Fuzzy locally first and second category sets.

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Abstract: In this paper we introduce a new concept of fuzzy set theory that is fuzzy locally first category and fuzzy locally second category, fuzzy locally residual set. Several properties are also discussed with illustrate suitable examples.

Keywords: Fuzzy set, fuzzy locally open set, fuzzy locally closed set, fuzzy locally dense set, fuzzy locally nowhere dense set, fuzzy locally first category, fuzzy locally second category, fuzzy locally residual set.

1. Introduction

In order to deal with uncertainties, the idea of fuzzy sets and fuzzy set operations was introduced by L. A. Zadeh [10]. The notion of fuzzy topological space had been defined by C. L. Chang [3]. The notion of first category was introduced by R. L. Baire in 1899. The fuzzy nowhere dense set, fuzzy first and second category were introduced and studied by the authors in Dr. G. Thangaraj and Dr. G. Balasubramanian [8]. The fuzzy locally dense and fuzzy locally nowhere dense sets were introduced and studied by Dr. S. Anjalmose and A. Saravana [1]. In this paper we introduce a concept of fuzzy locally first category, fuzzy locally second category and fuzzy locally residual set. Several properties are also discussed with suitable examples.

2. Preliminaries

Definition 2.1: [3]

By a fuzzy topological space, a non - empty set X together with a fuzzy topology T (in the sense of Chang) and denote it by (X, T).

Let λ and μ be any two fuzzy sets in (X, T). Then we define $\lambda \vee \mu : X \rightarrow [0,1]$ and $\lambda \wedge \mu : X \rightarrow [0,1]$ as follows: $(\lambda \vee \mu)(x) = \text{Max}\{\lambda(x), \mu(x)\}$ and $(\lambda \wedge \mu)(x) = \text{Min}\{\lambda(x), \mu(x)\}$.
Let (X, T) be any fuzzy topological space and λ be any fuzzy set in (X, T). We define $Cl(\lambda) = \wedge\{\mu / \lambda \leq \mu, 1-\mu \in T\}$ and $int(\lambda) = \vee\{\mu / \mu \leq \lambda, \mu \in T\}$. For any fuzzy set λ in a fuzzy topological space (X, T).

Definition 2.2: [8]

A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy G_δ - set in (X, T) if $\lambda = \wedge_{i=1}^{\infty}$ (λ_i), where $\lambda_i \in T$ for $i \in I$.

Definition 2.3: [8]

A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy F_σ - set in (X, T) if $\lambda = \vee_{i=1}^{\infty}$ (λ_i), where $1\text{-}\lambda_i \in T$ for $i \in I$.

Definition 2.4: [4]

A subset β of a fuzzy topological space X is called fuzzy locally closed set if $\beta = \alpha \wedge \delta$, where α is a fuzzy- open set and δ is fuzzy-closed set.

The complement of fuzzy-locally closed set is called fuzzy-locally open set.

Definition 2.5: [5]

A fuzzy set λ in a fuzzy Topological space (X; T) is called fuzzy -dense if there exists no fuzzy -closed set in (X; T) such that $\lambda < \mu < 1$.

Definition 2.6: [8]

A fuzzy set λ in a fuzzy topological space (X,T) is called fuzzy nowhere dense if there exists no non-zero fuzzy open set μ in (X,T) such that $\mu < cl(\lambda)$. That is, int cl(λ) = 0.

Definition 2.7: [1]

A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy locally dense if there exists no fuzzy locally closed set μ in (X, T) such that $\lambda < \mu < 1$.

Definition 2.8: [1]

A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy locally nowhere dense if there exists no non-zero fuzzy locally-open set μ in (X,T) such that $\mu <$ l-cl(λ). That is, l-int l-cl(λ) = 0.

Definition 2.9: [7]

A fuzzy topological space (X, T) is called a fuzzy resolvable space if there exists a fuzzy dense set λ in (X,T) such that cl(1-λ) = 1, otherwise (X, T) is called a fuzzy irresolvable space.

Definition 2.10: [9]

A fuzzy topological space (X, T) is called a fuzzy almost resolvable space if $\vee_{i=1}^{\infty} (\lambda_i) = 1$, where the fuzzy sets (λ_i)'s in (X, T) are such that $int(\lambda_i) = 0$. Otherwise (X, T) is called a fuzzy almost irresolvable space.

Definition 2.11: [5]

A fuzzy topological space (X, T) is called a fuzzy nodec space if every fuzzy nowhere dense set in (X, T) is a fuzzy closed set in (X, T).

Definition 2.12: [6]

A fuzzy topological space (X, T) is called a fuzzy P-space if countable intersection of fuzzy open sets in (X, T) is fuzzy open. That is, every non-zero fuzzy G_δ-set in (X, T) is fuzzy open in (X, T).

Definition 2.13: [2]

A fuzzy topological space (X, T) is called a fuzzy submaximal space if for each fuzzy set λ in (X, T), such that $cl(\lambda) = 1$, then $\lambda \in T$ in (X, T).

3. Fuzzy locally first category and fuzzy locally second category.

Definition 3.1:

A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy locally first category set if λ = $V_{i=1}^{\infty}(\lambda_i)$, where λ_i's are fuzzy locally nowhere dense sets in (X, T). Any other locally fuzzy set in (X, T) is said to be fuzzy locally second category.

Example 3.1:

Let X = {a, b, c}. The fuzzy sets λ, μ, and β are defined on X as follows:

λ : X →[0,1] defined as λ(a) = 0; λ(b) = 0.2; λ(c) = 0.5, μ:X →[0,1] defined as μ(a) = 0.1; μ(b) = 0.2; μ(c) = 0.7, β : X →[0,1] defined as β(a) = 0.3; β(b) = 0.5; β(c) = 0.9.

Then T = {0, λ, μ, β, 1} is a fuzzy topology on X.

Now the fuzzy sets λ∧(1-λ) = λ , λ∧(1-μ) = γ (say), λ∧(1-β) = δ (say), μ∧(1-λ) = η(say), μ∧(1- μ) = ζ(say), μ∧(1- β) = χ(say), β∧(1- λ) = κ(say), β∧(1- μ) = ν(say), β∧(1- β) = ι(say). Therefore the fuzzy sets λ, γ, δ, η, ζ, χ, κ, ν, ι are fuzzy locally closed sets, then 1-λ, 1-γ, 1-δ, 1-η, 1-ζ, 1- χ, 1- κ, 1-ν, 1-ι are fuzzy locally open sets.

The fuzzy sets λ, γ, δ, η, ζ, χ, κ, ν, ι are fuzzy locally nowhere dense sets, since

l-int l-cl(λ) = 0, l-int l-cl(γ) = 0, l-int l-cl(δ) = 0, l-int l-cl(η) = 0, l-int l-cl(ζ) = 0, l-int l-cl(χ) = 0, l-int l-cl(κ) = 0, l-int l-cl(ν) = 0, l-int l-cl(ι) = 0.

Now [λ ∨ γ ∨ δ ∨ η ∨ ζ ∨ χ ∨ κ ∨ ν ∨ ι] = ι is fuzzy locally first category set. Any other fuzzy locally set of (X, T) is fuzzy locally second category set.

Proposition 3.1:

If λ be a fuzzy locally first category set in a fuzzy topological space (X, T), then 1-λ is called a fuzzy locally residual set in (X, T).

In Example 3.1, 1- ι is fuzzy residual set in (X, T). Since ι is fuzzy locally first category set in (X, T).

Theorem 3.1:[1]

If λ ≤ μ and μ is a fuzzy locally nowhere dense set in a fuzzy topological space (X, T), then λ is fuzzy locally nowhere dense set in (X, T).

Proposition 3.2:

If λ ≤ μ and μ is a fuzzy locally first category set in a fuzzy topological space (X, T), then λ is also a fuzzy locally first category set in (X, T).

Proof:

Let μ be a fuzzy locally first category set in (X, T). Then $\mu = \vee_{i=1}^{\infty}(\mu_i)$, where μ_i's are fuzzy locally nowhere dense set in (X, T). Now λ∧μ gives λ∧μ = λ implies that λ∧μ ≤ μ, by theorem 3.1, λ∧μ is fuzzy locally nowhere dense set in (X, T). That is $\lambda = \vee_{i=1}^{\infty}(\lambda \wedge \mu_i)$, and $\lambda \wedge \mu_i$'s are fuzzy locally nowhere dense sets in (X, T) gives λ is also a fuzzy locally first category set in (X, T).

Theorem 3.2:[1]

If λ is a fuzzy locally nowhere dense set in a fuzzy topological space (X, T), then 1 – λ is a fuzzy locally dense set in (X, T).

Proposition 3.3:

If λ be a fuzzy locally first category set in a fuzzy topological space (X, T), then $1 - \lambda = \wedge_{i=1}^{\infty}(\mu_i)$, where $cl(\mu_i) = 1$.

Proof:

Let λ be a fuzzy locally first category set in (X, T). Then $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where λ_i's are fuzzy locally nowhere dense sets in (X, T). Now $1 - \lambda = 1 - (\vee_{i=1}^{\infty}(\lambda_i)) = \wedge_{i=1}^{\infty}(1 - \lambda_i)$. Since λ_i's are fuzzy locally nowhere dense sets in (X, T). By Theorem 3.2, $1 - \lambda_i$'s are fuzzy locally dense sets in (X, T). Let us put $\mu_i = 1 - (\lambda_i)$. Then we have $1 - \lambda = \wedge_{i=1}^{\infty}(\mu_i)$ where $cl(\mu_i) = 1$.

Proposition 3.4:

If λ ≤ μ and λ is a fuzzy locally residual set in a fuzzy topological space (X, T), then μ is also a fuzzy locally residual set in (X, T).

Proof:

Let λ be a fuzzy locally residual set in (X, T). Then by proposition 3.3, $1 - \lambda = \wedge_{i=1}^{\infty}(\mu_i)$, where $cl(\mu_i) = 1$ implies that $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where λ_i' are fuzzy locally nowhere dense in (X, T). $1 - \lambda$ is a fuzzy locally first category set in (X, T). Let β = 1- λ is a fuzzy locally first category set in (X, T). Now λ ≤ μ implies that 1- β ≤ μ and hence$\beta \geq 1 - \mu$, since β is a fuzzy locally first category set in (X, T), by proposition 3.2, 1- μ is a fuzzy locally first category set in (X, T). Hence μ is a fuzzy locally residual set in (X, T).

Proposition 3.5:

If λ and μ is a Fuzzy locally nowhere dense sets in a fuzzy topological space (X, T), then λ∨μ need not be fuzzy locally nowhere dense set in (X, T). Consider the following example.

Example 3.2:

Let X = {a, b, c}. The fuzzy sets λ, μ, and γ are defined on X as follows:

λ : X →[0,1] defined as λ(a) = 0; λ(b) = 1; λ(c) = 1, μ:X →[0,1] defined as μ(a) = 1; μ(b) = 0; μ(c) = 1, γ : X →[0,1] defined as γ (a) = 1; γ (b) = 1; γ (c) = 0.

Then T = {0, λ, μ, γ, 1} is a fuzzy topology on X.

The fuzzy locally open sets in (X, T) are λ, μ, γ. Now l-int l-cl(1- λ) = 0, l-int l-cl(1- μ) = 0, l-int l-cl(1- γ) = 0. Therefore the fuzzy sets l-int l-cl[(1-λ)∨(1-μ)] ≠ 0.

Hence union of fuzzy locally nowhere dense set need not be fuzzy locally nowhere dense set.

Proposition 3.6:

If λ be a fuzzy locally first category set in a fuzzy topological space (X,T), then int(λ) is need not be empty interior.

Proof:

Let λ be a fuzzy locally first category set in (X, T). Then $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where λ_i's are fuzzy locally nowhere dense sets in (X, T). Now Int(λ) = int($\vee_{i=1}^{\infty}(\lambda_i)$), implies that Int(λ) ≠ 0. Since by proposition 3.5, union of fuzzy locally nowhere dense sets need not be fuzzy locally nowhere dense set and also interior is not equal to zero.

4. Fuzzy locally first category space and some fuzzy topological spaces.

Definition 4.1:

A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy locally first category space if $V_{i=1}^{\infty}(\lambda_i) = 1$, where λ_i's are fuzzy locally nowhere dense sets in (X, T). A topological space which is not of fuzzy locally first category space is said to be fuzzy locally second category space.

Example 4.1:

Let X = {a, b, c}. The fuzzy sets λ, μ, and γ are defined on X as follows:

λ : X →[0,1] defined as λ(a) = 0; λ(b) = 1; λ(c) = 1, μ:X →[0,1] defined as μ(a) = 1; μ(b) = 0; μ(c) = 1, γ : X →[0,1] defined as γ (a) = 1; γ (b) = 1; γ (c) = 0.

Then T = {0, λ, μ, γ, 1} is a fuzzy topology on X.

The fuzzy locally open sets in (X, T) are λ, μ, γ. Now l-int l-cl(1- λ) = 0, l-int l-cl(1- μ) = 0, l-int l-cl(1- γ) = 0. Therefore the fuzzy sets 1-λ, 1-μ, $1 - \gamma$ are fuzzy locally nowhere dense sets. Now [(1-λ)∨(1-μ)∨($1 - \gamma$)] =1, hence the topological space (X, T) is fuzzy locally first category space.

Proposition 4.1:

A fuzzy locally first category space need not be fuzzy sub maximal space.

Proof:

In example 4.1, the fuzzy topological space (X, T) is fuzzy locally first category space but not of fuzzy sub maximal space. Since the fuzzy open sets in (X, T) are not of fuzzy closed, that is cl(λ) $\neq$ 1, cl(μ) $\neq$ 1 and cl(γ) $\neq$ 1.

Hence a fuzzy locally first category space need not be fuzzy sub maximal space.

Proposition 4.2:

A fuzzy locally first category space need not be fuzzy resolvable space.

Proof:

In example 4.1, the fuzzy topological space (X, T) is fuzzy locally first category space but not of fuzzy resolvable space. Since the fuzzy sets λ, μ, γ in (X, T) are fuzzy dense but $cl(1-\lambda) \neq 1, cl(1-\ \mu)\ \neq 1, cl(1-\gamma) \neq 1$.

Hence a fuzzy locally first category space need not be fuzzy resolvable space.

Proposition 4.3:

A fuzzy locally first category space need not be fuzzy nodec space. Consider the following example.

Example 4.2:

Let X = {a, b, c}. The fuzzy sets λ, μ, and γ are defined on X as follows:

λ : X →[0,1] defined as λ(a) = 0; λ(b) = 1; λ(c) = 1, μ:X →[0,1] defined as μ(a) = 1; μ(b) = 0; μ(c) = 1, γ : X →[0,1] defined as γ (a) = 1; γ (b) = 1; γ (c) = 0. α : X → [0,1] defined as α (a) = 0.9; α (b) = 0; α (c) = 0.

Then T = {0, λ, μ, γ, 1} is a fuzzy topology on X.

The fuzzy locally open sets in (X, T) are λ, μ, γ. Now l-int l-cl(1- λ) = 0, l-int l-cl(1- μ) = 0, l-int l-cl(1- γ) = 0. Therefore the fuzzy sets 1-λ, 1-μ, $1-\gamma$ are fuzzy locally nowhere dense sets.

Now [(1-λ)∨(1-μ)∨($1-\gamma$)] =1, hence the topological space (X, T) is fuzzy locally first category space. Now int cl(α) = 0, therefore the fuzzy sets 1-λ, 1-μ, $1-\gamma$ are fuzzy nowhere dense, fuzzy closed in (X, T) but α is fuzzy nowhere dense, not a fuzzy closed in (X, T).

Hence a fuzzy locally first category space need not be fuzzy nodec space.

Proposition 4.4:

A fuzzy locally first category space need not be fuzzy P-space.

Proof:

In example 4.1, the fuzzy topological space (X, T) is fuzzy locally first category space but not of fuzzy p-space. Since the intersection of fuzzy open sets λ *and* μ are not of fuzzy open in (X, T), that is $\lambda \wedge \mu$ is not open in (X, T).

Hence a fuzzy locally first category space need not be fuzzy P-space.

Proposition 4.5:

Let (X, T) be a fuzzy locally first category space, then (X, T) is fuzzy almost resolvable space.

Proof:

In example 4.1, the fuzzy topological space (X, T) is fuzzy locally first category space also fuzzy almost resolvable space. Since int $(1-\lambda) = 0$, int $(1-\mu) = 0$, int $(1-\gamma) = 0$ and $[(1-\lambda) \vee (1-\mu) \vee (1-\gamma)] = 1$.

Hence a fuzzy locally first category space is also fuzzy almost resolvable space.

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