# INTUITIONISTIC FUZZY METRIC SPACE AND SUB-SEQUENTIAL CONTINUOUS MAPPINGS

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## ABSTRACT

**Abstract :** The present paper deals with common fixed point theorems in intuitionistic fuzzy metric space employing the notion of sub-sequentially continuity. Our result extends and improves recent results of Singh & Jain [19] in the sense that all maps involved in the theorems are discontinuous even at common fixed point.

**Keywords :** Common fixed point, intuitionistic fuzzy metric space, weakly compatible mappings, semi-compatible mappings, sub-sequentially continuous mappings.

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## 1. INTRODUCTION

After Zadeh [21] introduced the concept of fuzzy sets in 1965, many authors have extensively developed the theory of fuzzy sets and its applications. Specially to mention, fuzzy metric spaces were introduced by Deng [4], Erceg [7], Kaleva and Seikkala [12], Kramosil and Michalek [13]. In this paper we use the concept of fuzzy metric space introduced by Kramosil and Michalek [13] and modified by George and Veeramani [8] to obtain Hausdorff topology for this kind of fuzzy metric space. Recently Singh et. al. [19] introduced the notion of semi-compatible maps in fuzzy metric space and compared this notion with the notion of compatible map, compatible map of type $(\alpha)$, compatible map of type $(\beta)$ and obtain some fixed point theorems in complete fuzzy metric space in the sense of Grabiec [6].

Atanassov [2] introduced and studied the concept of intuitionistic fuzzy set. In 2004, the notion of intuitionistic fuzzy metric space defined by Park [18] is a generalization of fuzzy metric space due to George and Veeramani [8]. Actually, Park's concept is useful in modeling some phenomena where it is necessary to study relationship between two probability functions. It has a direct physics motivation in the context of the two slit experiment as foundation of E-infinity of high energy physics, recently studied by El Naschie in [5, 6]. Afterwards, using the idea of Intuitionistic Fuzzy set, Alaca et al. [1] defined the notion of Intuitionistic Fuzzy Metric space, as Park [18] with the help of continuous t–norms and continuous t–conorms, as a generalization of fuzzy metric space due to Kramosil and Michalek [14]. Further Coker [3], Turkoglu [20] and others have been expansively developed the theory of Intuitionistic Fuzzy set and applications. After generalizing the Jungck's [11] common fixed point theorem in intuitionistic fuzzy metric, Turkoglu et al. [20] introduced the notion of Cauchy sequences in intuitionistic fuzzy metric space

space and proved the intuitionistic fuzzy version of Pant's theorem [17] by giving the definition of weakly commuting and R-weakly commuting mappings in intuitionistic fuzzy metric space.

In the present paper we prove fixed point theorems in complete intuitionistic fuzzy metric space by replacing continuity condition with a weaker condition called subsequential continuity. Employing the notion of subsequential continuity of mappings we can widen the scope of many interesting fixed point theorems in intuitionistic fuzzy metric space.

For the sake of completeness, we recall some definitions and known results in Fuzzy metric space.

## 2. PRELIMINARIES

**Definition 2.1.** [1] A binary operation $*$ : $[0,1] \times [0, 1] \to [0,1]$ is continuous t-norm if $*$ satisfies the following conditions :

(i) $*$ is commutative and associative;

(ii) $*$ is continuous;

(iii) $a * 1 = a$ for all $a \in [0,1]$;

(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all a, b, c, d $\in [0,1]$.

Examples of t-norm are $a * b = \min\{a, b\}$ and $a * b = ab$.

**Definition 2.2.** [1] A binary operation $\Diamond$: $[0,1] \times [0,1] \to [0,1]$ is continuous t-conorm if $\Diamond$ satisfies the following conditions :

(i) $\Diamond$ is commutative and associative;

(ii) $\Diamond$ is continuous;

(iii) $a \Diamond 0 = a$ for all $a \in [0,1]$;

(iv) $a \Diamond b \leq c \Diamond d$ whenever $a \leq c$ and $b \leq d$

for all a, b, c, d $\in [0,1]$.

Examples of t-conorm are $a \Diamond b = \max\{a, b\}$ and $a \Diamond b = \min\{1, a+b\}$.

**Remark 2.1.** [1] The concepts of triangular norms (t-norms) and triangular co-norms (t-conorms) are known as axiomatic skeletons that we use for characterizing fuzzy intersections and unions respectively.

**Definition 2.3.** [1] A 5-tuple $(X, M, N, *, \Diamond)$ is said to be an intuitionistic fuzzy metric space (IFM-space) if X is an arbitrary set, * is a continuous t-norm, $\Diamond$ is a continuous t-conorm and M, N are fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions :

(i) $M(x, y, t) + N(x, y, t) \leq 1$ for all $x, y \in X$ and $t > 0$;

(ii) $M(x, y, 0) = 0$ for all $x, y \in X$;

(iii) $M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;

(iv) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t > 0$;

(v) $M(x, y, t) * M(y, z, s) \leq M(x, z, t+s)$ for all $x, y, z \in X$ and $s, t > 0$;

(vi) for all $x, y \in X$, $M(x, y, .) : [0, \infty) \to [0,1]$ is left continuous;

(vii) $\lim_{t \to \infty} M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$;

(viii) $N(x, y, 0) = 1$ for all $x, y \in X$;

(ix) $N(x, y, t) = 0$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;

(x) $N(x, y, t) = N(y, x, t)$ for all $x, y \in X$ and $t > 0$;

(xi) $N(x, y, t) \lozenge N(y, z, s) \geq N(x, z, t + s)$ for all $x, y, z \in X$ and $s, t > 0$;

(xii) for all $x, y \in X$, $N(x, y, .) : [0,\infty) \to [0,1]$ is right continuous ;

(xiii) $\lim_{t\to\infty} N(x, y, t) = 0$ for all x, y in X.

Then (M, N) is called an intuitionistic fuzzy metric on X. The functions M(x,y,t) and N(x,y,t) denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

**Remark 2.2.** [1] Every fuzzy metric space $(X, M, *)$ is an intuitionistic fuzzy metric space of the form $(X, M, 1 - M, *, \lozenge)$ such that t-norm * and t-conorm $\lozenge$ are associated, i.e,

$$x \lozenge y = 1 - [(1 - x) * (1 - y)] \quad \text{for all } x, y \in X.$$

**Example 2.1.** [1] Let (X, d) be a metric space. Define t-norm $a * b = \min\{a,b\}$ and t -conorm $a \lozenge b = \max\{a, b\}$ and for all $x, y \in X$ and $t > 0$,

$$M_d(x,y,t) = \frac{t}{t + d(x,y)}, \quad N_d(x,y,t) = \frac{d(x,y)}{t + d(x,y)}.$$

Then $(X, M, N, *, \lozenge)$ is an IFM-space and the intuitionistic fuzzy metric (M, N) induced by the metric d is often referred to as the standard intuitionistic fuzzy metric.

**Remark 2.3.** [1] In intuitionistic fuzzy metric space $(X, M, N, *, \lozenge)$, M(x, y, .) is non-decreasing and N(x, y, .) is non-increasing for all $x, y \in X$.

**Definition 2.4.** [1] Let $(X, M, N, *, \lozenge)$ be an intuitionistic fuzzy metric space. Then

(a) a sequence $\{x_n\}$ in X is said to be Cauchy sequence if, for all $t > 0$ and $p > 0$,

$$\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1 \quad \text{and} \quad \lim_{n\to\infty} N(x_{n+p}, x_n, t) = 0.$$

(b) a sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, for all $t > 0$,

$$\lim_{n\to\infty} M(x_n, x, t) = 1 \quad \text{and} \quad \lim_{n\to\infty} N(x_n, x, t) = 0.$$

Since $*$ and $\lozenge$ are continuous, the limit is uniquely determined from (v) and (xi) respectively.

**Definition 2.5.** [1] An intuitionistic fuzzy metric space $(X, M, N, *, \lozenge)$ is said to be complete if and only if every Cauchy sequence in X is convergent.

**Lemma 2.1.** [2] Let $(X, M, N, *, \lozenge)$ be an intuitionistic fuzzy metric space. If there exists $k \in (0,1)$ such that

$$M(x,y,kt) \geq M(x,y,t) \text{ and } N(x,y,kt) \leq N(x,y,t) \text{ for } x, y \in X.$$

Then $x = y$.

**Definition 2.6.**[1] Two maps A and B from an intuitionistic fuzzy metric space $(X, M, N, *, \lozenge)$ into itself are said to be compatible if

$$\lim_{n\to\infty} M(ABx_n, BAx_n, t) = 1 \quad \text{and} \quad \lim_{n\to\infty} N(ABx_n, BAx_n, t) = 0$$

for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x \quad \text{for some } x \in X.$$

**Definition 2.7.** [15] Two maps A and B from an intuitionistic fuzzy metric space (X, M, N, *, ◊) into itself are said to weakly compatible if they commute at their coincidence points.

**Remark 2.4.** [15] Weak compatible maps are more general than compatible maps.

Motivated by [19], we define the following :

**Definition 2.6.**[1] Two maps A and B from an intuitionistic fuzzy metric space (X, M, N, ∗, ◊) into itself are said to be semi-compatible if

$$\lim_{n\to\infty} M(ABx_n, Bx, t) = 1 \quad \text{and} \quad \lim_{n\to\infty} N(ABx_n, Bx, t) = 0$$

for all t > 0, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x \quad \text{for some } x \in X.$$

**Definition 2.8.** Two self maps A and S of an intuitionistic fuzzy metric space are called reciprocal continuous if $\lim_{n\to\infty} ASx_n = At$ and $\lim_{n\to\infty} SAx_n = St$ for some t in X whenever $\{x_n\}$is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t.$$

**Definition 2.9.** [16] Two self maps A and S of an intuitionistic fuzzy metric space are said to be subsequentially continuous if and only if there exists a sequence $\{x_n\}$ in X such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some t in X and satisfy

$\lim_{n\to\infty} ASx_n = At$ and $\lim_{n\to\infty} SAx_n = St$.

Clearly, if A and S are continuous then they are obviously sub-sequentially continuous. The next example shows that there exist sub-sequential continuous pairs of mappings which are neither continuous nor reciprocally continuous.

**Example 2.2.** Let X = R, endowed with metric d. Define t-norm a ∗ b = min{a,b} and t -conorm a ◊ b = max{a, b} and for all x, y ∈ X and t > 0,

$$M_d(x,y,t) = \frac{t}{t+d(x,y)}, \quad N_d(x,y,t) = \frac{d(x,y)}{t+d(x,y)}.$$

Define the self mappings A, S as follows :

$$A(x) = \begin{Bmatrix} 2, & x < 3 \\ x, & x \geq 3 \end{Bmatrix} \quad \text{and} \quad S(x) = \begin{Bmatrix} 2x-4, & x \leq 3 \\ 3, & x > 3 \end{Bmatrix}.$$

Consider a sequence $x_n = 3 + \frac{1}{n}$ ; then,

$$A(x_n) = \left(3 + \frac{1}{n}\right) \to 3, S(x_n) = 3, SA(x_n) = S\left(3 + \frac{1}{n}\right) = 3 \neq S(3) = 2, \text{ as } n \to \infty.$$

Thus A and S are not reciprocally continuous but, if we consider a sequence $x_n = 3 - \frac{1}{n}$, then

$$A(x_n) = 2, S(x_n) = 2\left(3 - \frac{1}{n}\right) - 4 = \left(2 - \frac{2}{n}\right) = 2 \text{ as } n \to \infty$$

$$AS(x_n) = A\left(2 - \frac{2}{n}\right) = 2 = A(2), SA(x_n) = S(2) = 0 = S(2) \text{as } n \to \infty$$

Therefore, A and S are sub sequentially continuous.

**Remark 2.5.** [10] If A and S are continuous or reciprocally continuous then they are obviously sub sequentially continuous, but converse is not true.

## 3. MAIN RESULTS.

In the following theorem we replace the continuity condition by weaker notion sub-sequential continuous to get more general form of result 4.1, 4.2 and 4.9 of [19] in intuitionistic fuzzy metric space.

**Theorem 3.1.** Let A, B , S and T be self maps on a complete intuitionistic fuzzy metric space (X, M, N, $*$, $\Diamond$) where * is a continuous t-norm and $\Diamond$ is a continuous t –conorm satisfying :

(3.1) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$;

(3.2) (B, T) is weak compatible;

(3.3) for all $x, y \in X$ and $t > 0$, $M(Ax, By, t) \geq \Phi(M(Sx, Ty, t))$,

$$N(Ax, By, t) \leq \Phi(N(Sx, Ty, t)),$$

where $\Phi : [0,1] \to [0, 1]$ is a continuous function such that

$\Phi(1) = 1$, $\Phi(0) = 0$ and $\Phi(a) > a$ for each $0 < a < 1$.

If (A, S) is semi-compatible pair of sub-sequential continuous maps then A, B, S and T have a unique common fixed point.

**Proof.** Let $x_0 \in X$ be any arbitrary point. Then for which there exists $x_1, x_2 \in X$ such that $Ax_0 = Tx_1$ and $Bx_1 = Sx_2$. Thus we can construct a sequences $\{y_n\}$ and $\{x_n\}$ in X such that

$$y_{2n+1} = Ax_{2n} = Tx_{2n+1},\ y_{2n+2} = Bx_{2n+1} = Sx_{2n+2} \text{ for } n = 0, 1, 2, 3, \ldots .$$

By contractive condition, we get

$$\begin{aligned} M(y_{2n+1}, y_{2n+2}, t) &= M(Ax_{2n}, Bx_{2n+1}, t) \\ &\geq \Phi(M(Sx_{2n}, Tx_{2n+1}, t)) \\ &= \Phi(M(y_{2n}, y_{2n+1}, t)) \\ &> M(y_{2n}, y_{2n+1}, t) \text{ and} \end{aligned}$$

$$\begin{aligned} N(y_{2n+1}, y_{2n+2}, t) &= N(Ax_{2n}, Bx_{2n+1}, t) \\ &\leq \Phi(N(Sx_{2n}, Tx_{2n+1}, t)) \\ &= \Phi(N(y_{2n}, y_{2n+1}, t)) \\ &< N(y_{2n}, y_{2n+1}, t). \end{aligned}$$

Similarly, we get

$$M(y_{2n+2}, y_{2n+3}, t) > M(y_{2n+1}, y_{2n+2}, t) \text{ and}$$

$$N(y_{2n+2}, y_{2n+3}, t) < N(y_{2n+1}, y_{2n+2}, t).$$

In general,

$$\begin{aligned} M(y_{n+1}, y_n, t) &\geq \Phi(M(y_n, y_{n-1}, t)) \\ &> M(y_n, y_{n-1}, t) \text{ and} \end{aligned}$$

$$N(y_{n+1}, y_n, t) \leq \Phi(N(y_n, y_{n-1}, t))$$

$< N(y_n, y_{n-1}, t)$.

We claim that $l = 1$.

If $l < 1$ then $M(y_{n+1}, y_n, t) \geq M(y_n, y_{n-1}, t)$ and

$$N(y_{n+1}, y_n, t) \leq M(y_n, y_{n-1}, t).$$

On letting $n \to \infty$, we get

$$\lim_{n\to\infty} M(y_{n+1}, y_n, t) \geq \Phi(\lim_{n\to\infty} M(y_n, y_{n-1}, t)) \text{ and}$$

$$\lim_{n\to\infty} N(y_{n+1}, y_n, t) \leq \Phi(\lim_{n\to\infty} N(y_n, y_{n-1}, t))$$

i.e. $l \geq \Phi(l) = l$, a contradiction.

Now for any positive integer p ,

$$M(y_n, y_{n+p}, t) \geq M(y_n, y_{n+1}, t/p) * M(y_{n+1}, y_{n+2}, t/p) * \ldots$$

$$* M(y_{n+p-1}, y_{n+p}, t/p) \text{ and}$$

$$N(y_n, y_{n+p}, t) < N(y_n, y_{n+1}, t/p) * N(y_{n+1}, y_{n+2}, t/p) * \ldots$$

$$* N(y_{n+p-1}, y_{n+p}, t/p).$$

Letting $n \to \infty$, we get

$$\lim_{n\to\infty} M(y_n, y_{n+p}, t) \geq 1 * 1 * 1 * \ldots * 1 = 1 \text{ and}$$

$$\lim_{n\to\infty} N(y_n, y_{n+p}, t) \leq 1 * 1 * 1 * \ldots * 1 = 1.$$

Thus,

$$\lim_{n\to\infty} M(y_n, y_{n+p}, t) = 1 \text{ and}$$

$$\lim_{n\to\infty} N(y_n, y_{n+p}, t) = 1.$$

Thus $\{y_n\}$ is a Cauchy sequence in X. Since X is complete metric space $\{y_n\}$ converges to a point z (say) in X. Hence the subsequences $\{Ax_{2n}\}$, $\{Sx_{2n}\}$, $\{Tx_{2n+1}\}$ and $\{Bx_{2n+1}\}$ also converge to z.

Now since A and S are sub-sequential continuous and semi-compatible then we have $\lim_{n\to\infty} ASx_{2n} = Az$, $\lim_{n\to\infty} SAx_{2n} = Sz$ and

$$\lim_{n\to\infty} M(ASx_{2n}, Sz, t) = 1 \text{ and}$$

$$\lim_{n\to\infty} N(ASx_{2n}, Sz, t) = 0.$$

Therefore, we get $Az = Sz$. Now we will show $Az = z$. For this suppose $Az \neq z$. Then by contractive condition, we get

$$M(Az, Bx_{2n+1}, t) \geq \Phi(M(Sz, Tx_{2n+1}, t)) \text{ and}$$

$$N(Az, Bx_{2n+1}, t) < \Phi(N(Sz, Tx_{2n+1}, t)).$$

Letting $n \to \infty$, we get

$$M(Az, z, t) \geq \Phi(M(Az, z, t)) > M(Az, z, t) \text{ and}$$

$$N(Az, z, t) \le \Phi(N(Az, z, t)) < N(Az, z, t),$$

a contradiction, thus $z = Az = Sz$. Since $A(X) \subseteq T(X)$, there exists $u \in X$ such that $z = Az = Tu$. Putting $x = x_{2n}$, $y = u$ in (3), we get

$$M(Ax_{2n}, Bu, t) \ge \Phi(M(Sx_{2n}, Tu, t)) \text{ and}$$

$$N(Ax_{2n}, Bu, t) \le \Phi(N(Sx_{2n}, Tu, t)).$$

Letting $n \to \infty$, we get

$$M(z, Bu, t) \ge \Phi(M(z, z, t)) = \Phi(1) = 1 \text{ and}$$

$$N(z, Bu, t) \le \Phi(N(z, z, t)) = \Phi(1) = 1,$$

i.e. $z = Bu = Tu$ and the weak-compatibility of (B, T) gives $TBu = BTu$, i.e. $Tz = Bz$. Again by contractive condition and assuming $Az \ne Bz$, we get $Az = Bz = z$. Hence, finally we get

$z = Az = Bz = Sz = Tz$, i.e. $z$ is a common fixed point of A, B, S and T. The uniqueness follows from contractive condition. This completes the proof.

Now we prove an another common fixed point theorem with different contractive condition :

**Theorem 3.2.** Let A, B, S and T be self maps on a complete fuzzy metric space $(X, M, N, *, \Diamond)$ where $*$ is a continuous t-norm and $\Diamond$ is a continuous t –conorm satisfying:

(3.4) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$,

(3.5) (B, T) is weak compatible,

(3.6) for all $x, y \in X$ and $t > 0$,

$$M(Ax, By, t) \ge \Phi\{\min(M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), M(Ax, Ty, t))\} \text{ and}$$

$$N(Ax, By, t) \le \Phi\{\max(N(Sx, Ty, t), N(Ax, Sx, t), N(By, Ty, t), N(Ax, Ty, t))\},$$

where $\Phi : [0,1] \to [0,1]$ is a continuous function such that

$$\Phi(1) = 1, \Phi(0) = 0 \text{ and } \Phi(a) > a \text{ for each } 0 < a < 1.$$

If (A, S) is semi-compatible pair of sub-sequential continuous maps then A, B, S and T have a unique common fixed point.

**Proof.** Let $x_0 \in X$ be any arbitrary point. Then for which there exists $x_1, x_2 \in X$ such that $Ax_0 = Tx_1$ and $Bx_1 = Sx_2$. Thus we can construct sequences $\{y_n\}$ and $\{x_n\}$ in X such that

$$y_{2n} = Ax_{2n} = Tx_{2n+1}, \quad y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \text{ for } n = 0, 1, 2, 3, \ldots.$$

By contractive condition, we get

$$\begin{aligned} M(y_{2n+1}, y_{2n+2}, t) &= M(Ax_{2n}, Bx_{2n+1}, t) \\ &\ge \Phi\{\min(M(Sx_{2n}, Tx_{2n+1}, t), M(Ax_{2n}, Sx_{2n}, t), \\ &\qquad M(Bx_{2n+1}, Tx_{2n+1}, t), M(Ax_{2n}, Tx_{2n+1}, t))\} \\ &= \Phi\{\min(M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n-1}, t), \\ &\qquad M(y_{2n+1}, y_{2n}, t), M(y_{2n}, y_{2n}, t))\} \\ &= \Phi\{\min(M(y_{2n-1}, y_{2n}, t), M(y_{2n+1}, y_{2n}, t))\} \end{aligned}$$

$$= \Phi\{M(y_{2n-1}, y_{2n}, t)\}$$

$$> M(y_{2n-1}, y_{2n}, t) \text{ and}$$

$$N(y_{2n+1}, y_{2n+2}, t) = N(Ax_{2n}, Bx_{2n+1}, t)$$

$$\leq \Phi\{\max(N(Sx_{2n}, Tx_{2n+1}, t), N(Ax_{2n}, Sx_{2n}, t),$$

$$N(Bx_{2n+1}, Tx_{2n+1}, t), N(Ax_{2n}, Tx_{2n+1}, t))\}$$

$$= \Phi\{\max(N(y_{2n-1}, y_{2n}, t), N(y_{2n}, y_{2n-1}, t),$$

$$N(y_{2n+1}, y_{2n}, t), N(y_{2n}, y_{2n}, t))\}$$

$$= \Phi\{\max(N(y_{2n-1}, y_{2n}, t), N(y_{2n+1}, y_{2n}, t))\}$$

$$= \Phi\{N(y_{2n-1}, y_{2n}, t)\}$$

$$< N(y_{2n-1}, y_{2n}, t).$$

Similarly, we get

$$M(y_{2n+2}, y_{2n+3}, t) > M(y_{2n+1}, y_{2n+2}, t) \text{ and}$$

$$N(y_{2n+2}, y_{2n+3}, t) < N(y_{2n+1}, y_{2n+2}, t).$$

In general,

$$M(y_{n+1}, y_n, t) \geq \Phi(M(y_n, y_{n-1}, t)) > M(y_n, y_{n-1}, t) \text{ and}$$

$$N(y_{n+1}, y_n, t) \leq \Phi(N(y_n, y_{n-1}, t)) < N(y_n, y_{n-1}, t).$$

Then by the same technique of above theorem, we can easily show that $\{y_n\}$is a Cauchy sequence in X. Since X is complete metric space $\{y_n\}$ converges to a point z (say) in X. Hence, the subsequences $\{Ax_{2n}\}$, $\{Sx_{2n}\}$, $\{Tx_{2n+1}\}$ and $\{Bx_{2n+1}\}$ also converge to z.

Now since A and S are sub-sequential continuous and semi-compatible then we have

$$\lim_{n\to\infty} ASx_{2n} = Az, \lim_{n\to\infty} SAx_{2n} = Sz, \text{ and } \lim_{n\to\infty} M(ASx_{2n}, Sz, t) = 1 \text{ and } \lim_{n\to\infty} N(ASx_{2n}, Sz, t) = 0.$$

Therefore, we get $Az = Sz$. Now we will show $Az = z$. For this suppose $Az \neq z$. Then by (3.5), we get a contradiction, thus $Az = z$. Hence by similar techniques of above theorem, we can easily show that z is a common fixed point of A, B, S and T i.e. $z = Az = Bz = Sz = Tz$. Uniqueness of fixed point can be easily verify by contractive condition. This completes the proof.

We now give an example which not only illustrate our Theorem 2.1 but also shows that the notion of sub-sequential continuity of maps is weaker than the continuity of maps.

**Example 3.1.** Let $(X, d)$ be usual metric space where $X = [2, 20]$ with $M(x,y,t) = \dfrac{t}{t+|x-y|}$ and $N(x,y,t) = \dfrac{|x-y|}{t+|x-y|}$ for $x, y \in X$, $t > 0$. We define mappings A, B, S and T by

$$A2 = 2, \quad Ax = 3 \text{ if } x > 2$$

$$S2 = 2, \quad Sx = 6 \text{ if } x > 2$$

$$Bx = 2 \text{ if } x = 2 \text{ or } > 5, \quad Bx = 6 \text{ if } 2 < x \leq 5$$

$$Tx = 2, \; Tx = 12 \text{ if } 2 < x \leq 5, \; Tx = \frac{(x+1)}{3} \text{ if } x > 5.$$

Then A, B, S and T satisfy all the conditions of the above theorem with $\Phi(a) = \frac{7a}{(3a+4)}$ > a where a = 1/1 + d(Sx, Ty)/t and have a unique common fixed point x = 2. It may be noted that in this example A(X) = {2, 3} $\subseteq$ T(X) = [2, 7] $\cup$ {12} and B(X) = {2, 6} $\subseteq$ S(X) = {2, 6}.

Also A and S are sub-sequential continuous compatible mappings. But neither A nor S is continuous not even at fixed point x = 2. The mapping B and T are non-compatible but weak-compatible since they commute at their coincidence points. To see B and T are non-compatible, let us consider the sequence $\{x_n\}$ in X defined by $\{x_n\} = \left\{5 + \frac{1}{n}\right\}$; n ≥ 1. Then, $\lim_{n\to\infty} Tx_n = 2$, $\lim_{n\to\infty} Bx_n = 2$, $\lim_{n\to\infty} TBx_n = 2$ and $\lim_{n\to\infty} BTx_n = 6$. Hence B and T are non-compatible.

**Remark 3.1.** The maps A, B, S and T are discontinuous even at the common fixed point x = 2.

**REFERENCES**


1. Alaca, C., Turkoglu, D. and Yildiz, C., Fixed points in Intuitionistic fuzzy metric spaces, Chaos, Solitons & Fractals 29(5), (2006), 1073–1078.
2. Atanassov, K., Intuitionistic fuzzy sets, Fuzzy Sets and System 20 (1986), 87 – 96.
3. Coker, D., An introduction to intuitionistic fuzzy topological spaces, Fuzzy Sets and System 88(1997), 81–89.
4. Deng, Z. K., Fuzzy pseduo-metric spaces, J. Math. Anal. Appl. 86(1982), 191–207.
5. El Naschie, M. S., ‘t Hooft ultimate building blocks and space-time an infinite di-mensional set of transfinite discrete points, Chaos, Solitons and Fractals 25 (2005), 521-524.
6. El Naschie, M. S., The two slit experiment as the foundation of E-infinity of high energy physics, Chaos, Solitons and Fractals 25 (2005), 509-514.
7. Erceg, M. A., Metric spaces in fuzzy set theory, J. Math. Anal. Appl. 69(1979), 205–230.
8. George, A. and Veeramani, P., On some results in fuzzy metric spaces, Fuzzy Sets and System 64(1997), 395–399.
9. Grabiec, M., Fixed point in fuzzy metric spaces, Fuzzy Sets and Systems 27(1988), 385–389.
10. Jain, A., Badshah, V.H., Gupta, V.K. and Sharma, A., Fixed points for subsequential continuous mappings in fuzzy metric space, International Journal of Applied Physics and Mathematics 4(5), (2014), 424-431.
11. Jungck, G., Commuting mappings and fixed points, Amer. Math. Monthly 83(1976), 261–263.
12. Kaleva, O. and Seikkala, S., On fuzzy metric spaces, Fuzzy Sets and Systems 12(1984), 215–229.
13. Kramosil, I. and Michalek, J., Fuzzy metric statistical metric spaces, Kybernetica 11(1975), 326–334.

14. Kramosil, O. and Michalek, J., Fuzzy metric and statistical metric spaces, Kybernetica 11 (1975), 336-344.
15. Kumar, S., Vats, R.K., Singh, V. and Garg, S.k., Some Common Fixed Point Theorems in Intuitionistic Fuzzy Metric Spaces, Int. Journal of Math. Analysis, 4(26), (2010), 1255 – 1270.
16. Manro, S., Bouhadjera, H. and Singh, S., A common fixed point theorem in intuitionistic fuzzy metric space by using sub-compatible maps, Int. J. Contemp. Math. Sciences 5(55), (2010), 2699 – 2707.
17. Pant, R.P., Common fixed points of non-commuting mappings, J. Math. Anal. Appl. 188(1994), 436–440.
18. Park, J.H., Intuitionistic fuzzy metric spaces, Chaos, Solitons & Fractals 22 (2004), 1039-1046.
19. Singh, B. and Jain, S., Semi-compatible and fixed point theorems in fuzzy metric space, Chungcheong Math. Soc. 18(2005), 1–22.
20. Turkoglu, D., Alaca, C., Cho, Y.J., and Yildiz, C., Common fixed point theorems in intuitionistic fuzzy metric space, J. Appl. Math. and Computing 22 (2006), 411-424.
21. Zadeh, L. A., Fuzzy sets, Inform and Control 8 (1965), 338-353.