{1/2}$ if and only if it is both $T\breve{\theta}$-$\mathcal{J}$ and ${}_gT\breve{\theta}$-$\mathcal{J}$.

Proof

Necessity. Follows from Definition 2.3 (i) and Propositions 5.20.

Sufficiency. Assume that X is both $T\breve{\theta}$-$\mathcal{J}$ and ${}_gT\breve{\theta}$-$\mathcal{J}$. Let K be a g-cld set of X. Then K is $\breve{\theta}$-$\mathcal{J}$-cld, since X is a ${}_gT\breve{\theta}$-$\mathcal{J}$. Again since X is a $T\breve{\theta}$-$\mathcal{J}$, K is a cld set in X and so X is a $T_{1/2}$.

Definition 5.27

A space X is called a $\alpha\, T\breve{\theta}$-$\mathcal{J}$-space if every α g-cld set in it is $\breve{\theta}$-$\mathcal{J}$-cld.

Example 5.28

In the Example 5.3. Then X is a α T$\breve{\theta}$-$\mathcal{J}$-space and the space X in the Example 5.2 is not a α T$\breve{\theta}$-$\mathcal{J}$-space.

Proposition 5.29

Every α T_b-space is α T$\breve{\theta}$-$\mathcal{J}$-space but not conversely.

Proof

Follows from Definition 2.3 (iii).

The converse of Proposition 5.29 need not be true as seen from the following example.

Example 5.30

In the Example 5.3. Then X is a α T$\breve{\theta}$-$\mathcal{J}$-space but not a α T_b-space.

Proposition 5.31

Every α T$\breve{\theta}$-$\mathcal{J}$-space is a α T_d-space but not conversely.

Proof

Let Xbe an α T$\breve{\theta}$-$\mathcal{J}$-space and let K be an α g-cld set of X. Then K is a $\breve{\theta}$-$\mathcal{J}$-cld subset of X and by Remark 5.12, A is g-cld. Therefore X is an α T_d-space.

The converse of Proposition 5.31 need not be true as seen from the following example.

Example 5.32

In the Example 5.3. Then X is a α T_d-space but not a α T$\breve{\theta}$-$\mathcal{J}$-space.

Theorem 5.33

If X is a α T$\breve{\theta}$-$\mathcal{J}$-space, then every singleton subset of X is either α g-cld or $\breve{\theta}$-$\mathcal{J}$-open.

Proof

Similar to Theorem 5.24.

The converse of Theorem 5.33 need not be true as seen from the following example.

Example 5.34

In the Example 5.2. Then the sets {p} and {r} are α g-cld in X and the set {q} is $\breve{\theta}$-$\mathcal{J}$-open. But the space X is not a α T$\breve{\theta}$-$\mathcal{J}$-space.

Proposition 5.35

Every $*_sT_p$-space and T_p*-space is T$\breve{\theta}$-$\mathcal{J}$-space but not conversely.

Proof

Follows from Definition 2.3 (vi) and (v).

The converse of Proposition 5.35 need not be true as seen from the following example.

Example 5.36

In the Example 5.2. Then we have $GSP\,C(X) = \{\phi, \{p\}, \{r\}, \{p, q\}, \{p, r\}, \{q, r\}, X\}$ and $G^{*}P\,C(X) = \{\phi, \{p\}, \{r\}, \{p, r\}, \{q, r\}, X\}$. Thus Xis neither a $*_{S}T_{p}$-space nor a $T_{p}*$-space. Then X is $T\breve{\theta}$-$\mathcal{I}$-space.

REFERENCES

[1] A. Acikgoz, T. Noiri and S. Yuksel, On α-I-continuous and α-I-open functions, Acta Math.Hungar., 105(1-2)(2004), 27-37.

[2] R. Devi, K. Balachandran and H. Maki, Semi-generalized closed maps and generalized semi-closed maps, Mem. Fac. Kochi Univ. Ser. A.Math.,14(1993), 41-54.

[3] A. I. El-Maghrabi, and A. A. Nasef, Between semi-closed and GS-closed sets, Journal of Taibah University for Science, 2(2009), 78-87.

[4] T. R. Hamlett and D. Jonkovic, Ideals in General Topology, Lecture notes in pure and Appl. Math., 123(1990), 115-125.

[5] D. Jankovic and T. R. Hamlett, New topologies from old via ideals, Amer. Math. Monthly, 97(4)(1990), 295-310.

[6] K. Kuratowski, Topology, Vol. I. New York, Academic Press (1966).

[7] S. Jafari and N. Rajesh, Generalized closed sets with respect to an ideal, European J. Pure Appl. Math., 4(2)(2011), 147-151.

[8] N. Levine, Generalized closed sets in topology, Rend. Circ Mat. Palermo (2), 19(1970), 89-96.

[9] H. Maki, R. Devi and K. Balachandran, Associated topologies of generalized α-closed sets and α-generalized closed sets, Mem. Fac. Sci. Kochi Univ. Ser. A. Math., 15(1994), 51-63.

[10] K. Nono, R. Devi, M. Devipriya, K. Muthukumarasamy and H. Maki, On g#α-closed sets and the digital plane, Bull. Fukuoka Univ. Ed. part III, 53(2004), 15-24.

[11] O. Njastad, On some classes of nearly open sets, Pacific J. Math., 15 (1965), 961-970.

[12] N. Palaniappan and K. C. Rao, Regular generalized closed sets, Kyungpook Math. J., 33(1993),211-219.

[13] V. Renukadevi, Note on IR-closed and AIR-sets, Acta Math. Hungar., 122(4)(2009), 329-338.

[14] Stone, M. H.: Applications of the theory of Boolean rings to general topology, Trans Amer. Math. Soc., 41 (1937), 374-381.

[15] P. Sundaram and M. Rajamani, Some decompositions of regular generalized continuous maps in topological spaces, Far East J. Math. Sci., special volume, Part II, (2000), 179-188.

[16] M. K. R. S. Veera Kumar, g*-preclosed sets, Acta Ciencia Indica, Vol. XXVIIIM, (1) (2002), 51-60.

[17] M. K. R. S. Veerakumar, On $\hat{g}$ -closed sets in topological spaces, Bull. Allah. Math. Soc.,18(2003), 99-112.

[18] M. K. R. S. Veerakumar, Between closed sets and g-closed sets, Mem. Fac. Sci. Kochi. Univ.Ser. A. Math., 21(2000), 1-19.

[19] K. Viswanathan and J. Jayasutha, Some decompositions of continuity in ideal topological spaces, Eur. J. Math. Sci., 1(1)(2012), 131-141.